Dynamic equilibrium
with randomly arriving players

Pierre Bernhard,* and Marc Deschamps†

May, 30, 2019

Abstract

There are real strategic situations where nobody knows *ex ante* how many players there will be in the game at each step. Assuming that entry and exit could be modelled by random processes whose probability laws are common knowledge, we use dynamic programming and piecewise deterministic Markov decision processes to investigate such games. We study these games in discrete and continuous time for both finite and infinite horizon. While existence of dynamic equilibrium in discrete time is proved, our main aim is to develop algorithms. In the general nonlinear case, the equations provided are rather intricate. We develop more explicit algorithms for both discrete and continuous time linear quadratic problems.
KeywordsNash equilibrium, Dynamic programming, Piecewise Deterministic Markov Decision Process, Cournot oligopoly.

1 Introduction

1.1 Motivation

In the working paper [Bernhard and Deschamps(2016)], we provide several examples of game situations where an *a priori* unknown number of identical players might show up at *a priori* unknown time instants. One obvious example would be that of new competitors entering the same market. Their being identical is an over-simplification, but since the number of possible entrants is not bounded *a priori*, to get a problem with finite data, one would need to define a finite number of classes

*UCA-INRIA Sophia Antipolis, 2004 route des Lucioles, BP93, 06902 Sophia Antipolis cedex, Pierre.Bernhard@inria.fr

†Université de Bourgogne Franche Comté, CRESE EA3190, and BETA CNRS, GREDEG CNRS, OFCE-SciencesPo, France. Marc.Deschamps@univ-fcomte.fr

they may belong to, a complexity we would rather avoid at this stage. (See, e.g. [Tembine(2010)] and [Biard and Deschamps(2020)].)

1.2 Random number of players in the literature

Classical game theory models belong to the "fixed-n" paradigm and can not be applied to this kind of problems. To the best of our knowledge there exists currently in game theory three ways to manage uncertainty on the number of players in the game. First it could be modelled by assuming that there is a common knowledge on the number of potential players and a stochastic process chooses which ones will be active players (see [Levin and Ozdenoren(2004)] for example). A second way to model such games is to use population games, as Poisson games, where the number of players in the game is supposed to be drawn from a random variable, whose probability distribution is commonly known (as in [Myerson(1998b)] and [Myerson(1998a)]). Finally, a third way is to use games with a large number of players modeled as games with infinitely many players, see [Khan and Sun(2002)] for a survey, or mean field games [Lasry and Lions(2007), Caines(2015)] for instance. (Evolutionary games, which also involve an infinite number of players, are not really games where an equilibrium is sought.)

However, as we discuss in [Bernhard and Deschamps(2017b)], the kind of games we mentioned in our examples could not be analyzed with such tools. Indeed our diagnostic is that where the number of players is random, there is no time involved, and therefore no concept of entry. Typical examples are [Levin and Ozdenoren(2004)] for auction theory, and [De Sinopoli et al.(2014)] for Poisson games. Where there is a time structure in the game, the number of players is fixed, such as in stochastic games, see [Neyman and Sorin(2003)], or in generalized secretary problems, see [Ferguson(2005)]. In the literature on entry equilibrium, such as [Samuelson(1985), Breton et al.(2010)], the players are the would-be entrants, the number of which is known.

One notable exception is [Kordonis and Papavassilopoulos(2015)] which explicitly deals with a dynamic game with random entry. In this article, the authors describe a problem more complicated than ours on at least three counts: 1/ There are two types of players: a major one, an incumbent, who has an infinite horizon, and identical minor ones that enter at random and leave after a fixed time T (although the authors mention that they can also deal with the case where T is random), 2/ Each player has its own state and dynamics. Yet, the criteria of the players only depend on a mean value of these states, simplifying the analysis, and opening the way for a simplified analysis in terms of mean field in the large number of minor players case, and 3/ All the dynamics are noisy. (See however, our paragraph 2.1.3). It is simpler than ours in that it does not attempt to foray away from the discrete time, linear dynamics, quadratic payoff (L.Q.) case. Admittedly, our results in the nonlinear case are rather theoretical and remain difficult to use beyond the L.Q. case. But we do deal with the continuous time case also. Due to the added complexity, the solution proposed is much less explicit than what we offer in the linear quadratic problem. Typically, the authors solve the two maximization problems with opponents' strategies fixed and state that if the set of strategies is "consistent", i.e. solves the fixed point problem inherent in a Nash equilibrium, then it is the required equilibrium. The algorithm proposed to solve the fixed point problem is the natural Picard iteration. A convergence proof is only available in a very restrictive case.

Let us also mention that Poisson clocks deciding of events in games with a fixed number of players have been used for quite a long time. This is typically so in "revision games", where players may change their strategies at such times. See, e.g., [Lovo and Tomala(2015), Gensbittel et al.(2016)] and the bibliography therein.

1.3 Overview

Contrary to our first article [Bernhard and Deschamps(2017b)], here we seek a dynamic equilibrium, using the tools of dynamic programming (discrete time) and piecewise deterministic Markov decision processes, or piecewise deterministic control systems (continuous time), see [Sworder(1969), Rishel(1975)], or, for more recent references, [Vermes(1985), Haurie and Moresino(2000)]).

In the discrete time case (section 2), the resulting discrete Isaacs equation obtained is rather involved. As usual, it yields an explicit algorithm in the finite horizon, linear-quadratic case via a kind of discrete Riccati equation. The infinite horizon problem is briefly considered. It seems to be manageable only if one limits the number of players present in the game. In that case, the linear quadratic problem seems solvable via essentially the same algorithm, although we have no convergence proof, but only very convincing numerical evidence.

We then consider the continuous time case (section 3), with a Poisson arrival process. While the general Isaacs equation obtained is as involved as in the discrete time case, the linear quadratic case is simpler, and, provided again that we bound the maximum number of players allowed in the game, it yields an explicit algorithm. It takes a sign hypothesis not very realistic for an economic application to get in addition a convergence proof to the solution of the infinite horizon case.

In both discrete and continuous time, we briefly examine the case where players may leave the game. And examples of LQ games are provided in separate reports ([Bernhard and Deschamps(2018)]) as Cournot oligopolies with sticky prices.

The paper concludes with a summary of findings and limitations, and two appendices containing some auxiliary results supporting the developments in the main part of the paper.

2 Discrete time

2.1 The problem

2.1.1 Players, dynamics and payoffs

Time t is an integer. An horizon $T \in \mathbb{N}$ is given, and we will write $\{1, 2, \ldots, T\} = \mathbb{T}$, thus $t \in \mathbb{T}$. A state space X is given. A dynamic system in X may be controlled by an arbitrary number of agents or players. That number m varies with time. We let $m(t)$ be that number at time t and t_n be the arrival time of the n-th agent or player.

Note concerning the notation *We use lower indices to denote players, and upper indices to denote quantities pertaining to the total number of players in the game. An exception is* S^m *which is the cartesian power set* $\mathsf{S} \times \mathsf{S} \times \cdots \times \mathsf{S}$ m *times.*

In the simplest form, the agents arrive as a Bernoulli process with variable probability; i.e. at each time step there may arrive only one player, and this happens with a probability p^m when m players are present[1], independently of previous arrivals and of the past and present state history $x(\cdot)$, and independent of past and present strategies as well. We call t_n the arrival time of the n-th player, $s_n \in \mathsf{S}$ its decision (or control). We will later on discuss the case of multiple arrivals at the same time step and of random departures.

We distinguish the *finite case* where X and S are finite sets, from the *infinite case* where they are infinite. In that case, they are supposed to be topologic spaces, S compact.

We use the notation:

$$\begin{aligned} s^m &= (s_1^m, s_2^m, \ldots, s_m^m) \in \mathsf{S}^m\,, \quad v^{\times m} = (\overbrace{v, v, \ldots, v}^{m\,\text{times}})\,, \\ s^{m\setminus n} &= (s_1^m, \ldots, s_{n-1}^m, s_{n+1}^m, \ldots, s_m^m)\,, \\ \{s^{m\setminus n}, s\} = \{s, s^{m\setminus n}\} &= (s_1^m, \ldots, s_{n-1}^m, s, s_{n+1}^m, \ldots, s_m^m) \end{aligned}$$

The dynamics are ruled by the state equation in X:

$$x(t+1) = f^{m(t)}(t, x(t), s^{m(t)}(t))\,, \quad x(0) = x_0\,. \tag{1}$$

[1] In accordance with the above note, in the notation p^m, m is an upper index just meaning that this probability may depend on m, **not** *an exponent denoting a power*.

A double family of stepwise payoffs, for $n \leq m \in \mathbb{T}$ is given: $L_n^m : \mathbb{T} \times \mathsf{X} \times \mathsf{S}^m \to \mathbb{R} : (t, x, s^m) \mapsto L_n^m(t, x, s^m)$, as well as a discount factor $r \leq 1$. The overall payoff Π_n^e of player n, which it seeks to maximize, is the mathematical expectation of the discounted sum of its stepwise payoffs:

$$\Pi_n^e(t_n, x(t_n), \{s^m\}_{m \geq n}) = \mathbb{E} \sum_{t=t_n}^{T} r^{t-t_n} L_n^{m(t)}(t, x(t), s^{m(t)}(t)) . \tag{2}$$

Moreover, all players are assumed to be identical. Specifically, we assume that

1. The functions f^m are invariant by a permutation of the s_n,
2. the functions L_n^m enjoy the properties of a game with identical players as described in Appendix A. That is: a permutation of the s_n produces an identical permutation of the L_n^m.

Finally, in the infinite case, the functions f^m and L^m are all assumed continuous.

2.1.2 Pure strategies and equilibria

We have assumed that the current number of players in the game at each step is common knowledge. We therefore need to introduce $m(t)$-dependent controls: denote by $S_n \in \mathsf{S}_n = \mathsf{S}^{T-n+1}$ a complete n-th player's instantaneous strategy, deciding what it will do depending on the number m of players present, i.e. an application $\{n, \ldots, T\} \to \mathsf{S} : m \mapsto s_n^m$. We recall the notation for a strategy profile: $s^m = (s_1^m, s_2^m, \ldots, s_m^m) \in \mathsf{S}^m$. We also denote by S^m a decision profile: $S^m = (S_1, S_2, \ldots, S_m)$. It can also be seen as a family $S^m = (s^1, s^2, \ldots, s^m)$. The set of elementary controls in S^t is best represented by Table 1 where $s_n^m(t)$ is the control used by player n at time t if there are m players in the game at that time. A partial strategy profile $(S_1, \ldots, S_{n-1}, S_{n+1}, \ldots, S_m)$ where S_n is missing, will be denoted by $S^{m \setminus n}$. An open-loop profile of strategies is characterized by a sequence $S(\cdot) : \mathbb{T} \ni t \mapsto S^t(t)$. A partial open-loop strategy profile where $S_n(\cdot)$ is missing will be denoted by $S^{\setminus n}(\cdot)$.

The payoff $\Pi_n^e(t_n, x(t_n), S(\cdot))$ is a mathematical expectation conditioned on the pair $(t_n, x(t_n))$, which is a random variable independent of $S_n(\cdot)$.

Definition 2.1 *An open loop dynamic pure Nash equilibrium is a family history* $\widehat{S}(\cdot)$ *such that*

$$\begin{aligned} &\forall n \in \mathbb{T}, \forall (t_n, x(t_n)) \in \mathbb{T} \times \mathsf{X}, \forall S_n(\cdot) : \{t_n, \ldots, T\} \to \mathsf{S}_n , \\ &\Pi_n^e(\{\widehat{S}^{\setminus n}(\cdot), S_n(\cdot)\}) \leq \Pi_n^e(\widehat{S}(\cdot)) . \end{aligned} \tag{3}$$

	Player				
$m(t)$	1	2	$\cdots$	t	
1	$s_1^1(t)$				$s^1(t)$
2	$s_1^2(t)$	$s_2^2(t)$			$s^2(t)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$		$\vdots$
t	$s_1^t(t)$	$s_2^t(t)$	$\ldots$	$s_t^t(t)$	$s^t(t)$
	$S_1(t)$	$S_2(t)$	$\ldots$	$S_t(t)$	

Table 1: Representation of $S^t(t)$, the section at time t of an open-loop profile of strategies $S(\cdot)$. In the rightmost column: the names of the lines, in the last line: the names of the columns.

The equilibrium will be called uniform *if at all times, all players present in the game use the same decision, i.e., with our notations, if, for all t, for all m, $\hat{s}^m(t) = \hat{s}(t)^{\times m}$ for some sequence $\hat{s}(\cdot)$.*

Remark 2.1 *A game with identical players may have non uniform pure equilibria, and even have pure equilibria but none uniform. However, if it has a unique equilibrium, it is a uniform equilibrium (see appendix A).*

However, we will be interested in *closed loop* strategies. In that respect, we stress the following:

Remark 2.2 *In the sense of automata theory, the* state *of the control system at hand is the pair (x, m). Closed loop strategies must therefore be based on this composite state.*

Specifically, in *state feedback* strategies, we assume that each player is allowed to base its control at each time step t on the current time, the current state $x(t)$ and the current number $m(t)$ of players in the game. We therefore allow families of state feedbacks indexed by the number m of players:

$$\varphi^m = (\varphi_1^m, \varphi_2^m, \ldots, \varphi_m^m) : \mathbb{T} \times \mathsf{X} \to \mathsf{S}^m$$

and typically let

$$s_n^m(t) = \varphi_n^m(t, x(t)) .$$

We denote by $\Phi_n \in \mathcal{F}_n$ a whole family $(\varphi_n^m(\cdot, \cdot), m \in \{n, \ldots, T\})$ (the complete strategy choice of a player n), Φ a complete strategy profile, $\Phi^{\setminus n}$ a partial strategy profile specifying their strategy Φ_ℓ for all players except player n. A closed loop strategy profile Φ generates through the dynamics and the entry process a random open-loop strategy profile $S(\cdot) = \Gamma(\Phi)$. With a transparent abuse of notation, we write $\Pi_n^e(\Phi)$ for $\Pi_n^e(\Gamma(\Phi))$.

Definition 2.2 *A closed loop dynamic pure Nash equilibrium is a profile* $\widehat{\Phi}$ *such that*

$$\forall n \in \mathbb{T}\,, \forall (t_n, x(t_n)) \in \mathbb{T} \times \mathsf{X}\,, \forall \Phi_n \in \mathcal{F}_n\,, \quad \Pi_n^e(\{\widehat{\Phi}^{\setminus n}, \Phi_n\}) \le \Pi_n^e(\widehat{\Phi})\,. \quad (4)$$

It will be called uniform *if it holds that* $\hat{\varphi}^m = \hat{\varphi}^{\times m}$ *for some* $\hat{\varphi}$*.*

We further notice that using state feedback strategies (and dynamic programming) will naturally yield time consistent and subgame perfect strategies.

2.1.3 Mixed strategies and disturbances

For the sake of simplicity, we will focus on pure strategies hereafter. But of course, a pure Nash equilibrium may not exist. In the discrete time case investigated here, we can derive existence results if we allow mixed strategies.

Let $\mathcal{S}$ be the set of probability distributions over S. Replacing S by $\mathcal{S}$ in the definitions of open-loop and closed-loop strategies above yields equivalent open-loop and closed-loop behavioral mixed strategies. By behavioral, we mean that we use sequences of random choices of controls and not random choices of sequences of controls. See [Bernhard(1992)] for a more detailed analysis of the relationship between various concepts of mixed strategies for dynamic games.

In case the strategies are interpreted as mixed strategies, $s^{m(t)}(t)$ in equations (1) and (2) are random variables, and the pair $(m(\cdot), x(\cdot))$ is a (controlled) Markov chain. But since anyhow, $m(\cdot)$ is already a Markov chain even with pure strategies, the rest of the analysis is unchanged.

We might go one step further and introduce disturbances in the dynamics and the payoff. Let $\{w(\cdot)\}$ be a sequence of independent random variables in $\mathbb{R}^\ell$, and add the argument $w(t)$ in both f^m and L_n^m. All results hereafter in the discrete time problem remain unchanged (except for formula (9) where one term must be added). We keep with the undisturbed case for the sake of simplicity of notation, and because in the continuous time case, to be seen later, it spares us the Ito terms in the equations.

2.2 Isaacs equation

2.2.1 Finite horizon

We use dynamic programming, and therefore Isaacs equation in terms of a family of Value functions $V_n^m : \mathbb{T} \times \mathsf{X} \to \mathbb{R}$. It will be convenient to associate to any such family the family W_n^m defined as

$$W_n^m(t,x) = (1-p^m)V_n^m(t,x) + p^m V_n^{m+1}(t,x)\,, \tag{5}$$

and the Hamiltonian functions

$$H_n^m(t,x,s^m) := L_n^m(t,x,s^m) + rW_n^m(t+1, f_n^m(t,x,s^m))\,. \tag{6}$$

We write Isaacs equation for the general case of a non uniform equilibrium, but the uniform case will be of particular interest to us.

Theorem 2.1 *An subgame perfect equilibrium $\widehat{\Phi} = \{\hat{\varphi}_n^m\}$ exists, if and only if there is a family of functions V_n^m satisfying the following Isaacs equation, which makes use of the notation* (5), (6)*:*

$$\forall n \le m \in \mathbb{T}\,,\ \forall (t,x) \in \{0,\dots,T\} \times \mathsf{X}\,,\ \forall s \in \mathsf{S}\,,$$

$$V_n^m(t,x) = H_n^m(t,x,\hat{\varphi}^m(t,x)) \ge H_n^m(t,x,\{\hat{\varphi}^{m\setminus n}(t,x), s\})\,,$$

$$\forall m \in \mathbb{T}\,,\ \forall x \in \mathsf{X}\,,\ V_n^m(T+1,x) = 0\,.$$

And then, the equilibrium payoff of player n joining the game at time t_n at state x_n is $V_n^n(t_n,x_n)$. If the equilibrium is uniform, i.e. for all $n \le m$, $\hat{\varphi}_n^m = \hat{\varphi}_1^m$, then $V_n^m = V_1^m$ for all m,n (and we may call it V^m).

Proof This is a classical dynamic programming argument. We notice first that the above system can be written in terms of conditional expectations given (m,x) as

$$\forall n \le m \in \mathbb{T}\,,\ \forall (t,x) \in \{0,\dots,T\} \times \mathsf{X}\,,\ \forall s \in \mathsf{S}\,,$$

$$\begin{aligned} V_n^m(t,x) &= \mathbb{E}^{m,x}\Big[L_n^m(t,x,\hat{\varphi}^m(t,x)) \\ &\quad + rV_n^{m(t+1)}\big(t+1, f^m(t,x,\hat{\varphi}^m(t,x))\big)\Big] \\ &\ge \mathbb{E}^{m,x}\Big[L_n^m(t,x,\{\hat{\varphi}^{m\setminus n}(t,x), s\}) \\ &\quad + rV_n^{m(t+1)}\big(t+1, f^m(t,x,\{\hat{\varphi}^{m\setminus n}(t,x), s\})\big)\Big] \end{aligned}$$

$$\forall m \in \mathbb{T}\,,\ \forall x \in \mathsf{X}\,,\ V_n^m(T+1,x) = 0\,.$$

Assume first that all players use the strategy $\hat{\varphi}$. Fix an initial time t_n (which may or may not be the arrival time of the n-th player) a state x_n and an initial m. Assume all players use their control $\hat{\varphi}_n(t,x(t))$, and consider the random process $(m(t),x(t))$ thus generated. For brevity, write $\hat{s}^m(t) := \hat{\varphi}^m(t,x(t))$. Write the equality in theorem 2.1 at all steps of the stochastic process $(m(t),x(t),\hat{s}^{m(t)}(t))$:

$$V_n^m(t,x(t)) = \mathbb{E}^{m(t),x(t)}\left[L_n^{m(t)}\big(t,x(t),\hat{s}^{m(t)}(t)\big) + rV_n^{m(t+1)}\big(t+1,x(t+1)\big)\right].$$

Multiply by r^{t-t_n}, take the *a priori* expectation of both sides and use the theorem of embedded conditional expectations, to obtain

$$\mathbb{E}\Big[-r^{t-t_n}V_n^{m(t)}(t,x(t))+r^{t-t_n}L_n^{m(t)}\big(t,x(t),\hat{s}^{m(t)}(t)\big) \\ +r^{t+1-t_n}V_n^{m(t+1)}(t+1,x(t+1))\Big]=0\,.$$

Sum these equalities from t_n to T and use $V_n^m(T+1,x)=0$ to obtain

$$-V_n^m(t_n,x_n)+\mathbb{E}\left[\sum_{t=t_n}^{T}r^{t-t_n}L_n^{m(t)}(t,x(t),\hat{s}^{m(t)}(t))\right]=0\,,$$

hence the claim that the payoffs of all players from (t_n,x_n,m) is just $V_n^m(t_n,x_n)$, and in particular the payoff of player n as in the theorem.

Assume now that player n deviates from $\hat{\varphi}_n$ according to any sequence $s_n(\cdot)$. Exactly the same reasoning, but using the inequality in the theorem, will lead to $V_n^m(t_n,x_n)\geq\Pi_n^e$. We have therefore shown that the conditions of the theorem are sufficient for the existence of a subgame perfect equilibrium.

Finally, assume that the subgame perfect equilibrium exists. Let $V_n^m(t,x)$ be defined as the payoff to player n in the subgame starting with m players at (t,x). The equality in the theorem directly derives from the additivity of the mathematical expectation. And if at one (m,t,x) the inequality were violated, for the subgame starting from that situation, a control $s_n(t)=s$ would yield a higher expectation for player n, which is in contradiction with the fact that $\widehat{\Phi}$ generates an equilibrium for all subgames.

Concerning a uniform equilibrium, observe first that (for all equilibria), for all m,n, for all $x\in\mathsf{X}$, $V_n^m(T+1,x)=0$. Assume that $V_n^m(t+1,x)=V_1^m(t+1,x)$. Observe that then, on the right hand side of Isaacs equation, only L_n^m depends on n. let π be a permutation that exchanges n and 1. By hypothesis, $L_n^m(t,x,\hat{\varphi}^{\pi[m]}(t,x))=L_1^m(t,x,\hat{\varphi}^m)$. But for a uniform equilibrium, it also holds that $\hat{\varphi}^{\pi[m]}(t,x)=\hat{\varphi}^m(t,x)$. Hence $V_n^m(t,x)=V_1^m(t,x)$. ∎

Isaacs equation in the theorem involves a sequence of Nash equilibria of the Hamiltonian. In general, stringent conditions are necessary to ensure existence of a pure equilibrium. However, our hypotheses ensure existence of a mixed equilibrium (see, e.g. [Ekeland(1974)] and [Bernhard(1992)]). And since the equation is constructive via backward induction, we infer

Corollary 2.1 *A dynamic subgame perfect Nash equilibrium in behavioral strategies exists in the finite horizon discrete time game with randomly arriving players.*

A natural approach to using the theorem is via dynamic programming (backward induction). Assume that we have discretized the set of reachable states in N_t

points at each time t. (Or $x \in \mathsf{X}$, a finite set) The theorem brings the determination of a subgame perfect equilibrium set of strategies to the computation of $\sum_t t \times N_t$ Nash equilibria (one for each value of m at each (t,x)). A daunting task in general. However, the search for a *uniform* equilibrium may be much simpler. On the one hand, there is now a one-parameter family of functions $V^m(t,x)$, and, in the infinite case, if all functions are differentiable (concerning W_n^m this is *not* guaranteed by regularity hypotheses on f^m and L_n^m) and if the equilibrium is interior, the search for each static Nash equilibrium is brought back to solving an equation of the form (32):

$$\partial_{s_1} L_1^m(t,x,s^{\times m}) + r\partial_x W^m(t+1, f^m(t,x,s^{\times m}))\, \partial_{s_1} f^m(t,x,s^{\times m}) = 0\,.$$

We will see that in the linear quadratic case that we will consider, this can be done.

2.2.2 Infinite horizon

We consider the same problem as above, with both f^m and L_n^m independent of time t. We assume that the L_n^m are uniformly bounded by some number L, and we let the payoff of the n-th player in a (sub)game starting with n players at time t_n and state $x(t_n) = x_n$ be

$$\Pi_n^e(t_n, x_n; S(\cdot)) = \mathbb{E} \sum_{t=t_n}^{\infty} r^{t-t_n} L_n^{m(t)}(x(t); s^{m(t)}(t))\,. \tag{7}$$

We look for a subgame perfect equilibrium set of strategies $\hat\varphi_n^m(x)$. Isaacs equation becomes an implicit equation for a bounded infinite family of functions $V_n^m(x)$. Using the time invariant form of equations (5) and (6), we get:

Theorem 2.2 *Let $r < 1$. Then, a subgame perfect equilibrium $\widehat{\Phi}$ of the infinite horizon game exists if and only if there is a two-parameter infinite family of uniformly bounded functions $V_n^m(x)$ satisfying the following Isaacs equation:*

$$\forall n \le m \in \mathbb{N}\,,\ \forall x \in \mathsf{X}\,,\ \forall s \in \mathsf{S}\,,$$

$$V_n^m(x) = H_n^m(x, \hat\varphi^m(x)) \ge H_n^m(x, \{\hat\varphi^{m\setminus n}(x), s\})\,.$$

Then, the equilibrium payoff of player n joining the game at state x_n is $V_n^n(x_n)$. If the equilibrium is uniform, $V_n^m = V_1^m$ for all n, m.

Proof The proof proceeds along the same lines as in the finite horizon case. In the summation of the sufficiency proof, there remains a term $r^{T-t_n} V^m(x(T))$ that goes to zero as T goes to infinity, because the functions V^m have been assumed to be bounded. And this is indeed necessary since the bound assumed on the L_n^m implies that the Value functions are bounded by $L/(1-r)$. ▮

We restrict our attention to uniform equilibria, so that we have a one-parameter family of Value functions V^m. But it is infinite. To get a feasible algorithm, we make the following assumption:

Hypothesis 2.3 *There is a finite $M \in \mathbb{N}$ such that $p^M = 0$.*

Thanks to that hypothesis, there is a finite number M of Value functions to consider. There remains to find an algorithm to solve for the fixed points bearing on the family $\{V^m(x)\}_m$ for all $x \in \mathsf{X}$. We offer the *conjecture* that the mapping from the family $\{V^m(t+1,\cdot)\}_m$ to the family $\{V^m(t,\cdot)\}_m$ in the finite horizon Isaacs equation is a contraction in an appropriate distance. If so, then it provides an algorithm of "iteration on the Value" to compute the $V^m(x)$ of the infinite horizon problem. (We will offer a different conjecture in the linear quadratic case.)

Remark 2.3 *Hypothesis 2.3 is natural in case the payoff is decreasing with the number of players and there is a fixed entry cost. Otherwise, it may seem artificial and somewhat unfortunate. Yet, we may notice that for any numerical implementation, we are obliged to consider only a bounded (since finite) set of x. We are accustomed to doing so, relying upon the assumption that very large values of x will be reached very seldom, and play essentially no role in the computation. In a similar fashion, we may think that very large values of $m(t)$ will be reached for very large t, which, due to the discount factor, will play a negligible role in the numerical results. This is an unavoidable feature of numerical computations, not really worse in our problem than in classical dynamic programming.*

2.3 Entering and leaving

2.3.1 Methodology

It would be desirable to extend the theory to a framework where players may also leave the game at random. However, we must notice that although our players are identical, the game is not anonymous. As a matter of fact, players are labelled by their rank of arrival, and their payoffs depend on that rank. We must therefore propose exit mechanisms able to take into account *who* leaves the game. Before doing so, we agree on the fact that once a player has left the game, it does not re-enter. (Or if it does, this new participation is considered as that of another player.) Let T_n be the exit time of the player of rank n, a random variable. We now have

$$\Pi_n^e(t_n, x(t_n), S(\cdot)) = \mathbb{E}\sum_{t=t_n}^{T_n} r^{t-t_n} L_n^{m(t)}(t, x(t), s^{m(t)}(t)) \,.$$

In defining the controls of the players, we may no longer have $n \le m \le t$ as previously, and Table 1 must be modified accordingly. Let $N(m)$ be the maximum possible rank of players present when there are m of them, and $M(n)$ the maximum possible number of players present when player n is. Then $s^m(t) = \{s_n^m\}_{n\le N(m)}$ and $S_n(t) = \{s_n^m(t)\}_{m\le M(n)}$. And of course, a choice of $s_n^m(t)$ means the decision that player of rank n chooses at time t if there are m players present at that time, *including itself.*

We also stress that the probabilities of entry (or exit) are functions such as p^m of the current number of players present, and not of the rank of entry.

When a player leaves the game, from the next time step on it will not get any payoff. Thus, we may just consider that for it, the Value functions $V_n^m(t+1,x)$ are null. To take this into account we determine the probabilities $\mathbb{P}^{m,k}$ that there be k players at the next time step *and that the focal player has not left*, knowing that there are m players present at the current step. We can now state:

Theorem 2.4 *Theorem 2.1 above and its proof remain unchanged upon substituting*

$$W_n^m = \sum_k \mathbb{P}^{m,k} V_n^k$$

to equation (5).

Notice that In the Bernoulli entry-only version of the problem, we may set $\mathbb{P}^{m,m+1} = p$ and $\mathbb{P}^{m,m} = (1-p)$.)

We propose several entry and exit mechanisms as examples.

2.3.2 A joint scheme

In this scheme, there is a probability q^m that one player leaves the game at the end of a step where there are m players present. (And of course, $q^0 = 0$.) Moreover, we add the dictum that should one player actually leave, which one leaves is chosen at random with uniform probability among the players present. As a consequence, each player present has a probability q^m/m to leave the game at (the end of) each time step. Let $m(t) = m$, then the probabilities that a given player among the m present at step t still be present at time $t+1$ and that $m(t+1)$ take different values is given by the following table:

$m(t+1)$	probability
$m+1$	$\mathbb{P}^{m,m+1} = p^m(1-q^m)\,,$
m	$\mathbb{P}^{m,m} = p^m q^m \dfrac{m-1}{m} + (1-p^m)(1-q^m)$
$m-1$	$\mathbb{P}^{m,m-1} = (1-p^m) q^m \dfrac{m-1}{m}\,.$

2.3.3 Individual schemes

The previous scheme is consistent with our entry scheme. But it might not be the most realistic. We propose two other schemes.

In the first, each player, once it has joined the game, has a probability q of leaving the game at each time step, independently of the other players and of the past and current arrival sequence. We need powers of p and q. So, to keep the sequel readable, we take them constant, and upper indices in the table below are powers. It is only a matter of notation to take them dependent on m. In computing the probability that a given number of players has left, we must remember that those must be chosen among the other $m-1$ players, and that the focal player must have remained. The corresponding table of probabilities is now

$m(t+1)$	probability
$m+1$	$\mathbb{P}^{m,m+1} = p(1-q)^m\,,$
$1 < k \le m$	$\mathbb{P}^{m,k} = \frac{(m-1)!}{(m-k)!(k-2)!} q^{m-k}(1-q)^{k-1}\left[\frac{(1-p)(1-q)}{k-1} + \frac{pq}{m-k+1}\right],$
1	$\mathbb{P}^{m,1} = (1-p)(1-q)q^{m-1}\,.$

A more coherent scheme, but one that drives us away from the main stream of this article, is the one where there is a finite pool of M agents who are eligible to enter the game. At each time step, each of them has a probability p of actually entering. Once into the game, each has a probability q of leaving at each time step, and if so, it re-enters the pool. In that case, we set

$$\mathcal{L}^{m,k} = \{\ell \in \mathbb{N} | \ell \ge 0\,,\ \ell \ge m-k\,,\ \ell \le m-1\,,\ \ell \le M-k\}$$

and we have, for all m, k less than or equal to M:

$$\mathbb{P}^{m,k} = \sum_{\ell \in \mathcal{L}^{m,k}} \begin{pmatrix} m-1 \\ \ell \end{pmatrix} \begin{pmatrix} M-m \\ k-m+\ell \end{pmatrix} p^{k-m+\ell}(1-p)^{M-k+\ell} q^{\ell}(1-q)^{m-\ell}\,.$$

2.3.4 Beyond the Bernoulli process

At this stage, it is not difficult to generalize our model to one where several players may join the game at each instant of time, provided that it remains a finite Markov chain. Introduce probabilities p^m_ℓ that ℓ players join the game when m players are already there. In a similar fashion, in the so called "joint scheme" above, we might have probabilities q^m_ℓ that ℓ players leave at the same time.

Set $p^m_j = 0$ for any $j < 0$. We then have

$$\mathbb{P}^{m,k} = \sum_{\ell=0}^{m-1} \frac{m-\ell}{m} q^m_\ell p^m_{k-m-\ell}\,. \tag{8}$$

2.4 Linear quadratic problem

2.4.1 The problem

We consider an academic example as follows: the state space is $\mathsf{X} = \mathbb{R}^d$, the control set $\mathsf{S} = \mathbb{R}^a$, the dynamics are defined by a sequence of square $d \times d$ matrices $A(t)$ and a sequence of $d \times a$ matrices $B(t)$ and

$$x(t+1) = A(t)x(t) + B(t)\sum_{n=1}^{m(t)} s_n(t)\,.$$

The payoff of player n is given in terms of two sequences of square matrices $Q^m(t)$ and $R(t)$, the first nonnegative definite, the second positive definite, as [2]

$$\Pi_n^e = \mathbb{E}\sum_{t=t_n}^{T} r^{t-t_n}\left[\|x(t)\|^2_{Q^{m(t)}(t)} - \|s_n(t)\|^2_{R(t)}\right]\,.$$

The idea is that the players, through their controls s_n, collectively produce some good x, worth $\|x(t)\|^2_{Q(t)}$ at time t. Either it is a public good, and they all benefit equally, then $Q^m = Q$, or it is company earnings, and they share equally the dividends, and then $Q^m = (1/m)Q$. But each of them pays its effort at each time step t an amount $\|s_n(t)\|^2_{R(t)}$. (It is only for notational convenience that we do not let R depend on m.)

This is not quite the classical linear quadratic problem, because the two terms in the payoff have different signs. As a consequence, to avoid an indefiniteness in the payoff, we must add the dictum that all players are constrained to using decision sequences $s_n(\cdot)$ of finite weighted norm, (or "energy") in the sense that

$$\sum_{t=t_n}^{T} r^{t-t_n}\|s_n(t)\|^2_{R(t)} < \infty\,.$$

2.4.2 Solution via the Riccati equation

As usual, we seek a solution with a quadratic Value function. We look for a uniform equilibrium, and a one-parameter family of Value functions of the form

$$V^m(t,x) = \|x\|^2_{P^m(t)}\,. \tag{9}$$

Notice first that, according to the terminal condition in theorem 2.1, for all $m \le T$, $P^m(T+1) = 0$. Assume, as a recursion hypothesis, that $V^m(t+1,x)$ is, for all

[2]We use prime for transposed and $\|x\|^2_Q = x'Qx$, $\|s\|^2_R = s'Rs$.

m, a quadratic form in x, i.e. that there exist symmetric matrices $P^m(t+1)$ such that

$$V^m(t+1,x) = \|x\|^2_{P^m(t+1)}\,.$$

Since for any controls of the others, each player may always use $s_n = 0$ and that way ensure itself a nonnegative payoff, it follows that $P^m(t+1)$ is nonnegative. Isaacs equation is now

$$\begin{aligned}V^m(t,x) = \max_s \Big\{ &\|x\|^2_{Q^m(t)} - \|s\|^2_{R(t)} \\ &+ r\Big[(1-p^m)\|A(t)x + (m-1)B(t)\hat{s} + B(t)s\|^2_{P^m(t+1)} \\ &\quad + p^m\|A(t)x + (m-1)B(t)\hat{s} + B(t)s\|^2_{P^{m+1}(t+1)}\Big]\Big\},\end{aligned}$$

the maximum in s being reached at $s = \hat{s}$. Let [3]

$$W^m(t+1) = r\left[(1-p^m)P^m(t+1) + p^m P^{m+1}(t+1)\right]. \qquad (10)$$

These are symmetric non-negative definite matrices, and $W^m(T+1) = 0$. Isaacs equation can be written

$$V^m(t,x) = \max_s \Big\{ \|x\|^2_{Q^m(t)} - \|s\|^2_{R(t)} + \|A(t)x + (m-1)B(t)\hat{s} + B(t)s\|^2_{W^m(t+1)} \Big\}.$$

The right hand side is a (non homogeneous) quadratic form in s, with a quadratic term coefficient

$$K^m(t) = B'(t)W^m(t+1)B(t) - R(t)\,. \qquad (11)$$

At time T, this is the negative definite matrix $-R(T)$. The optimum $s(T)$ is clearly zero, leading to $P^m(T) = Q^m(T)$, and therefore

$$K^m(T-1) = rB'(T-1)[(1-p^m)Q^m(T) + p^m Q^{m+1}(T)]B(T-1) - R(T-1)\,.$$

If this matrix is negative definite, there is a finite maximum in s. In the case where one $K^m(t)$ would fail to be at least nonpositive definite, s could be chosen so as to make the form arbitrarily large positive, the subgame starting from this node (m,t,x) could not have an equilibrium. In the case where one $K^m(t)$ would be nonpositive definite but singular, the quadratic form would have a finite maximum only if, for all x

$$s'K^m(t)s = 0 \Rightarrow s'B'(t)W^m(t+1)[A(t)x + mB(t)s] = 0.$$

[3]Notice that contrary to what we did in the nonlinear case, we include the factor r in W^m, to simplify later expressions

Now, $s'K^m(t)s = 0$ implies that $s'B'W^m(t+1)Bs = s'Rs$ and therefore is not identically zero. But the requirement would be that for those s in the kernel of K, $s'B'W^m(t+1)Ax = ms'Rs$ for all x. The left hand side of this last inequality is linear in x and could be constant only if it were zero. In conclusion, the quadratic form cannot have a finite maximum for all x, and therefore the game no subgame perfect equilibrium.

Assume therefore that all $K^m(t)$ for $t, m \le T-1$ are negative definite. Equating the derivative with respect to s to zero, and equating all controls, yields

$$[B'(t)W^m(t+1)B(t) - R(t)]\hat{s} + B'(t)W^m(t+1)[A(t)x + (m-1)B(t)\hat{s}] = 0\,,$$

or, rearranging

$$\left[mB'(t)W^m(t+1)B(t) - R(t)\right]\hat{s} = B'(t)W^m(t+1)A(t)x\,. \qquad (12)$$

If this equation has no solution for some (m, t), a uniform subgame perfect equilibrium cannot exist. If it has a solution, there exists a matrix $F^m(t)$ such that

$$\hat{s} = -F^m(t)x =: \hat{\varphi}_1^m(t, x). \qquad (13)$$

We write it

$$F^m(t) = \left[mB'(t)W^m(t+1)B(t) - R(t)\right]^{-1} B'(t)W^m(t+1)A(t)\,, \qquad (14)$$

knowing that indeed the inverse might have to be replaced by a pseudo inverse. Finally, placing this value of s on the right hand side, we find that $V^m(t, x)$ is indeed a quadratic form in x. Thus we have proven that (9) holds, with

$$\begin{aligned} P^m(t) = Q^m(t) &- F^{m\prime}(t)R^m(t)F^m(t) \\ &+ [A'(t) - mF^{m\prime}(t)B'(t)]W^m(t+1)[A(t) - mB(t)F^m(t)]\,, \end{aligned}$$

and after substituting $F^m(t)$ and reordering:

$$\begin{aligned} P^m(t) =& Q^m(t) + A'(t)W^m(t+1)A(t) - \\ & A'(t)W^m(t+1)B(t)[mB'(t)W^m(t+1)B(t) - R(t)]^{-1} \\ & [m^2B'(t)W^m(t+1)B(t) - (2m-1)R(t)] \\ & [mB'(t)W^m(t+1)B(t) - R(t)]^{-1}B'(t)W^m(t+1)A(t)\,, \end{aligned} \qquad (15)$$

$$\forall m \in \mathbb{T}\,, \quad P^m(T) = Q^m(T)\,. \qquad (16)$$

Recall that each matrix W^m involves P^{m+1}. But there cannot be more than T players at any time in the game (and T of them only at $T-1$, the final decision time). Therefore, starting with $P^T(T) = Q^T$ and computing the $P^m(t)$ backward, this is a constructive algorithm. We therefore end up with the following:

Theorem 2.5 *The finite horizon, linear quadratic problem admits a uniform subgame perfect equilibrium if and only if equations* (15) *have a solution over* $[0, T]$, *—i.e. all equations* (12) *have a solution—, leading to negative definite matrices* $K^m(t)$ *as defined by equation* (11). *If these conditions are satisfied, the unique uniform equilibrium is given by equations (9,10,13,14,15,16).*
Entering and leaving *Allowing one of our extended entry and exit mechanisms: the result holds upon replacing the definition* (10) *of* W^m *by*

$$W^m(t+1) = r \sum_{k=1}^{m+1} \mathbb{P}^{m,k} P^k(t+1)$$

with the relevant set of probabilities $\mathbb{P}^{m,k}$ *as given by equation* (8).

Infinite horizon We might want to consider the infinite horizon game with all system matrices constant. Notice first that the problem has a meaning only if the matrix A is stable (or at least has a spectral radius strictly less than $r^{-1/2}$). Otherwise, all players might just play $s_n(t) = 0$ and get an infinitely large payoff. But even so, and contrary to the case where Q would be nonpositive definite, we do not know the asymptotic behavior of the Riccati equations, let alone whether it has the necessary stabilizing properties to lead to a Nash equilibrium. (See [Mageirou(1976)].)

An example of use of this theory to a Cournot oligopoly with sticky prices is provided in [Bernhard and Deschamps(2018)].

3 Continuous time

3.1 The problem

3.1.1 Players, dynamics and payoff

We consider a game with randomly arriving (or arriving and leaving) players as in the previous section, but in continuous-time. The players arrive as a Poisson process of variable intensity, still independent from the past state or strategy histories. The interval lengths $t_{m+1} - t_m$ between successive arrivals are independent random variables obeying exponential laws[4] with intensity λ^m:

$$\mathbb{P}(t_{m+1} - t_m > \tau) = \mathrm{e}^{-\lambda^m \tau}$$

for a given sequence of positive λ^m. An added difficulty, as compared to the discrete time case, is that the number of possible arrivals is unbounded, even for the

[4] As in the discrete time case with the p^m, the m in λ^m is an upper index, **not an exponent**.

finite horizon problem. For that reason, the sequence λ^m is *a priori* infinite. But we assume that the λ^m are bounded by a fixed Λ. As a matter of fact, for any practical use of the theory, we will have to assume that the λ^m are all zero for m larger than a given integer M, thus limiting the number of players to M. Alternatively, for a finite horizon T, we may notice that for any M, the probability $\mathbb{P}(m(t) > M)$ is less than $(\Lambda T)^M/M!$ and therefore goes to zero as $M \to \infty$, and take argument to neglect very large m's.

The dynamic system is also in continuous time. The state space X is now the Euclidean space $\mathbb{R}^d$, or a subset of it, and the dynamics

$$\dot{x} = f^{m(t)}(t, x, s^{m(t)}), \quad x(0) = x_0 .$$

Standard regularity and growth hypotheses hold on the functions f^m to ensure the existence of a unique solution in X over $[0, T]$ to the dynamics for every m-tuple of measurable functions $s^m(\cdot) : [0, T] \to \mathsf{S}^m$.

A positive discount factor ρ is given, and the performance indices are given via

$$\mathcal{L}_n(t_n, x(t_n), \{s^m(\cdot)\}_{m\in\mathbb{N}}) = \int_{t_n}^{T} \mathrm{e}^{-\rho(t-t_n)} L_n^{m(t)}(t, x(t), s^{m(t)}(t))\,\mathrm{d}t$$

as

$$\Pi_n^e(t_n, x(t_n), \{s^m(\cdot)\}_{m\in\mathbb{N}}) = \mathbb{E}\mathcal{L}_n(t_n, x(t_n), \{s^m(\cdot)\}_{m\in\mathbb{N}}) . \tag{17}$$

The functions L_n^m are assumed to be continuous and uniformly bounded.

As in the discrete time case, we consider identical players, i.e. the functions f^m are invariant by a permutation of the s_n, and the functions L_n^m enjoy the properties of a game with identical players as detailed in Appendix A.

3.1.2 Strategies and equilibrium

We seek a state feedback equilibrium. Let $\mathcal{A}^m$ be the set of *admissible feedbacks* when m players are present. A control law $\varphi : [0, T] \times \mathsf{X} \to \mathsf{S}$ will be in $\mathcal{A}^m$ if, on the one hand, the differential equation

$$\dot{x} = f^m(t, x, \varphi(t, x)^{\times m})$$

has a unique solution for any initial data $(t_n, x_n) \in [0, T] \times \mathsf{X}$, and on the other hand, for every measurable $s(\cdot) : [0, T] \to \mathsf{S}$, the differential equation

$$\dot{x} = f^m(t, x(t), \{s(t), \varphi(t, x(t))^{\times m\setminus 1}\})$$

has a unique solution over $[0, T]$ for any initial data $(t_n, x_n) \in [0, T] \times \mathsf{X}$.

We define a state feedback pure equilibrium as in the previous section, namely via definition 2.2. Moreover, we shall be concerned only with uniform such equilibrium strategies.

3.1.3 Mixed strategies and disturbances

We would rather avoid the complexity of mixed strategies in continuous time (see, however, [Elliot and Kalton(1972)]), as experience teaches us that they are often unnecessary.

Adding disturbances to the dynamics and payoffs as in the discrete time problem is not difficult. But the notation needs to be changed to that of diffusions, and we would get extra second order terms in Isaacs equation, due to Ito calculus. All results carry over with the necessary adaptations. We keep with the deterministic set up for the sake of simplicity.

3.2 Isaacs equation

3.2.1 Finite horizon

The Isaacs equation naturally associated with a uniform equilibrium in this problem is as follows, where $\hat{s}$ stands for the argument of the maximum (we write V_t and V_x for the partial derivatives of V):

$$\forall (t,x) \in [0,T] \times \mathsf{X}\,, \quad (\rho + \lambda^m)V^m(t,x) - \lambda^m V^{m+1}(t,x) - V_t^m(t,x) -$$

$$\max_{s\in\mathsf{S}} \Big[V_x^m(t,x) f^m(t,x,\{s, \hat{s}^{\times m\setminus 1}\}) + L_1^m(t,x,\{s,\hat{s}^{\times m\setminus 1}\}) \Big] = 0\,, \tag{18}$$

$$\forall x \in \mathsf{X}\,, \quad V^m(T,x) = 0\,.$$

As already mentioned, even for a finite horizon, the number of players that may join the game is unbounded. Therefore, equation (18) is an infinite system of partial differential equations for an infinite family of functions $V^m(t,x)$. We will therefore make use of the hypothesis similar to hypothesis 2.3:

Hypothesis 3.1 *There exists an integer M such that $\lambda^M = 0$.*

As hypothesis 2.3 of the discrete time case, this is a natural hypothesis in case of a decreasing payoff and fixed finite entry cost, and akin to classical approximations of the Isaacs equation in dynamic programming algorithms.

Under that hypothesis, using the tools of piecewise deterministic Markov decision Processes, we have the following easy extension of [Fleming and Soner(1993)]:

Theorem 3.2 *A uniform subgame perfect equilibrium exists if and only if there exists a family of admissible feedbacks $\varphi^m \in \mathcal{A}^m$ and a family of bounded uniformly continuous functions $V^m(t,x)$ that are, for all $m \le M$, viscosity solutions of the partial differential equation* (18)*. Then, $s_n(t) = \hat{\varphi}^{m(t)}(t,x(t))$ is a uniform subgame perfect equilibrium, and the equilibrium payoff of player n joining the game at time t_n and state x_n is $V^n(t_n,x_n)$.*

A sketch of the proof is given in appendix B.1.

The question naturally arises of what can be said of the problem without hypothesis 3.1. To investigate this problem, we consider an "original problem" defined by its infinite sequence $\{\lambda^m\}_{m\in\mathbb{N}}$, assumed bounded :

$$\exists\Lambda > 0 : \quad \forall m \in \mathbb{N}\,,\; \lambda^m \leq \Lambda\,,$$

and a family of "modified problems" depending on an integer M, where we modify the sequence $\{\lambda^m\}$ at λ^M that we set equal to zero. (And therefore all λ^m for $m > M$ are irrelevant: there will never be more than M players.) The theorem above holds for all modified problems, whatever the M chosen. We call $V^{m|M}$ (a finite family) the solution of the corresponding equation (18). They yield the equilibrium value of the payoff Π^{eM} in the modified problems.

We propose in appendix B.2 arguments in favor of the following:

Conjecture 3.1 *As M goes to infinity, the equilibrium state feedbacks φ^M of the modified problems converge, in L^1 (possibly weighted by a weight $\exp(-\alpha\|x\|)$) toward an equilibrium feedback $\varphi^\star$ of the original problem, and the functions $V^{m|M}$ converge in C^1 toward the equilibrium value V^m. Consequently, theorem 3.2 holds for the original, unmodified problem.*

3.2.2 Infinite horizon

We assume here that the functions f^m and L_n^m are time invariant, and $\rho > 0$. We set

$$\Pi_n^e(t_n, x(t_n), \{s^m(\cdot)\}_{m\in\mathbb{N}}) = \mathbb{E}\int_{t_n}^{\infty} \mathrm{e}^{-\rho(t-t_n)} L_n^{m(t)}(x(t), s^{m(t)})\,\mathrm{d}t\,.$$

As expected, we get

Theorem 3.3 *Under hypothesis 3.1, a uniform subgame perfect equilibrium in infinite horizon exists if and only if there exists a family of admissible feedbacks $\hat\varphi^m \in \mathcal{A}^m$ and a family of bounded uniformly continuous functions $V^m(x)$ that are, for all m, viscosity solutions of the following partial differential equation, where $\hat s$ stands for $\hat\varphi^m(t,x)$ and the maximum is reached precisely at $s = \hat s$:*

$$\forall x \in \mathsf{X}\,, \qquad 0 = (\rho + \lambda^m)V^m(x) - \lambda^m V^{m+1}(x) - \tag{19}$$

$$\max_{s\in\mathsf{S}}\Big[V_x^m(x) f^m(x, \{s, \hat u^{\times m\setminus 1}\}) + L_1^m(x, \{s, \hat s^{m\setminus 1}\})\Big] \tag{20}$$

Then, $s_n(t) = \hat\varphi^{m(t)}(x(t))$ is a uniform subgame perfect equilibrium, and the equilibrium payoff of player n joining the game at state x_n is $V^n(x_n)$.

The proof involves extending equation (18) to the infinite horizon case, a sketch of which is provided in appendix B.1, relying on the boundedness of the functions V^m to ensure that $\exp(-\rho T)V^m(x(T))$ goes to zero as T increases to infinity. The rest of the proof is exactly as in the previous subsubsection.

The original problem without the bounding hypothesis 3.1 requires a different approach from that of the previous subsection, because in infinite horizon, it is no longer true that $\mathbb{P}(\Omega_M)$ is small. Indeed it is equal to 1 if the hypothesis does not hold and the λ^m have a lower bound.

3.3 Entering and leaving

As in the discrete time case, we may extend the theory to the case where the players may also leave. We consider that once a player has left, it does not re-enter. We let T_n be the exit time of player n. In the joint exit mechanism, the process that one of the m players present may leave is a Poisson process with intensity μ^m, and if one does, it is one of the players present with equal probability. In the individual scheme, each of the m players present has a Poisson exit process with probability μ^m.

This leads to probabilities $\mathbb{P}^{m,\ell}$ as in the following table:

scheme	$\mathbb{P}^{m,m-1}$	$\mathbb{P}^{m,m}$	$\mathbb{P}^{m,m+1}$
joint	$\frac{m-1}{m}\mu^m$	$\lambda^m+\mu^m$	λ^m
individual	μ^m	$\lambda^m+m\mu^m$	λ^m

(21)

We may now state

Theorem 3.4 *Allowing players to leave, according to either the joint or individual schemes, Isaacs' equation reads*

$$(\rho+\mathbb{P}^{m,m})V^m(t,x)-\mathbb{P}^{m,m+1}V^{m+1}(t,x)-\mathbb{P}^{m,m-1}V^{m-1}(t,x)-V^m_t(t,x)$$
$$-\max_{s\in\mathsf{S}}\Big[V^m_x(t,x)f^m(t,x,\{s,\hat{s}^{m\setminus 1}\})+L^m_1(t,x,\{s,\hat{s}^{m\setminus 1}\})\Big]=0\,,$$

Proof The easy proof is left to the reader.

3.4 Linear quadratic problem

3.4.1 Finite horizon

We turn to the non standard linear quadratic case, where the dynamics are given by piecewise continuous (or even measurable) time dependent matrices $A(t)$ and

$B(t)$ of dimensions, respectively $d \times d$ and $d \times a$ (both could be m-dependent)

$$\dot{x} = A(t)x + B(t)\sum_{n=1}^{m} s_n\,,$$

but where each payoff has two terms of opposite signs, given by a discount factor ρ, and two families of piecewise continuous symmetric matrices: nonnegative $d \times d$ matrices $Q^m(t)$ and positive definite $a \times a$ matrices $R(t)$, as

$$\Pi_n^e = \mathbb{E}\left[\int_{t_n}^{T} \mathrm{e}^{-\rho(t-t_n)}\left(\|x(t)\|^2_{Q^{m(t)}(t)} - \|s_n(t)\|^2_{R(t)}\right)\,\mathrm{d}t\right].$$

The economic interpretation is as in the discrete time case, sharing a resource $\|x(t)\|^2_{Q(t)}$ jointly produced at an individual cost $\|s_n(t)\|^2_{R(t)}$. It is for pure notational convenience that we do not let $R(t)$ depend on m, as we shall need its inverse $R^{-1}(t)$.

As in the discrete time case, we must restrict all players to finite weighted norm decisions:

$$\int_{t_n}^{T} \mathrm{e}^{-\rho(t-t_n)}\|s_n(t)\|^2_{R(t)}\,\mathrm{d}t < \infty. \tag{22}$$

We again seek a uniform solution with Value functions

$$V^m(t,x) = \|x\|^2_{P^m(t)}\,. \tag{23}$$

Isaacs equation now reads

$$\rho\|x\|^2_{P^m(t)} = \max_{s\in\mathsf{S}}\Big[\|x\|^2_{\dot{P}^m(t)} + 2x'P^m(t)(A(t)x + B(t)s + (m-1)B(t)\hat{s}) + \|x\|^2_{Q^m(t)} - \|s\|^2_{R(t)}\Big] + \lambda^m\big(\|x\|^2_{P^{(m+1)}(t)} - \|x\|^2_{P^m(t)}\big).$$

We drop explicit time dependences of the system matrices for legibility. We obtain

$$\hat{s} = R^{-1}B'P^m(t)x \tag{24}$$

and

$$\dot{P}^m - (\rho+\lambda^m)P^m + P^mA + A'P^m + (2m-1)P^mBR^{-1}B'P^m + Q^m + \lambda^mP^{m+1} = 0 \tag{25}$$

with the terminal condition

$$P^m(T) = 0\,. \tag{26}$$

As a corollary of theorem 3.2, we have proved the following:

Corollary 3.1 *If the Riccati equations* (25) (26) *all have a solution over* $[0, T]$*, the finite horizon linear quadratic problem has a unique uniform equilibrium given by equations (23,24,25,26).*

Note that in equation (25), the right hand side is locally Lipshitz continuous in P^m. Therefore, if it is finite dimensional, i.e. under hypothesis 3.1, it is locally integrable backwards. We infer:

Corollary 3.2 *Under hypothesis 3.1, there exists a positive* $T^\star$ *such that for all* $T < T^\star$*, the finite horizon linear quadratic problem has a unique uniform equilibrium.*

Entering and leaving We may of course deal with the case where players may leave the game in the same way as before, replacing in the Riccati equation (25) the term

$$\lambda^m\big(P^{m+1}(t) - P^m(t)\big)$$

by

$$\mathbb{P}^{m,m-1}P^{m-1}(t) - \mathbb{P}^{m,m}P^m(t) + \mathbb{P}^{m,m+1}P^{m+1}(t)\,.$$

the $\mathbb{P}^{m,k}$ being given by the table (21). The Riccati equations can no longer be integrated in sequence from $m = M$ down to $m = 1$. But they can still be integrated backward jointly, as a finite dimensional ordinary differential equation. As long as all $P^m(t)$ exist, the interpretation as Value functions still guarantees their nonnegativeness.

Several remarks are in order :

Remark 3.1

1. *If the maximum number of players that may enter the game is bounded, i.e. under hypothesis 3.1, this is an explicit algorithm to determine whether a uniform subgame perfect equilibrium can thus be computed and to actually compute it.*
2. *We only stated a sufficiency theorem. What happens if some of the Riccati equations diverge before* $t = 0$ *is a complicated matter, supposedly more complicated than for a simple zero-sum two-player differential game. And even in that case, we know that, under some non generic condition, a saddle point (a Nash equilibrium) may survive such a conjugate point. See [Bernhard(1979), Bernhard(1980)].*
3. *In the case where the* $Q^m(t)$ *would be nonpositive definite, and under hypothesis 3.1, one can prove that the Riccati equations do have a solution over* $[0, T]$*, proving the existence of the uniform subgame perfect equilibrium.*

3.4.2 Infinite horizon

We consider now the case where the system matrices A, B, Q^m, and R are constant, and the problem with payoffs

$$\Pi_n^e = \mathbb{E}\int_{t_n}^{\infty} \mathrm{e}^{-\rho(t-t_n)}[\|x(t)\|^2_{Q^{m(t)}} - \|s\|^2_R]\,\mathrm{d}t\,.$$

We still assume (3.1) and constrain all players to finite weighted norm decisions as in (22) with $T = \infty$. Furthermore, we assume that the matrix A has all its eigenvalues with real parts strictly smaller than ρ, to rule out the trivial case where all players could just play $s(t) = 0$ and get an infinitely large payoff, and that $Q^M > 0$. We may state the following:

Theorem 3.5 *If the M Riccati equations* (25) *when integrated backward from $P^m(0) = 0$, have a limit $\bar P^M$ as $t \to -\infty$, for a large enough ρ it holds that*

$$\bar P^M BR^{-1}B'\bar P^M - Q^M < 0\,, \tag{27}$$

and then, the strategy profile $\hat s_n(t) = R^{-1}B'\bar P^m(t)x(t)$ is a uniform subgame perfect pure Nash equilibrium.

Proof We aim to apply a modification of theorem 3.3. Notice first that if the limits exist, then the $\bar P^m$ solve the algebraic Riccati equations

$$-(\rho+\lambda^m)P^m+P^mA+A'P^m+(2m-1)P^mBR^{-1}B'P^m+Q^m+\lambda^mP^{m+1} = 0\,. \tag{28}$$

Therefore, the value functions $\|x\|^2_{\bar P^m}$ satisfy the stationary Isaacs equation of theorem 3.3.

Since the (variable) $P^m(t)$ are all positive definite, the $\bar P^m$ are nonnegative definite. But they are even positive definite. As a matter of fact, (28) shows that $x'\bar P^M x = 0$, which implies $\bar P^M x = 0$, is impossible since $\lambda^M = 0$ and $Q^M > 0$. Then, recursively, the same holds for all $m < M$. We also notice that using a standard comparison theorem for ordinary differential equations for $x'P^Mx$ with a constant x and equation (25), we see that the $P^m(t)$ are decreasing in ρ, and using (28) divided through by ρ, we see that their limits as $\rho \to \infty$ are zero. ($P^m(t)$ being decreasing with ρ is bounded.)

To apply the reasoning of theorem 3.3, because the integrand in the payoff is not uniformly bounded, we need first to verify that $\exp(-\rho t)\|x(t)\|^2_{\bar P^m}$ goes to zero when all players use their strategies $\hat s_n$. Since in infinite time, the number of players will almost surely reach M, the asymptotic behavior of the system is ruled by the control with $\bar P^M$. A direct calculation using (28) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\rho t}\|x(t)\|^2_{\bar P^M}\right] = x'[\bar P^MBR^{-1}B'\bar P^M - Q^M]x.$$

Apply a standard Lyapunov theory for linear systems to conclude that under the condition (27), the Value functions indeed all go to zero as $t \to \infty$.

We need also to check that this limit also holds when all players but one use the strategies $\hat{s}_n$, while the other one plays any admissible control $s(t)$. This will hold if the system where $M-1$ players play according to $\hat{s}$ and the last one zero is stable. Apply again a Lyapunov theory. We find that under this condition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\rho t}\|x(t)\|^2_{\bar{P}^M}\right] = -x'[\bar{P}^M B R^{-1} B' \bar{P}^M + Q^M]x.$$

Therefore, the system is indeed stabilized. ▮

Remark 3.2 *In the case where $Q^m \leq 0$, we can prove that indeed the Riccati equations have a limit as $t \to -\infty$. The stability condition is the same, but corresponds to the stability of the system with $M-1$ players using their Nash control.*

As in the discrete time case, in [Bernhard and Deschamps(2018)] we provide an example of application to an oligopoly with sticky prices, emphasizing the case of a monopolist such as a historical operator being informed that the government will issue a given number of new licenses.

4 Conclusion

In our previous article [Bernhard and Deschamps(2017b)] we investigated dynamic games with randomly arriving players and proposed a way to find a sequence of static equilibria for games in discrete time in finite and infinite horizons. With this one we resolve several limitations of the model therein since here we have: a true dynamic equilibrium, variable entry probability (or density), possibility of group entry (if there is a finite number of players) and some exit mechanisms, all done in discrete time and continuous time.

Here the tools of piecewise deterministic Markov decision processes have been extended to games with random player arrivals. We have chosen some specific problems within this wide class, namely identical players (there might be several classes of players as in, e.g. [Tembine(2010)]). We have emphasized Bernoulli arrival process in the discrete time case, Poisson in the continuous time case, with no exit. Yet, we have given a few examples of other schemes, with exit at random time also.

We have also considered a restricted class of linear quadratic problems as an illustration. All these might be extended to other cases. The present article shows clearly how to proceed. The question is to find which other cases are both interesting and, if possible, amenable to feasible computational algorithms.

In that respect, the unbounded number of players in the infinite horizon discrete time problem, and in all cases in continuous time, poses a problem, mainly computational in the former case, also mathematical in the later, because of the difficulty of dealing with an infinite set of partial differential equations. The computational problem, however, is not much different from that of discretizing an infinite state space.

The premise of the current article is squarely with an exogenous entry process. Yet, some form of endogenous influence on the process can be achieved by letting the entry probability depend on some endogenous data. A typical example in the case of a Cournot competition is shown in [Bernhard and Deschamps(2017a)].

Finally, we may point out the main weakness of our theory: our agents have no idiosyncratic state. As such, we could not, for example, deal with classes of agents, as *e.g.* in [Tembine(2010), Biard and Deschamps(2020)], or exit linked with the agent state, typically, as in [Kordonis and Papavassilopoulos(2015)], an exit after a fixed residence time.

Nevertheless we consider that our model can probably be used to better understand some real life economic problems and if not at least slightly extend economic modelling.

References

[Bernhard(1979)] Bernhard P (1979) Linear quadratic zero-sum two-person differential games: Necessary and sufficient condition. JOTA 27(1):51–69

[Bernhard(1980)] Bernhard P (1980) Linear quadratic zero-sum two-person differential games: Necessary and sufficient condition: Comment. JOTA 31(2):283–284

[Bernhard(1992)] Bernhard P (1992) Information and strategies in dynamic games. SIAM Journal on Control and Optimization 30(1):212–228

[Bernhard and Deschamps(2016)] Bernhard P, Deschamps M (2016) Dynamic equilibrium in games with randomly arriving players, working paper, CRESE # 2016-10, Université de Bourgogne-Franche Comté, France

[Bernhard and Deschamps(2017a)] Bernhard P, Deschamps M (2017a) Cournot oligopoly with randomly arriving producers. Working paper, CRESE # 2016-14, Université de Bourgogne-Franche Comté, France

[Bernhard and Deschamps(2017b)] Bernhard P, Deschamps M (2017b) On dynamic games with randomly arriving players. Dynamic Games And Applications 7:360–385

[Bernhard and Deschamps(2018)] Bernhard P, Deschamps M (2018) Cournot equilibrium with sticky prices and random entry

[Biard and Deschamps(2020)] Biard R, Deschamps M (2020) Quel impact pour les taxis si la loi supprimait toute différence réglementaire entre taxis et VTC ? CRESE University of Besançon, Working Paper 2020-03

[Breton et al.(2010)] Breton M, Sbragia L, Zaccour G (2010) A dynamic model for international environmental agreements. Environmental and Resource Economics 45:25–48

[Caines(2015)] Caines PE (2015) Mean field games. In: Ballieul J, Samad T (eds) Encyclopedia of Systems and Control, Springer, pp 706–712

[De Sinopoli et al.(2014)] De Sinopoli F, Meroni C, Pimienta C (2014) Strategic stability in Poisson games. Journal of Economic Theory 153:46–63

[Ekeland(1974)] Ekeland I (1974) Théorie des jeux et introduction à l'économie mathématique. Presses Universitaires de France

[Elliot and Kalton(1972)] Elliot RJ, Kalton NJ (1972) The existence of value in differential games of pursuit and evasion. Journal of Differential Equations 12:504–523

[Evans and Souganidis(1984)] Evans LC, Souganidis PE (1984) Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. Indiana University Mathematical Journal 33:773–797

[Ferguson(2005)] Ferguson ATS (2005) Selection by committee. In: Nowak A, Szajowski K (eds) Advances in Dynamic Games, no. 7 in Annals of the International Society of Dynamic Games, Birkhäuser, Boston, pp 203–209

[Fleming and Soner(1993)] Fleming WH, Soner HM (1993) Controlled Markov processes and viscosity solutions. Springer

[Friedman(1994)] Friedman A (1994) Differential games. In: Auman R, Sergiu H (eds) Handbook of Game Theory, Vol. 2, Elsevier, pp 781–789

[Gensbittel et al.(2016)] Gensbittel F, Lovo S, Renault J, Tomala T (2016) Zero sum revision games. `https://studies2.hec.fr/jahia/webdav/site/hec/shared/sites/tomala/acces_anonyme/GLRTsept01.pdf`

[Haurie and Moresino(2000)] Haurie A, Moresino F (2000) A stochastic programming approach to manufacturing flow control. IEEE Transactions on Automatic Control 32:907–919

[Khan and Sun(2002)] Khan MA, Sun Y (2002) Non-cooperative games with many players. In: Auman R, Hart S (eds) Handbook of game theory with economic applications, volume 3, Elsevier, chap 46, pp 1761–1808

[Kordonis and Papavassilopoulos(2015)] Kordonis I, Papavassilopoulos GP (2015) LQ Nash games with random entrance: An infinite horizon major player and minor players of finite horizon. IEEE Transactions on Automatic Control AC 60:1486–1500

[Kukushkin(1994)] Kukushkin NS (1994) Best response dynamics in finite games with additive aggregation. Games and Economic Behavior 48:94–110

[Laraki and Sorin(2015)] Laraki R, Sorin S (2015) Advances in dynamic games. In: Young P, Zamir S (eds) Handbook of Game Theory, Vol. 4, Elsevier, pp 27–93

[Lasry and Lions(2007)] Lasry JM, Lions PL (2007) Mean field games. Japanese Journal of Mathematics 2:229–260

[Levin and Ozdenoren(2004)] Levin D, Ozdenoren E (2004) Auctions with uncertain number of bidders. Journal of Economic Theory 118:229–251

[Lovo and Tomala(2015)] Lovo S, Tomala T (2015) Perfect equilibria in stochastic revision games. https://studies2.hec.fr/jahia/webdav/site/hec/shared/sites/tomala/acces_anonyme/SRG.pdf

[Mageirou(1976)] Mageirou EF (1976) Values and strategies for infinite time linear quadratic games. IEEE Transactions on Automatic Control AC-21:547–550

[Myerson(1998a)] Myerson RB (1998a) Extended Poisson games and the Condorcet jury theorem. Games and Economic Behavior 25:111–131

[Myerson(1998b)] Myerson RB (1998b) Population uncertainty and Poisson games. International Journal of Game Theory 27:375–392

[Neyman and Sorin(2003)] Neyman A, Sorin S (2003) Stochastic Games. NATO ASI series, Kluwer academic publishers

[Quincampoix(2009)] Quincampoix M (2009) Differential games. In: Meyers N (ed) Encyclopaedia of Complexity and System Science, Springer, pp 1948–1956

[Rishel(1975)] Rishel R (1975) Dynamic programming and minimum principles for systems with jump markov disturbances. SIAM Journal on Control 13:338–371

[Samuelson(1985)] Samuelson WF (1985) Competitive bidding with entry costs. Economic Letters 17:53–57

[Sworder(1969)] Sworder DD (1969) Feedback control of a class of linear systems with jump parameters. IEEE Transactions on Automatic Control AC-14:9–14

[Tembine(2010)] Tembine H (2010) Evolutionary networking games. In: Game Theory for Wireless Communications and Networking, Auerbach Publications, CRC Press, pp 133–158

[Vermes(1985)] Vermes D (1985) Optimal control of piecewise deterministic Markov process. Stochastics 14:165–207

A Games with identical players

By assumption, in the game considered here, all players are identical. To reflect this fact in the mathematical model, we need to consider permutations $\pi^m \in \Pi^m$ of the elements of $\{1, 2, \ldots, m\}$. We also recall the notation

$$\begin{aligned} s^{m \setminus n} &:= (s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_m)\,, \\ \{s^{m \setminus n}, s\} &:= (s_1, \ldots, s_{n-1}, s, s_{n+1}, \ldots, s_m) \end{aligned}$$

Furthermore, we denote

$$\begin{aligned} s^{\pi} = s^{\pi[m]} &:= \left(s_{\pi(1)}, s_{\pi(2)}, \ldots, s_{\pi(m)}\right)\,, \\ s^{\pi[m] \setminus \pi(n)} &:= \left(s_{\pi(1)}, \ldots, s_{\pi(n-1)}, s_{\pi(n+1)}, \ldots, s_{\pi(m)}\right)\,, \\ \{s^{\pi[m] \setminus \pi(n)}, s\} &:= \left(s_{\pi(1)}, \ldots, s_{\pi(n-1)}, s, s_{\pi(n+1)}, \ldots, s_{\pi(m)}\right)\,, \\ s^{\times m} &:= (s, s, \ldots, s) \in \mathsf{S}^m\,. \end{aligned}$$

Definition A.1 *An m-person game $\{J_n : \mathsf{S}^m \to \mathbb{R}\}$, $n = 1, \ldots, m$ will be called a* game with identical players *if, for any permutation π of the set $\{1, \ldots, m\}$, it holds that*

$$\forall n \leq m\,, \quad J_n(s_{\pi(1)}, \ldots, s_{\pi(m)}) = J_{\pi(n)}(s_1, \ldots, s_m)\,. \tag{29}$$

We shall write this equation as $J_n(s^{\pi[m]}) = J_{\pi(n)}(s^m)$.

An alternate definition of a game with identical players is given by the following:

Lemma A.1 *A game with identical players is defined by a function* $G : \mathsf{S} \times \mathsf{S}^{m-1} \to \mathbb{R}$ invariant by a permutation of the elements of its second argument, *i.e. such that,*

$$\forall s \in \mathsf{S}\,, \forall v^{m-1} \in \mathsf{S}^{m-1}\,, \forall \pi \in \Pi^{m-1}\,, \quad G(s, v^{m-1}) = G(s, v^{\pi[m-1]})\,. \tag{30}$$

And the J_n *are defined by*

$$J_n(s^m) = G(s_n, s^{m\backslash n}) \tag{31}$$

Proof It is clear that if the J_n are defined by (31) with G satisfying (30), they satisfy (29). Indeed, then

$$J_n(s^{\pi[m]}) = G(s_{\pi(n)}, s^{\pi[m]\backslash\pi(n)}) = G(s_{\pi(n)}, s^{m\backslash\pi(n)}) = J_{\pi(n)}(s^m)\,.$$

Conversely, assume that the J_n satisfy (29). Define

$$G(s_1, s^{m\backslash 1}) = J_1(s^m)\,.$$

Let $\pi_1 \in \Pi^{m-1}$, and π defined by $\pi(1) = 1$, and for all $j \geq 2$, $\pi(j) = \pi_1(j-1)$. (i.e. π is any permutation of Π^m that leaves 1 invariant.) It follows from (29) that

$$G(s_1, s^{m\backslash 1}) = J_1(s^m) = J_{\pi(1)}(s^m) = J_1(\{s_1, s^{\pi_1[m\backslash 1]} = G(s_1, s^{\pi_1[m\backslash 1]})\,.$$

Therefore G is invariant by a permutation of the elements of its second argument. Let now π be a permutation such that $\pi(1) = n$. We have

$$J_n(s^m) = J_{\pi(1)}(s^m) = J_1(s^\pi) = G(s_{\pi(1)}, s^{\pi\backslash\pi(1)}) = G(s_n, s^{m\backslash n})\,,$$

which is equation (31). And this proves the lemma. ▮

The main fact is that the set of pure Nash equilibria is invariant by a permutation of the decisions:

Theorem A.1 *Let* $\{J_n : \mathsf{S}^m \to \mathbb{R}\}$, $n = 1, \dots, m$ *be a game with identical players. Then if* $\hat{s}^m$ *is a Nash equilibrium, so is* $\hat{s}^{\pi[m]}$.

Proof Consider $J_n(\hat{s}^{\pi[m]})$, and then substitute some s to $\hat{s}_{\pi(n)}$ in the argument. Because $J_n(s^{\pi[m]}) = J_{\pi(n)}(s^m)$, it follows that

$$J_n(\{\hat{s}^{\pi[m]\backslash\pi(n)}, s\}) = J_{\pi(n)}(\{\hat{s}^{m\backslash\pi(n)}, s\}) \leq J_{\pi(n)}(\hat{s}^m) = J_n(\hat{s}^{\pi[m]})\,.$$

And this is true for all $n \leq m$, which proves the theorem. ▮

Example An example of the above reasoning is as follows. Let $m = 2$ and by hypothesis, $\forall (s_1, s_2)$, $J_1(s_2, s_1) = J_2(s_1, s_2)$. Let $(\hat{s}_1, \hat{s}_2)$ be a Nash equilibrium. Let us show that $(\hat{s}_2, \hat{s}_1)$ is also an equilibrium:

$$\forall s\,,\quad J_1(s, \hat{s}_1) = J_2(\hat{s}_1, s) \le J_2(\hat{s}_1, \hat{s}_2) = J_1(\hat{s}_2, \hat{s}_1).$$

Corollary A.1 *A pure Nash equilibrium of a game with identical players can be unique only if it is* uniform*, i.e. with all players using the same control:*

$$\exists \hat{s} \in \mathsf{S} : \forall n \le m\,, \quad \hat{s}^m_n = \hat{s}\,.$$

Existence of such a Nash equilibrium is not guaranteed, and even if it exists, it might not be the only one. However there is a simple way to look for one. Let us first assert the following fact:

Theorem A.2 *Let $\{J_n : \mathsf{S}^m \to \mathbb{R}\}, n = 1, \ldots, m$ be a game with identical players. If the function $s_1 \mapsto J_1(\{s^{m\setminus 1}, s_1\})$ is concave, so are all the functions $s_n \mapsto J_n(\{u^{m\setminus n}, s_n\})$.*

Proof Let $\tilde{s}^m = (s_n, s_2, \ldots, s_{n-1}, s_1, s_{n+1}, \ldots, s_m)$, and let $\pi^{1,n}$ be the permutation that swaps 1 and n. Then, $s^m = \tilde{s}^{\pi^{1,n}}$. Thus,

$$J_n(s^m) = J_n(\tilde{s}^{\pi^{1,n}}) = J_1(\tilde{s}^m) = J_1(s_n, \ldots)\,.$$

Now, J_1 is by hypothesis concave in its first argument, here u_n. Therefore J_n is concave in s_n. ▮

Finally, we shall use the corollary of the following theorem[5]:

Theorem A.3 *Let $\{J_n : \mathsf{S}^m \to \mathbb{R}\}, n = 1, \ldots, m$ be a game with identical players. Let $s \in \mathsf{S}$ and $s^{\times m} = (s, s, \ldots, s) \in \mathsf{S}^m$. Then*

$$\forall n \le m\,, \quad \mathrm{D}_n J_n(s^{\times m}) = \mathrm{D}_1 J_1(s^{\times m})\,.$$

Proof Observe first that obviously,

$$\forall n \le m\,, \quad J_n(s^{\times m}) = J_1(s^{\times m}).$$

Let now $\tilde{s}^m = (s + \delta s, s, \ldots, s)$, and as previously $\pi^{1,n}$ be the permutation that swaps 1 and n. Let us perturb the n-th control in $J_n(s^{\times m})$ by δs. We get

$$J_n(s, \ldots, s, s + \delta s, s, \ldots, s) = J_n(\tilde{s}^{\pi^{1,n}}) = J_1(\tilde{s}^m).$$

Therefore, the differential quotients involved in $\mathrm{D}_n J_n(s^{\times m})$ and $D_1 J_1(s^{\times m})$ are equal, hence the result. ▮

[5]Where we use Dieudonné's notation $\mathrm{D}_k J$ for the partial derivative of J with respect to its k-th variable

Corollary A.2 *If* $s_1 \mapsto J_1(s^m)$ *is concave, an interior solution* $\hat{s} \in \mathsf{S}$ *of the equation*

$$D_1 J_1(s^{\times m}) = 0 \tag{32}$$

yields a uniform Nash equilibrium $\hat{s}^{\times m}$*.*

B Continuous Isaacs equation

B.0 Viscosity solutions

We quickly recall the definition of the viscosity solution of a first order PDE. Let Z be an open domain of a Euclidean space with a smooth boundary ∂Z. Let $V : Z \to \mathbb{R} : z \mapsto V(z)$ be a real function on Z, and $F : Z \times \mathbb{R} \times Z \to \mathbb{R} : (z, V, q) \mapsto F(z, V, q)$ a real function, and $G : \bar{Z} \to \mathbb{R}$ a continuous real function on the closure $\bar{Z}$ of Z. Let V_z stand for the gradient of V. Consider equation

$$\forall z \in Z \; : \; F(z, V, V_z) = 0 \, , \quad \forall z \in \partial Z \, , \; V(z) = G(z) \, . \tag{33}$$

Definition B.1 *A continuous function* V *is said to be a* viscosity solution *of equation* (33) *if*

- *it satisfies the boundary condition,*
- *for all* C^1 *real function* ϕ *on* Z *and all* $z \in Z$ *such hat* $V - \phi$ *has a local minimum at* z*,* $F(z, V, \phi_z) \geq 0$*,*
- *for all* C^1 *real function* ϕ *on* Z *and all* $z \in Z$ *such hat* $V - \phi$ *has a local maximum at* z*,* $F(z, V, \phi_z) \leq 0$.

The definition above can alternatively be written in terms of the super- and sub-differential of ϕ.

Observe that the viscosity solutions of $F = 0$ and of $-F = 0$ do not coincide.

B.1 Modified, bounded m, problem

We first evaluate the following mathematical expectation, given t_m:

$$\mathcal{S}^m = \mathbb{E} \left[\int_{t_m}^{t_{m+1}} \mathrm{e}^{-\rho t} L^m(t, x(t), s^m(t)) \, \mathrm{d}t + \mathrm{e}^{-\rho t_{m+1}} V^{m+1}(t_{m+1}, x(t_{m+1})) \right] .$$

given that both $L^m(t)$ and $V^{m+1}(t)$ are taken equal to zero if $t > T$. We have

$$\mathcal{S}^m = \mathrm{e}^{-\lambda^m(T-t_m)} \int_{t_m}^{T} \mathrm{e}^{-\rho t} L^m(t, x(t), s^m(t))\,\mathrm{d}t + \\ \int_{t_m}^{T} \lambda^m \mathrm{e}^{-\lambda^m(\tau-t_m)} \left[\int_{t_m}^{\tau} \mathrm{e}^{-\rho t} L^m(t, x(t), s^m(t))\,\mathrm{d}t + \mathrm{e}^{-\rho\tau} V^{m+1}(\tau, x(\tau)) \right] d\tau\,.$$

Exchanging the order of summations in the double integral, changing the name of the integration variable in the second, it comes, after cancellation of the first term with one of those coming from the double integral:

$$\mathcal{S}^m = \int_{t_m}^{T} \mathrm{e}^{-\lambda_m(t-t_m)-\rho t} \big(L^m(t, x(t), s^m(t)) + \lambda^m V^{m+1}(t, x(t)) \big)\,\mathrm{d}t\,. \tag{34}$$

We turn to the Isaacs equation (18), and deal with it as if the Value functions V^m were of class C^1. Multiply both sides of the equation by $\exp(-\lambda(t - t_m) - \rho t)$ and rewrite it as

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\mathrm{e}^{-\lambda^m(t-t_m)-\rho t} V^m(t, x(t)) \Big) + \mathrm{e}^{-\lambda^m(t-t_m)-\rho t} L^m(t, x(t), s^m(t)) \\ + \lambda^m \mathrm{e}^{-\lambda^m(t-t_m)-\rho t} V^{m+1}(t, x(t)) \le 0\,,$$

being understood that the lagrangian derivative and L^m are evaluated at $s^m(t) = \{s(t), \hat{s}^{(m\setminus 1)}(t)\}$, and that the inequality becomes an equality for $s(t) = \hat{s}(t)$. Integrating from t_m to T, we recognize $\mathcal{S}^m$ and write

$$\mathrm{e}^{-\rho t_m} V^m(t_m, x(t_m)) \ge \mathrm{e}^{-(\lambda^m+\rho)T+\lambda^m t_m} V^m(T, x(T)) \\ + \mathbb{E} \left[\int_{t_m}^{t_{m+1}} L^m(t, x(t), s^m(t)) + \mathrm{e}^{-\rho t_{m+1}} V^{m+1}(t_{m+1}, x(t_{m+1})) \right].$$

In the finite horizon version, we have $V^m(T, x) = 0$, so that the first term on the right hand side cancels, and we are left with

$$\mathrm{e}^{-\rho t_m} V^m(t_m, x(t_m)) \ge \\ + \mathbb{E} \left[\int_{t_m}^{t_{m+1}} L^m(t, x(t), s^m(t)) + \mathrm{e}^{-\rho t_{m+1}} V^{m+1}(t_{m+1}, x(t_{m+1})) \right]$$

if player one, say, deviates alone from $\hat{u}^m(t)$, and equality if $u^m(t) = \hat{u}^{(m)}(t)$. In the infinite horizon case, use the fact that V^m is bounded to see that the same first term of the r.h.s. cancels in the limit as T goes to infinity.

With this last inequality, we proceed as in discrete dynamic programing: take the *a priori* expectation of both sides, sum for all $m \le M$, cancel the terms that appear on both sides of the sum and use $t_1 = 0$ (the first player starts at time 0) to get

$$V^1(0, x_0) \geq \mathbb{E} \int_0^T \mathrm{e}^{-\rho t} L^{m(t)}(t, x(t), s^m(t)) \,\mathrm{d}t = \Pi_1^e(0, x_0, s^m)\,,$$

for $s^m(t) = \{s(t), s^{(m\setminus 1)}(t)\}$, and equality if $s^m(t) = \hat{s}^{(m)}(t)$.

Having restricted our search to state feedback strategies and to a uniform equilibrium of identical players, and ignoring the intrinsic fixed point problem that for each (m, t, x) the maximizing control be precisely $\hat{\phi}^m(t, x)$ used by all other players, the inequality in definition 2.2 defines a unique maximization problem. As a consequence, in the case where the functions V^m are not globally C^1, both the necessary and the sufficiency characters with viscosity solutions are derived from this calculation in the same way as for one-player control problems. But a major difference with that case is that here, existence is far from granted. On the one hand, the fixed point for each (m, t, x) may not exist, and on the other hand, if it always does, it might not define an admissible strategy as characterized in paragraph 3.1.2. The situation is more complex for many player games than for two player games, where one can dispense with state feedback strategies. For these difficult technical matters, see [Evans and Souganidis(1984), Friedman(1994), Quincampoix(2009), Laraki and Sorin(2015)].

B.2 Unmodified unbounded m problem

We aim to extend theorem 3.2 to the unmodified problem where the number of players who may join the game before the time T is unbounded, and therefore equation (18) involves an infinite number of functions V^m. We simplify the notations as follows. Given two admissible state feedbacks ϕ and ψ, let

$$G(\phi, \psi) = \Pi_1^e(\{\phi, \psi^{\times m(t)\setminus 1}\})$$

and the same with upper index M (respectively N) be the corresponding quantity in the modified problem where $\lambda^M = 0$ (resp. $\lambda^N = 0$).

We make the following hypotheses which would need to be converted into hypotheses bearing on the data f^m and L^m of the problem, probably via the hamiltonian

$$H^m(t, x, p, s) = \max_{r \in \mathsf{S}} [\langle p, f(t, x, \{r, s^{\times m\setminus 1}\})\rangle + L(t, x, \{r, s^{\times m\setminus 1}\})]\,.$$

We endow the set of state feedbacks with the topology of L^1 and assume:

Hypothesis B.1

1. *The function $\phi \mapsto G(\phi, \psi)$ is, for all ψ quasi concave with a unique maximum and differentiable.*

2. *There exists a positive number β such that,*

$$\forall M \in \mathbb{N}\,,\ \forall \phi, \chi, \psi \in \mathcal{A}\,,\ \forall \mu \in [0,1]$$
$$G^M\big((1-\mu)\phi + \mu\chi, \psi)\big) \leq$$
$$(1-\mu)G^M(\phi,\psi) + \mu G^M(\chi,\psi) + \frac{\beta}{2}\mu(1-\mu)\|\phi - \chi\|^2\,.$$

   If $\phi \mapsto G^M(\phi, \psi)$ is of class C^2, this is equivalent to

$$\forall \phi, \chi, \psi \in \mathcal{A}\,, \quad |\langle \mathrm{D}_{11}G(\phi,\psi)\chi, \chi\rangle| \leq \beta\|\chi\|^2\,.$$

3. *For all M and ψ, the map $\phi \mapsto D_1G^M(\phi, \phi)$ is locally invertible in a neighborhood of zero with an inverse locally uniformly Lipshitz of modulus γ. If $(\phi, \psi) \mapsto G^M(\phi, \psi)$ is of class C^2, it suffices that the operator $\mathrm{D}_{11}G(\phi,\psi) + \mathrm{D}_{12}G(\phi,\psi)$ be onto, with an inverse uniformly bounded by a positive number γ.*

With this set of hypotheses, too abstract at this stage, we can prove conjecture 3.1. We first prove a simple lemma.

Let $\mathbb{P}$ be the probability structure induced by the entry process in the original problem, $\mathbb{E}$ the mathematical expectation in that probability law, $\mathbb{P}^M$ the probability law induced by the modified problem with $\lambda^M = 0$, and $\mathbb{E}^M$ the mathematical expectation in that law. We prove the following lemma.

Lemma B.1 *Let $X(\omega)$ be a bounded random variable measurable on the sigma-field generated by the entry process. $\mathbb{E}^M X$ converges to $\mathbb{E}X$ as M goes to infinity.*

Proof In the original problem, let Ω^M be the set of events for which $m(T) < M$ and Ω_M the complement: events such that $m(T) \geq M$. These sets belong to the sigma-field generated by the entry process. We have

$$\mathbb{E}(X) = \int_{\Omega^M} X(\omega)\,\mathrm{d}\mathbb{P}(\omega) + \int_{\Omega_M} X(\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

and similarly for $\mathbb{E}^M X$. Now, both laws coincide over Ω^M. Therefore

$$\begin{aligned} |\mathbb{E}X - \mathbb{E}^M X| &= \left| \int_{\Omega_M} X(\omega)\,\mathrm{d}\big(\mathbb{P}(\omega) - \mathbb{P}^M(\omega)\big) \right| \\ &\leq \sup_{\omega \in \Omega_M} |X(\omega)| \left(\mathbb{P}(\Omega_M) + \mathbb{P}^M(\Omega_M)\right)\,. \end{aligned}$$

Notice finally that $\mathbb{P}(\Omega^M) = \mathbb{P}^M(\Omega^M)$, and therefore for their complements:

$$\mathbb{P}(\Omega_M) = \mathbb{P}^M(\Omega_M) = \mathbb{P}(m(T) \geq M) < \frac{(\Lambda T)^M}{M!}$$

which goes to zero with M. As a consequence, $\mathbb{E}^M X$ converges to $\mathbb{E}X$ as M goes to infinity. ∎

Let $M < N$ be two integers. Let φ^M and φ^N be the equilibrium feedbacks of the modified problems G^M and G^N respectively. Using the lemma, we see that given a positive number ε, there exists an integer K such that for any M and N lager than K, and any φ,

$$|G^M(\varphi, \varphi^N) - G^N(\varphi, \varphi^N)| \leq \varepsilon\,.$$

It follows that

$$\forall \varphi\,, G^M(\varphi, \varphi^N) \leq G^N(\varphi, \varphi^N) + \varepsilon \leq G^N(\varphi^N, \varphi^N) + \varepsilon \leq G^M(\varphi^N, \varphi^N) + 2\varepsilon\,.$$

From the fact that $G^M(\varphi^N, \varphi^N)$ is close to the maximum in ϕ of $G^M(\phi, \varphi^N)$ and hypothesis 2, we may derive that

$$\|\mathrm{D}_1 G^M(\varphi^N, \varphi^N)\| \leq 2\sqrt{\beta\varepsilon}.$$

On the other hand, $\mathrm{D}_1 G^M(\varphi^M, \varphi^M) = 0$. From hypothesis 3 we conclude that

$$\|\varphi^N - \varphi^M\| \leq 2\gamma\sqrt{\beta\varepsilon}\,.$$

Hence the sequence $\{\varphi^M\}$ is Cauchy, and thus converges to some $\varphi^\star$. Hence the $V^{m|M}$ converge, and because all satisfy the P.D.E. they converge in C^1.

Hypotheses B.1 can be made mode concrete with the use of the theory of the second variation to calculate the second order derivatives. There is little point in developing that idea here.