Kalman: beyond the filter

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Foreword

This is the source manuscript of what became the article "Kalman 1960: the birth of modern system theory" published in *Mathematical Population Studies*, volume 26, pages 123–145. However, the published text is an extensive re-writing of the manuscript by an anonymous[1] copy editor who sought to improve the quality of the English language, over the 7 iterations of an unusually painful and contentious reviewing process. We could not avoid less satisfactory statements in the resulting text —also spoiled by the introduction on line of our footnotes—, let alone a dubious improvement of the English language. The text hereafter is our real contribution.[2]

Abstract

Rudolph E. Kalman is mainly known for the Kalman Filter, first published in 1960. We emphasize the fact that he published the same year two equally important contributions: linear state space system theory and linear quadratic optimal control. We give a synthetic account of these three domains and of their intertwining in the (somewhat later) theory of Linear Quadratic Gaussian (LQG) control. We provide an example of the use of an extended version of the linear quadratic optimal control theory in a problem of cooperation in mathematical population ecology.

1 Introduction

Rudolf Emil Kalman ("R.E.K.") passed away on July, 2nd, 2016. His name will remain tied to the most famous *Kalman filter*, which, by the end of his life, he would himself call "the KF". (We will follow his lead here.)

This is a mathematical method, which can be implemented as a set of algorithms, to estimate time varying variables from noisy and incomplete measurements, using a

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[1]but clearly French-speaking judging from his own English language mistakes

[2]As compared to our submitted manuscript, in order to stay closer to the historical formulations we replaced here $\|y\|_Q^2$ by $\|x\|_Q^2$ in the LQ optimal control problem, and dually deleted the weighting matrix D in the KF, and we enlarged the remark following theorem 6.

(noisy) model of the underlying dynamics.[3] Because it lets one estimate a variable which is not directly measured, it has sometimes been "commercialized" by computer scientists as a "software sensor". It, and its extensions, are now widely used in a great variety of domains, industrial and technological of course, but also in social, biological, and earth sciences, health systems, etc. It is a fact that even persons who have never heard of it have used it extensively, be it only in their GPS receivers.

But the point we want to make here is that the filter article [Kalman, 1960b] is only one of the three major papers of R.E.K. in 1960. The other two are the great article about Linear Quadratic optimal control [Kalman, 1960a] —a 1959 conference communication—, and the system theory article [Kalman, 1960c], to be followed by [Kalman, 1962] —dated January 1961 in the proceedings— and [Kalman, 1963]. To control theoreticians, these were earth-shaking contributions, definitively transforming control theory and linear system theory, well beyond filtering and prediction.[4]

Bolstered by the advent of the digital computer, these theories were extensively put to use via the "Automatic Synthesis Program" in the Apollo lunar landing program which started at the same time. Seldom has a brand new piece of theory been embraced so quickly by the practitioners in such an important endeavor.[5]

We will give here an ordered exposition of Kalman's contribution of the early 60's, focusing on control theory, and end up with a small example of its use in a small model in mathematical population ecology.

2 System theory

2.1 A paradigmatic change

2.1.1 State of the theory before 1960

To appreciate the importance of R.E.K.'s contribution to systems and control, one must have a crude idea of what was the state of these fields before 1960. The theory was confined to linear time-invariant (LTI) systems, and essentially to so-called "monovariable" ones, i.e. where the signals considered were scalar.

A (linear) dynamical system is a device which is excited by a time varying signal, the *input*, and simultaneously produces a time varying signal, the *ouput*. To the mathematicians, a linear system is therefore a linear operator, transforming a time function called input into another time function called output. Because this happens in real time, some properties are required: causality (the current output does not depend on future inputs), and some others (relating notably to behavior "at infinity"), and linearity and time invariance to be amenable to the theory then available.

[3]Do not say that it belongs to "estimation theory", as R.E.K. strongly objected to that qualification, arguing that this phrase had come to name a body of non-science. He later denied its being based upon probabilities, and refused its definition as (only) an algorithm, because it is a complete theory.

[4]Another pair of important articles on stability of linear dynamical systems also appeared in 1960: [Kalman and Bertram, 1960a, Kalman and Bertram, 1960b], complementing the other control articles. We will not discuss them here.

[5]Another example, though, may be the KF's predecessor: the Wiener filter [Wiener, 1949], largely conceived in 1940 for the anti-aircraft radar, and embargoed until 1949 for its military sensitivity.

Mathematicians had developed an extremely elegant and powerful way of manipulating such transformations, through the use of an esoteric mathematical transformation of the signals : the *Laplace transform*, which led to the representation of the system via a "transfer function" (a ratio of polynomials of a complex variable). An important consideration is that it is straightforward to derive the transfer function from a linear differential equation describing the system. It had, inter alia, the property of transforming cascade of systems (the output of the first one becoming the input of the second one) into simple products of their transfer functions, making possible the analysis of *feedback* systems, where the output is re-introduced as a component of the input of the same system, a necessary ingredient of any servomechanism. It also led to the Wiener filter, then the standard tool in signal processing.

These methods were intimately tied to the *frequency response* of the system: its behavior if excited by sinusoidal signals of various frequencies. Engineers had developed both powerful analytical and graphical tools and a deep understanding of the frequency content of signals and its meaning for the analysis and control of LTI systems.

A clever trick of Wiener filtering was to consider a noisy signal to be "filtered" (freed from its noisy content) as the output of a linear system excited by a noise with adequate statistical properties. This was used by R.E.K. in the KF, so that he once told one of the authors of this article: "Take the Kalman filter, which, as everybody knows, was invented by Wiener…". This author would not have dared that quip.

2.1.2 Innovations of 1960

R.E.K. chooses to represent the transformation of inputs into outputs by the mediation of an internal *state* of the system: in his case a vector of several real variables that also vary with time according to a forced first order differential or difference equation. Hence the name of an *internal* description of the system (the classical one becoming *external*.) In that representation, the input acts on the dynamics of the state, and the state instantly produces an output. Since the state itself is a vector, and all relations considered are linear, matrices and linear algebra are at play. And this mathematical apparatus lends naturally itself to the consideration of vector-valued inputs and outputs. Moreover, if the matrices defining the system are time varying, the system is no longer time invariant. If part of realization theory, the heart of linear system theory, still concerns LTI systems, both filtering and what we want to stress here: control, can naturally be extended to non-time invariant systems.

The notion of state of a system was known imprecisely as "*a set of numbers from which the entire future behavior of the plant may be determined provided that the future inputs of the plant are known*" (quoted from [Kalman and Koepcke, 1958])[6]. Typically, positions and velocities in a mechanical system, intensities in inductors and charges of capacitors in an electrical system. The intimate link with first order differential equations was obviously recognized. But their almost exclusive use was in deriving from them the transfer function of the (LTI) system, a straightforward process. The direct use of the differential (or difference) equations in optimization was becoming more frequent in the late fifties under the influence of Bellman's Dynamic Programming

[6][Kalman, 1960c] adds the precision that it is the *smallest* such set. This was to be made precise later via the (earlier) concept of Nerode equivalence class of formal languages and automata theory.

[Bellman, 1957]. R.E.K.'s bold move was to make it the core definition of a linear system and to investigate in depth its properties.

With the internal description, the extension to non-time invariant systems is possible because the tools developed dispense with the Laplace transform (the frequency analysis). They manipulate the signals as time functions, hence also the name of *time domain* analysis, as opposed to the *frequency domain*. For control theory, they delve into calculus of variations, and for filtering into Markov processes.

The advent of the digital computer and of direct digital control also led R.E.K. and later researchers to systematically develop a discrete-time theory along the continuous-time one, at first as a theory of *sampled data systems* i.e. looking at a continuous-time system at discrete instants of time. It turned out that the parallel was particularly natural and elegant in the new theory.[7]

The transfer function of a time invariant system in internal form is obtained via a simple algebraic formula. The converse: finding the internal representation of a system given in external form, is a deeper question, involving a detailed analysis of the mathematical nature of a linear system in internal form, the topic of realization theory.

2.2 Realization theory

Kalman's contribution to system theory started with his article "On the general theory of control systems" [Kalman, 1960c]. But the real founding article, which we shall follow here, is [Kalman, 1962] (or the journal article [Kalman, 1963]).

Definition 1 *A* realization *of a linear (or affine) input-output transformation is a representation in internal form as* (1)(2) *or* (3)(4) *below.*

Let $x \in \mathbb{R}^n$ be the state of the system (n is called the dimension of the realization), $u \in \mathbb{R}^m$ be the input, or control, and $y \in \mathbb{R}^p$ the output. To avoid some trivialities, we assume that $m \leq n$ and $p \leq n$. We use Newton's notation for time derivatives: $\dot{x} = \mathrm{d}x/\mathrm{d}t$. A continuous-time system is of the form[8]

$$\dot{x}(t) = Fx(t) + Gu(t)\,, \tag{1}$$

$$y(t) = Hx(t)\,, \tag{2}$$

and in discrete time

$$x(t+1) = Fx(t) + Gu(t)\,, \tag{3}$$

$$y(t) = Hx(t)\,. \tag{4}$$

Notice that some would like to extend the output equation adding a term $+Ju(t)$. R.E.K. himself argued against in general system theory, with good arguments.

[7] It should be mentioned, though, that with an admirable prescience, Kolmogorov [Kolmogorov, 1941] had developed, independently from Wiener, a discrete-time version of the Wiener —or Wiener-Kolmogorov— filter.

[8] Essentially all the current literature replaces (F, G, H) by (A, B, C). We keep R.E.K.'s notation as he really meant it ! He noted a system as (H, F, G).

For the sake of completeness, let us mention that if all three matrices F, G, and H are constant, the system is LTI, and its transfer function is

$$\mathcal{H}(s) = H(sI - F)^{-1}G\,.$$

It follows from Cramer's theorem that this is a matrix of (strictly proper) rational fractions of s, with as their common denominator, the characteristic polynomial of F, and thus for poles its eigenvalues.

Even without referring to the transfer function, it is clear that the transformation from input to output induced by these equations is not altered if we make a change of basis in the state space, or equivalently if we use for new state $\xi = Tx$ with T an invertible matrix. The continuous-time system becomes

$$\begin{aligned}\dot{\xi} &= TFT^{-1}\xi + TGu\,,\\ y &= HT^{-1}\xi\,.\end{aligned}$$

Hence changing (H,F,G) into (HT^{-1},TFT^{-1},TG) is unessential, representing the same system, with the same transfer function if it is time-invariant. This points to a weakness of the new representation: it is non unique for the same input-output system. And other non-uniqueness may show up, as the following trivial one, where we introduce a higher dimensional vector, say z made of x and a vector ξ of arbitrary dimension:

$$z = \begin{pmatrix} x \\ \xi \end{pmatrix}, \qquad \begin{aligned}\dot{z} &= \begin{pmatrix} F & 0 \\ A & B \end{pmatrix} z + \begin{pmatrix} G \\ C \end{pmatrix} u\\ y &= \begin{pmatrix} H & 0 \end{pmatrix} z\,.\end{aligned}$$

A, B and C are arbitrary matrices. They play no role, since ξ does not influence y, neither directly nor via x. Therefore, realizations of different dimensions may represent the same input-output system. In the case above, it is trivial, but assume that a change of basis such as the previous one mixes x and ξ, and the excess dimension may be more difficult to detect. Moreover, other cases may appear.

The solution of this problem involves two fundamental concepts. The first is *controllability*. A state is controllable if there exists a control function $u(\cdot)$ that drives the system from this state as initial state to the origin. The system is said to be *completely controllable* if every state is controllable. We will here cheat somewhat with history by using instead *reachability*. A state is reachable if there exists a control function that drives the system from the origin to that state. A system is *completely reachable* if every state is reachable. The two concepts are equivalent for continuous-time systems, but not for discrete-time systems, unless the matrices $F(t)$ are invertible for all t. Hence a system is completely reachable if the application $u([t_0,t_1]) \mapsto x(t_1)$, which is linear if $x(t_0) = 0$, is onto for some $t_1 > t_0$.

The second fundamental concept is that of *observability*. A state $x_0 \neq 0$ is *unobservable* if the ouput of the "free" system, i.e. with $u(\cdot) = 0$, initialized at that state, is $y([t_0,t_1]) = 0$ for any $t_1 \geq t_0$. The system is *completely observable* if no state is unobservable. Hence the system is completely observable if the application $x(t_0) \mapsto y([t_0,t_1])$, which is linear if $u(\cdot) = 0$, is one to one for some $t_1 > t_0$.

The article [Kalman, 1960a] also gives efficient criteria to check these properties. In the case of time-invariant systems the Kalman criteria are in terms of the rank of composite matrices:

Theorem 1

$$(F\ G) \text{ \textit{completely reachable}} \Leftrightarrow \operatorname{rank}[G\ FG\ F^2G \dots F^{n-1}G] = n\,,$$

$$(H\ F) \text{ \textit{completely observable}} \Leftrightarrow \operatorname{rank}\begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix} = n$$

The article also gives simple criteria for non time-invariant systems. They are less algebraic, more analytic, but they share the following striking property (stated in two equivalent forms below), called *duality*: (we use prime for transposed)

$$(F\ G) \text{ completely reachable} \Leftrightarrow (G'\ F') \text{ completely observable}$$

$$(H\ F) \text{ completely observable} \Leftrightarrow (F'\ H') \text{ completely reachable}$$

We will see that this duality reaches into optimal control and filtering. R.E.K. himself gave a detailed analysis of the duality between the Wiener filter and the linear quadratic regulator [Kalman, 1960c][9]. But the whole extent of duality in the LQG theory to be seen below remains difficult to explain, the more so that it mysteriously extends into modern $\mathcal{H}_\infty$-optimal control (see [Başar and Bernhard, 1995]).

We use these concepts in realization theory with the help of the following formal definition:

Definition 2 *A realization completely reachable and completely observable is called* canonical*.*

The main theorem is as follows:

Theorem 2 *A realization is minimal (has a state space of minimum dimension) if and only if it is canonical. And then it is unique up to a change of basis in the state space.*

The article [Kalman, 1962] further shows that the state space of any linear system in internal form, even not time invariant (i.e. a system such as (1),(2) with matrices H, F, and G depending continuously on time) can be decomposed canonically as the direct sum of four (variable if the system is not LTI) subspaces as in the left hand diagram of figure 1 borrowed from [Kalman, 1962], or more classically in block diagram form as in the right hand one borrowed from [Kalman, 1963], and reproduced in all textbooks since. A, B, C, and D are the subspaces corresponding respectively to states reachable but unobservable, reachable and observable, unreachable and unobservable, unreachable but observable. Using a basis adapted to that decomposition yields a canonical

[9] Defining the observability of *costates* in the dual space of the state space. Duality clearly has to do with the fact that a linear operator between linear spaces is onto if and only if its adjoint operator is one to one, and it is one to one if and only if its adjoint operator is onto. As pointed out in [Kalman, 1960c], it is also related to the known duality between the differential equations $\dot{x} = Fx$ and $\dot{p} = -F'p$ or the difference equations $x(t+1) = Fx(t)$ and $p(t) = F'p(t+1)$ which leave the inner product $p'x$ invariant.

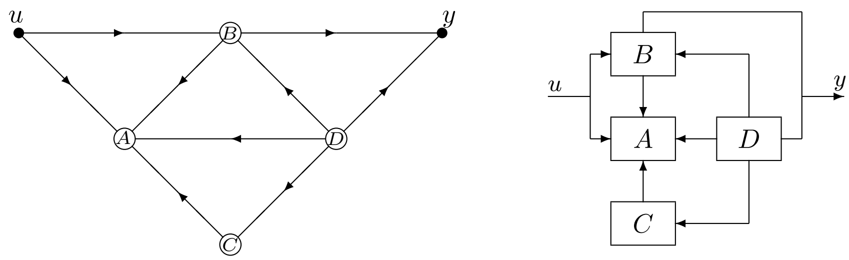

Figure 1: The canonical decomposition of a linear system in internal form

decomposition of the system matrices that exhibits these properties as follows:

$$F = \begin{bmatrix} F_{AA} & F_{AB} & F_{AC} & F_{AD} \\ 0 & F_{BB} & 0 & F_{BD} \\ 0 & 0 & F_{CC} & F_{CD} \\ 0 & 0 & 0 & F_{DD} \end{bmatrix}, \quad G = \begin{bmatrix} G_A \\ G_B \\ 0 \\ 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & H_B & 0 & H_D \end{bmatrix}.$$

The subspaces B, C, and D are not uniquely defined, but the decomposition of the system matrices is, up to changes of basis within each of the four subspaces.

2.3 Compensator design: pole placement approach

The KF was the first *observer*[10], either in discrete time as in [Kalman, 1960c] (but without the control term)

$$\hat{x}(t+1) = F\hat{x}(t) + Gu(t) + K[y(t) - H\hat{x}(t)]\,, \tag{5}$$

or in continuous time, as in [Kalman and Bucy, 1961]:

$$\dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + K[y(t) - H\hat{x}(t)]\,, \tag{6}$$

providing an estimate $\hat{x}(t)$ of the state, optimal in the sense that it minimizes the expected squared L^2 norm of the *error signal* $\tilde{x}(t) = x(t) - \hat{x}(t)$. The natural idea, proposed as early as [Kalman, 1960c] for monovariable discrete-time systems, is to associate such an observer with a control law

$$u(t) = -C\hat{x}(t)\,.$$

There remains to choose the gains K and C. This can be done via the following results.

Writing the overall system in terms of $(x, \tilde{x})$ instead of $(x, \hat{x})$, it is a simple matter to prove the *principle of separation of the dynamics* ([Luenberger, 1964]):

[10]Called "observing system" in [Kalman, 1960c] which proposes an "optimal" one in terms of the number of time steps necessary to exactly recover the state in a discrete time system. The term "observer" was coined by [Luenberger, 1964], which extends the concept, in a less explicit form that lacks the simplicity displayed here and the rather crucial stability argument invoked in paragraph 4.1.2.

Theorem 3 (Separation of dynamics) *: The eigenvalues of the dynamic matrix of the closed loop observer-controller are the union of the eigenvalues of the "controller" $F-GC$ and those of the "observer" $F-KH$.*

A later result [Wonham, 1967][11] is the following extension to multi-input systems of the *pole shifting theorem*, known in 1960 (and used in [Luenberger, 1964]) for single input systems:

Theorem 4 *If the pair (F,G) is completely reachable, then given any monic n-th degree polynomial $p(z)$, there exists a matrix C such that the characteristic polynomial of $F-GC$ be p. Dually, if the pair (H,F) is completely observable, the characteristic polynomial of $F-KH$ can be assigned to any desired one by the choice of K.*

Hence a purely system theoretic argument in favor of the proposed control structure, and a means of choosing C and K (see paragraph 4.1.2 below).

A further important remark is that in the discrete-time case, hence also in the sampled data problem of any digital control, the observer is a one step predictor: It gives the estimate $\hat{x}(t{+}1)$ of $x(t{+}1)$ with the data $y(\tau)$, $\tau \leq t$. Hence one has one time step to compute the control $u(t+1) = -C\hat{x}(t{+}1)$, the gain C being pre-computed off line.

3 Linear Quadratic Gaussian (LQG) theory

3.1 Linear Quadratic (LQ) optimal control

The topic covered here is partially investigated in [Kalman, 1960c], but the definitive article is [Kalman, 1960a]. We will essentially adopt its notation.

The introduction of [Kalman, 1960a] states, we quote: *"This problem dates back, in its modern form, to Wiener and Hall at about 1943."* It also quotes, although in rather denigrating terms, [Newton Jr et al., 1957] as the state of the art at that time. Therefore in its infinite horizon (optimal regulator) form, it was not new. But the solutions offered were in terms comparable to those of the Wiener filter, i.e. spectral factorization, and did not easily lead to efficient algorithms, particularly so for "multivariable" problems, and could not be extended to finite horizon problems.[12]

3.1.1 Finite horizon problem

The new approach started with the investigation of a finite horizon optimal control problem, i.e. not time invariant. It involves quadratic forms that we denote as follows: for any positive definite or semi-definite $\ell \times \ell$ matrix M and z a ℓ-vector,

$$\langle z, Mz\rangle = z'Mz = \|z\|_M^2 \,.$$

The problem investigated is as follows[13]:

[11] Other proofs were quickly given as comments of this article by [Heymann, 1968] and [Davison, 1968].

[12] Actually, a particular finite-horizon nonhogeneous scalar-control linear-quadratic optimization problem is solved in [Merriam III, 1959], with the correct Riccati equation and the linear equations for the non homogeneous terms, although difficult to recognize.

[13] In [Kalman, 1960a], the problem is first posed and investigated in a more general form.

Linear quadratic optimal control problem Given the system (1), with all system matrices possibly (piecewise) continuous functions of time, and $x(t_0) = x_0$, and given the symmetric $n \times n$ matrix $A \geq 0$, and the symmetric (piecewise) continuous respectively $n \times n$ and $m \times m$ matrix functions, $Q(t) \geq 0$ and $R(t) > 0$, find, if it exists, the control law that minimizes the performance index

$$V(x_0, t_0, t_1; u(\cdot)) = \|x(t_1)\|_A^2 + \int_{t_0}^{t_1} \left[\|x(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2 \right] \mathrm{d}t \,. \qquad (7)$$

Kalman's approach to the problem followed Carathéodory's [Carathéodory, 1935].[14] The solution is as follows. Define a symmetric matrix function $P(t)$ as the solution, if it exists, of the matrix Riccati equation (where all matrices are time dependent)

$$-\dot{P} = PF + F'P - PGR^{-1}G'P + Q \,, \qquad P(t_1) = A \,. \qquad (8)$$

(By Cauchy's theorem, there exists a solution on some open time interval (t_2, t_1). However existence of a solution over the time interval $[t_0, t_1]$ is by no means guaranteed a priori, as the solution might diverge to infinity before reaching down t_0.)

The full theorem is as follows:

Theorem 5

1. *The Riccati equation* (8) *has a solution* $P(t) \geq 0$ *over* $[t_0, t_1]$ *for every* $t_0 < t_1$.
2. *The solution of the linear quadratic optimal control problem is given in state feedback form by*

$$u(t) = -C(t)x(t) \,, \qquad C(t) = R(t)^{-1}G'(t)P(t) \,, \qquad (9)$$

3. *and the optimal value of the performance index is*

$$V^0(x_0, t_0, t_1) = \|x_0\|_{P(t_0)}^2 \,. \qquad (10)$$

Important remark: The Riccati equation (8) and the optimal feedback gain (9) are the duals of the KF's Riccati equation and gain. (See section 3.2)

3.1.2 Optimal regulator (infinite horizon) problem

The challenge in this theory is to investigate the infinite horizon problem. For the sake of simplicity, we only give here its LTI version, the only one amenable to the previous, "classical" at that time, theory, and the most widely used. But [Kalman, 1960a] also gives the solution for a non time-invariant problem.

[14] While [Kalman and Koepcke, 1958] and [Kalman, 1960c] explicitly use [Bellman, 1957] and (discrete) dynamic programming, a strong incentive for using the state space representation, [Kalman, 1960a] does not quote it for the continuous-time problem, using instead Carathéodory's theory. Symmetrically, [Bellman, 1957] does not refer to [Carathéodory, 1935]. Yet, continuous dynamic programming is essentially a re-discovery of Carathéodory's theory, Bellman's *return function* being the control equivalent of Carathéodory's *principal function*.

Optimal regulator problem Given the time invariant linear system (1) with initial state $x(0) = x_0$, a $p \times p$ symmetric matrix $Q \geq 0$ and a $m \times m$ symmetric matrix $R > 0$, find, if it exists, the control that minimizes the performance index

$$V(x_0; u(\cdot)) = \int_0^\infty \left[\|x(t)\|_Q^2 + \|u(t)\|_R^2\right] \mathrm{d}t\,.$$

This investigation requires the introduction of both controllability and observability. Indeed, in its introduction, R.E.K. states "*The principal contribution of the paper lies in the introduction and exploitation of the concepts of controllability and observability*". In retrospective, he might also have quoted the Riccati equation (8). We shall use here its algebraic version, where all matrices are now constant:

$$PF + F'P - PGR^{-1}G'P + Q = 0\,. \tag{11}$$

The full theorem is as follows:

Theorem 6

1. *If the pair (F, G) is completely controllable,*[15] *then*
   (a) *the solution $P(t)$ of the Riccati equation* (8) *has a limit $\bar{P}$ as $t \to -\infty$, which solves the algebraic Riccati equation* (11)*,*
   (b) *the solution of the optimal regulator problem in state feedback form is*

   $$u(t) = -Cx(t)\,, \qquad C = R^{-1}G'\bar{P}\,, \tag{12}$$

   (c) *and the optimal value of the performance index is*

   $$V^0(x_0) = \|x_0\|_{\bar{P}}^2\,.$$

2. *If furthermore the pair $(Q^{1/2}, F)$ is completely observable,*[16] *$\bar{P}$ is positive definite and the system governed by the law* (12) *is asymptotically stable.*

The optimal regulator thus obtained is a realization (internal representation) of the solution offered by [Newton Jr et al., 1957]. The relationship between the algebraic Riccati equation and spectral factorization, at the heart of the old solution, was further investigated by [Willems, 1971] and [Bernhard and Cohen, 1973]. Of course, the duality pointed out in the finite horizon problem holds here, making the optimal regulator dual to the stationary KF, i.e. to a realization of the Wiener filter.

3.1.3 Discrete-time case

Articles [Kalman, 1960b] and [Kalman, 1960c] provide the equivalent discrete-time results[17]. The system is (3)(4), and the performance index is:

$$V(x_0, t_0, t_1; \{u(\cdot)\}) = \|x(t_1)\|_A^2 + \sum_{t=t_0}^{t_1-1} \left[\|x(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2\right]\,. \tag{13}$$

[15] *stabilizable* i.e. $\exists D : F - GD$ stable, suffices.

[16] For the stability result, $(Q^{1/2}, F)$ *detectable*, i.e. $\exists L : F - LQ^{1/2}$ stable, suffices.

[17] First approached in [Kalman and Koepcke, 1958, Appendix], dealing with sampled data control of a continuous-time system. But the treatment there is not completely satisfactory.

The Riccati differential equation is replaced by the so-called discrete Riccati equation (the system matrices may all be time dependent) or its "algebraic" version (with all system matrices constant) where $P(t) = P(t+1) = \bar{P}$:

$$P(t) = F'P(t{+}1)F - F'P(t{+}1)G(G'P(t{+}1)G + R)^{-1}G'P(t{+}1)F + Q,$$
$$P(t_1) = A\,. \tag{14}$$

The optimal feedback control is

$$u(t) = -C(t)x(t)\,, \qquad C(t) = (G'P(t+1)G + R)^{-1}G'P(t+1)F\,. \tag{15}$$

Both the finite and infinite horizon results follow exactly as for the continuous-time case, with the same controllability and observability conditions.

3.2 The Kalman Filter

For the sake of completeness, and to stress duality, we quickly review the famous KF, [Kalman, 1960b] and [Kalman and Bucy, 1961]. Existence and stability properties for both the finite horizon and infinite horizon cases are directly derived from those of the dual LQ optimal control problem.

3.2.1 Discrete-time

We start with the discrete-time problem, after [Kalman, 1960b].[18] Let us consider a discrete time linear system excited by *white noise* and a known control $u(\cdot)$:

$$x(t+1) = F(t)x(t) + G(t)u(t) + v(t)\,, \quad x(t_0) = x_0\,,$$
$$y(t) = H(t)x(t) + w(t)\,,$$

where $(v(t), w(t))$ is a gaussian random variable with zero mean and known covariance, independent of all the $(v(\tau), w(\tau))$ for $\tau \neq t$. In the simplest case, $v(t)$ is also independent of $w(t)$, but this is not necessary for the theory to hold. A possible non-zero cross correlation between them is dual to the presence of a cross term $x'Su$ in the quadratic performance index of LQ control, which we will not introduce here. The noise is therefore characterized by its covariance matrix (with $\delta_{t,\tau}$ the Kronecker symbol):

$$\mathbb{E}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} (v'(\tau)\ w'(\tau)) = \begin{pmatrix} V(t) & 0 \\ 0 & W(t) \end{pmatrix} \delta_{t,\tau}\,.$$

The initial state is also given as a gaussian random variable of known distribution:

$$\mathbb{E}(x(t_0)) = \hat{x}_0\,, \qquad \mathbb{E}(x(t_0) - \hat{x}_0)(x(t_0) - \hat{x}_0)' = \Sigma_0\,.$$

The problem is to compute the conditional mathematical expectation:

$$\hat{x}(t) = \mathbb{E}(x(t)|y(\tau), \tau < t)\,. \tag{16}$$

[18][Kalman, 1960b] has no added noise in the measurement equation, nor control.

The solution is of the form (5) initialized at $\hat{x}(t_0) = \hat{x}_0$, where the gain K is given via the error covariance matrix $\Sigma(t) = \mathbb{E}(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))'$, solution of the discrete Riccati equation (17) dual of (14), and by the formula (18) dual of (15):

$$\Sigma(t+1) = F\Sigma(t)F' - F\Sigma(t)H'(H\Sigma(t)H' + W)^{-1}H\Sigma(t)F' + V\,, \tag{17}$$

$$K(t) = F\Sigma(t)H'(H\Sigma(t)H' + W)^{-1}\,. \tag{18}$$

The time invariant, infinite horizon case is the internal form of the Kolmogorov filter.

3.2.2 Continuous-time

Given a continuous-time system in internal form excited by both dynamic and measurement "white noises" as above, but in continuous-time, with (in terms of the Dirac δ as in [Kalman and Bucy, 1961], probabilists now have a different way of stating things)

$$\mathbb{E}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix}(v'(\tau)\ w'(\tau)) = \begin{pmatrix} V(t) & 0 \\ 0 & W(t) \end{pmatrix}\delta(t-\tau)\,,$$

the conditional expectation (16) sought is the solution of the continuous-time observer (6) with the dual formulas from LQ control:

$$\dot{\Sigma} = F\Sigma + \Sigma F' - \Sigma H'W^{-1}H\Sigma + V\,, \qquad \Sigma(t_0) = \Sigma_0\,.$$

and

$$K(t) = \Sigma(t)H'W^{-1}\,.$$

The time invariant, infinite horizon case coincides with the internal representation of the Wiener filter which, as the Kolmogorov filter, was given in external form.

3.3 The separation theorem

The control laws (9) or (12) or (15) assume that the state $x(t)$ is exactly measured. But the underlying assumption of this whole theory is that only the output $y(t)$ is measured. The natural idea, then, is to associate a KF with the optimal LQ control law in an "optimum" observer controller $u(t) = -C\hat{x}(t)$. This idea was proposed in the two 1960 papers [Kalman, 1960c] and [Kalman, 1960a]. In the latter, it is hinted that the duality principle makes it legitimate to associate the KF estimate and the optimal LQ gain, anticipating the separation theorem. But this was not quite sorted out at that time. What follows somehow ties the loose ends of 1960 with later results.

One may recover optimality through the *separation theorem* initially proved in [Joseph and Tou, 1961] in discrete-time[19] [20], and for a more general (non quadratic) performance index and continuous time by [Wonham, 1968][21]. The continuous-time problem is much more difficult. It turns out that part of the problem is… to precisely

[19] With no observation noise, as in [Kalman, 1960b], and, dually, no control cost.

[20] Early "certainty equivalence" results, in some particular cases with perfect state information, and without the system theoretic formulation, appeared in the economic literature: [Simon, 1956] and [Theil, 1957].

[21] A simpler proof, more specific to the LQG case, was also due to [Faurre, 1968].

state the problem. This involves continuous Brownian motions, Ito calculus, filtrations[22] and measurability. We will not attempt to state it in modern rigorous terms, but be content with the engineering form of the early sixties. The aim is to have firm grounds to devise a feedback dynamic compensator. (See paragraph 4.1.3 below.)

Linear Quadratic Gaussian (LQG) stochastic optimal control problem. Given a linear system in internal form with additive gaussian white random disturbances in the dynamics and output equations, find, if it exists, a control law where $u(t)$ only depends on past outputs $y(\tau)$, $\tau < t$, that minimizes the mathematical expectation of a quadratic performance index among all such control laws.

The answer is the *separation and certainty equivalence theorem*, true for the discrete-time and continuous-time, finite horizon and stationary infinite horizon problems:

Theorem 7 *The solution of the LQG stochastic optimal control problem exists and is obtained by replacing the state $x(t)$ by the KF estimate $\hat{x}(t)$ in the LQ deterministic optimal control state feedback law. (But positive terms must be added to formula* (10) *for the optimal criterion value.)*

4 Applications

4.1 Compensator design in engineering

4.1.1 Linear and linearized control systems

Already in "pre-kalmanian" times, the LTI theory was put to use in a variety of physical systems. Some had reasonable linear physical models, and when considered in steady state, were time invariant. But most industrial systems such as transportation systems, energy systems, etc., have a natural nonlinear model. The engineering practice, then, is to define a desired or *nominal* output trajectory, and take as the output of the control system the *error signal*, i.e. the difference between the actual and the desired outputs. The objective of the control system is then to keep this error signal close to zero via a *dynamic compensator* : a dynamic system whose input is the measured error signal, and the output the control input of the to-be-controlled system. (Hence a feedback system, as understood by Wiener.)

In order to achieve this goal, one builds a linear model as the linearization of the nonlinear model for small deviations around the nominal trajectory. This can be done either from an analytic nonlinear model linearized by a first order expansion, or from experiments using further parts of the theory (such as the consideration of cross correlations between input and output pseudo-random small deviations).

This being done, the aim of the control system is too keep the error variables, approximated as the variables in the control model[23], close to zero in spite of disturbances

[22] We quote R.E.K. in a conference on applications of the KF in Hydrogeology, Hydraulics, and Water Resources (1978): "There are three types of filters: (i) those which keep tea leaves from falling into the tea cup, (ii) those we are talking about today, (iii) those which are so fancy that only topologists use them."

[23] Here lies the "paradox of linear control theory": the error stays close to zero because the control is effective, the control is effective because the linear approximation is good, the linear approximation is good because the error stays close to zero.

in the dynamics, measurement errors, lack of direct measurement of some important variables, not to mention modeling errors and biasses.[24]

4.1.2 Observer-controller: algebraic approach

Keeping a steady state variable close to zero is achieved by forcing the system to be sufficiently stable. In that process, consideration of the poles of the transfer function, therefore the eigenvalues of the internal description of the overall system, is of the essence, since, for continuous-time systems, their real parts, which must be negative to insure stability, give a measure of the degree of stability while their imaginary parts yield a measure of the oscillatory character of the response of the system. (In discrete time, their modulus must be less than one to insure stability.)

As mentioned in subsection 2.1.1, engineers had developed more sophisticated tools than just the inspection of the poles of the transfer function. But these remain of paramount importance. Hence the use of the separation of dynamics and pole shifting theorems, choosing separately the poles of the observer and those of the controller. A rule of thumb being that the observer must be an order of magnitude faster than the controller. Trial and errors with the localization of the poles using simulation models (both linearized and nonlinear if available) would allow one to construct an efficient control device.

4.1.3 Quadratic synthesis

However, with the advancement of modeling science, largely driven by the advent of the computational power of digital computers, the dimension of the models used increased to a point where simple methods based on poles location were not practical.[25]

Moreover, some problems such as automatically landing an airplane, controlling an industrial baking cycle by heating and cooling an oven, etc. are intrinsically finite horizon, non LTI problems, with sometimes a great emphasis on terminal error control. These problems are beyond the reach of algebraic methods.

Engineers may have a fairly precise idea of the origin and sizes of the disturbances in the dynamics and error sources in the measurements, let alone modeling biasses. This provides a sound basis for computing an observer gain via the KF.

Concerning the controller, one computes a control gain via a quadratic performance index and the Kalman optimal gain. The process of trial and errors in tuning it is performed on the weighting matrices of the performance index, and is made easy and efficient through the interpretation of the gain as minimizing this performance index, so that one knows how the different optimal state and control variables will respond to a change in the weighting matrices. This process, known as *quadratic synthesis*, is what made the new theory so popular among engineers, to the point of being the main tool used in designing the Apollo control systems.

[24]These do not have the statistical characteristics of “noises”, and were at the inception of “robust control” theories, and most noticeably for our purpose, $\mathcal{H}_\infty$-optimal control by [Zames, 1981].

[25]The rigid body dynamics of a landing airplane are described by a 12th order system. Adding engines and control surfaces dynamics, and in modern airliners flexible modes, leads way beyond that figure.

4.2 Population ecology

The use of Kalman filtering in estimating a population, such as in a census, is not new. But in keeping with the focus of this article, we provide an example of the use of LQ optimal control in a crude model of cooperation in an animal population, showing how the individual quest of self interest may sustain a collaborative behavior. Our dynamic model attempts to capture in the simplest possible way the qualitative characteristics of the situation to be investigated. It gives us the opportunity to exhibit several extensions of the LQ theory as presented above, mainly:

- nonhomogeneous terms in the problem statement,
- time-discounted criterion,
- several player Nash equilibrium rather than one-person optimization.

The model is as follows.

A large but finite population of animals dwell in an area where they have two actions: improving the foraging environment, and foraging. (Think, e.g., of a colony of beavers constructing a dam.) The food intake they obtain for a given foraging effort is an increasing function of the environment quality, but is hindered by the total foraging effort as the resource becomes scarcer. The environment quality has an exponential decay if left unattended, but is enhanced by the environment improving effort, although with a decreasing marginal effect. It may be slightly degraded by the foraging effort.

We investigate the equilibrium (Nash) behavior among "selfish" individuals either over a season $[0, T]$ or over their lifetime with an exponentially distributed random death time. Indeed, it is known ([Vincent and Brown, 2006][Sandholm, 2010, chap. 6]) that many models of evolutionary dynamics converge toward a Nash equilibrium, mainly so when, as here, it is unique.

We therefore introduce the (scalar) variables

- x the environment quality,
- U_i the environment improving effort of individual i, and $u_i = \sqrt{U_i}$ the decreasing marginal effect of the effort,
- v_i the foraging effort of individual i,
- K the efficiency factor of any individual's foraging effort in terms of energy intake,

and the (positive) parameters

- n the number of individuals in the population,
- f the natural rate of decay of the environment quality,
- g and h the coefficients of the total efforts $\sum_k u_k$ and $\sum_k v_k$ respectively in the quality dynamics,

- q and r the costs of unit foraging and environment improving efforts respectively per unit time, in terms of energy consumption,
- a and b coefficients determining K as an affine function (see below), and $\alpha = a - q$, assumed positive,
- ρ the coefficient of the exponential death law.

The mathematical model is as follows:

$$\dot{x} = -fx + g\sum_k u_k - h\sum_k v_k\,,$$

$$K = a + x - b\sum_k v_k\,,$$

We write the performance index that each individual seeks to maximize in the same form for the seasonal or life-time optimization. In the first case, ρ may (optionally) be taken equal to zero (if the possibility of death during the season may be ignored), in the second case, $T = \infty$ and the stationary theory applies:

$$J_i = \int_0^T \mathrm{e}^{-\rho t}(Kv_i - qv_i - ru_i^2)\,\mathrm{d}t\,.$$

The detailed analysis of this problem is given in the appendix. In short, the nonhomogeneous character is dealt with by introducing a nonhomogeneous Value function, the exponential discounting by having the same discount factor in the Value function, and the Nash equilibrium by solving an optimization problem for one individual assuming that all others use the "optimal" (equilibrium) strategy.

The end result is that indeed, cooperation is sustained. We introduce two coefficients P and p (variable in the finite horizon case, fixed in the infinite horizon case), and find that the equilibrium strategies are:

$$u^\star = \frac{g}{r}(Px + p)\,,$$
$$v^\star = \frac{1}{(n+1)b}[(1 - 2hP)x - 2hp + \alpha]$$

For suitable values of the parameters, and in particular if n is large enough, these formulas yield positive controls, with positive coefficients P and $(1 - 2hP)$, as well as stable dynamics and positive values for the environment quality x.

5 Going on

R.E.K.'s contributions of 1960-1961 were a powerful stimulus for system theory and control research. Many researchers followed suit, both to get further theoretical advances (such as those hinted at in sections 3.3, 4.2.2 and much more), and to develop efficient algorithms concretely implementing those theoretical results.

Algebraic system theory attracted many researchers such as [Wonham, 1967] and remained R.E.K.'s main research area, using sophisticated algebraic tools and ideas from automata theory. (See, e.g. [Kalman, 1965, Kalman et al., 1969, Kalman, 1972]). The linear quadratic theory of control and observation was deeply renewed by the theory of $\mathcal{H}_\infty$-optimal control, initiated by [Zames, 1981] in the frequency domain "external" description, and later transferred in a Kalman-like time domain formulation (see [Başar and Bernhard, 1995]), with a minimax, probability-free treatment of uncertainties, where the same duality shows up, in a more complex setup, and a bit mysteriously.

The basic theory quickly found its way into all engineering control textbooks, and later in textbooks of many other domains (see, e.g. [Weber, 2011]). The algorithms were coded into publicly available software packages such as Matlab or Scilab, and they have been used in a wide range of application domains, well beyond the industrial and transportation systems of the early times, encompassing all branches of engineering as well as natural and bio-medical sciences.

One finds some examples of the use of linear plus quadratic performance indices in the investigation of animal cooperation. See e.g. [Brown and Vincent, 2008]. Yet, the *dynamic* LQ optimization theory is essentially absent from the literature on population dynamics. Indeed, most models used are nonlinear (see e.g. [Clark and Mangel, 2000, Pastor, 2008]), like the classical logistic growth model. But, as mentioned in paragraph 4.1.1, this is also true in other domains where this theory has found many applications. We have shortly discussed above a possible use in population ecology with a small example exhibiting, at least, some of the easy extensions of the theory.

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A A model in population ecology

A.1 The model

We reproduce the crude model developed in the main text: A large but finite population of animals dwell in an area where they have two actions: improving the foraging environment, and foraging. The food intake they obtain for a given foraging effort is an increasing function of the environment quality, but is hindered by the total foraging effort as the resource becomes scarcer. The environment quality has an exponential decay if left unattended, but is enhanced by the environment improving effort, although with a decreasing marginal effect. It may be slightly degraded by the foraging effort.

We investigate the Nash equilibrium behavior among "selfish" individuals, maximizing their energy intake either over a season $[0, T]$ or over their lifetime with an exponentially distributed random death time.

We therefore introduce the (scalar) variables

- x the environment quality,
- $U_i = u_i^2$ the environment improving effort of individual i,
- v_i the foraging effort of individual i,
- K the efficiency factor of any individual's foraging effort in terms of energy intake,

and the (positive) parameters

- n the number of individuals in the population,
- f the natural rate of decay of the environment quality,
- g and h the coefficients of the total efforts $\sum_k u_k$ and $\sum_k v_k$ respectively in the quality dynamics,
- q and r the costs of unit foraging and environment improving efforts respectively per unit time, in terms of energy consumption,
- a and b coefficients determining K as an affine function (see below), and $\alpha = a - q$, assumed positive,
- ρ the coefficient of the exponential death law.

The mathematical model is as follows:

$$\dot{x} = -fx + g \sum_k u_k - h \sum_k v_k \,,$$

$$K = a + x - b \sum_k v_k \,,$$

We write the performance index in the same form for the seasonal or life-time optimization. In the first case, ρ may (optionally) be taken equal to zero, in the second case, $T = \infty$ and the stationary theory applies:

$$J_i = \int_0^T \mathrm{e}^{-\rho t}(Kv_i - qv_i - ru_i^2)\,\mathrm{d}t\,.$$

A.2 The Nash equilibrium

A.2.1 The Hamilton-Jacobi-Carathéodory-Isaacs-Bellman (HJCIB) equation

We tacitly assumed that all individuals are identical. Therefore they share a unique Bellman (or rather Isaacs Value) function, which we seek of the form $V(t,x) = \exp(-\rho t)W(t,x)$ with W a nonhomogeneous quadratic function of x:

$$W(t,x) = P(t)x^2 + 2p(t)x + \pi(t)\,,$$

Thus, the HJCIB equation reads

$$-\dot{W} + \rho W = \max_{u_i,v_i}\Bigg\{ 2(Px+p)\left[-fx(t) + g\sum_{k=1}^{n} u_k(t) - h\sum_{k=1}^{n} v_k\right] + v_i\left(\alpha + x(t) - b\sum_{k=1}^{n} v_k(t)\right) - ru_i^2\Bigg\}.$$

We easily derive the equations for the maximizing policies $u^\star$ and $v^\star$ as

$$ru^\star = g(Px+p)\,,$$
$$(n+1)bv^\star = (1-2hP)x + \alpha - 2hp\,.$$

Placing these back in the HJCIB equation and equating the coefficients of like powers of x, we obtain

$$-\dot{P} = \left[(2n-1)\frac{g^2}{r} + \frac{4n^2h^2}{(n+1)^2b}\right]P^2 - 2\left[f + \frac{\rho}{2} + \frac{(n^2+1)h}{(n+1)^2b}\right]P + \frac{1}{(n+1)^2b}\,,$$
$$-\dot{p} = \left[-f - \rho + (2n-1)\frac{g^2}{r}P + \frac{4n^2h^2P - (n^2+1)h}{(n+1)^2b}\right]p + \frac{1-(n^2+1)hP}{(n+1)2b}\alpha\,,$$
$$-\dot{\pi} = -\rho\pi + \left[(2n-1)\frac{g^2}{r} - \frac{2h\alpha}{b}\right] + \frac{(\alpha+2nhp)^2}{(n+1)^2b}\,.$$

These differential equations are to be integrated backward from zero terminal conditions at T, an easy task to carry out numerically. Observe that the equation for P is decoupled from the following ones. Once P is computed, the equation for p is linear, and $\exp(-\rho t)\pi(t)$ may be obtained as a mere integral.

A.2.2 Analysis, finite horizon

The equation for P reads as a Riccati equation $\dot{P} = -\Pi(P)$ where Π is a second degree polynomial with two positive roots if they are real. Close examination of Π's discriminant shows that it is always positive if

$$f + \frac{\rho}{2} \geq \sqrt{\frac{(2n-1)g^2}{(n+1)^2 br}},$$

(observe that the right hand side above goes to zero as n increases) and positive otherwise if h is larger than a limit value that decreases as n increases. At terminal time T, P reaches 0 with a negative slope, and as $t \to -\infty$, $P(t)$ converges toward the smallest (real) root $\bar{P}$ of Π if it exists. It diverges to $+\infty$ if not. (The possibility of P diverging, contrary to theorem 6, stems from the presence of a cross term xv in the criterion.) Therefore $P(t)$ remains positive, and smaller than $\bar{P}$ when it exists. It easy to check that $\bar{P}$ goes to zero as $1/n^2$ when n increases to infinity. Furthermore, examination of

$$\Pi\left(\frac{1}{(n^2+1)h}\right) = \frac{(2n-1)g^2}{(n^2+1)^2 r} \times \frac{1}{h^2} - 2\frac{f+\frac{\rho}{2}}{n^2+1} \times \frac{1}{h} - \frac{1}{(n+1)^2 b}$$

shows that for h larger than a limit $h^\star$ (which goes to zero as n goes to ∞), this quantity is negative, ensuring a) that Π has real roots, and b) that $1-(n^2+1)hP$ and a fortiori $1-2hP$, are positive for all t. Looking at the equation for $-\dot{p}$, this in turn ensures that p is also positive for all t and smaller than a limit value $\bar{p}$ which goes to zero as $1/n$ as $n \to \infty$. Hence, for n large enough, if x is positive, so are $u^\star$ and $v^\star$.

Finally, the environment quality dynamics under the equilibrium strategies are:

$$\dot{x} = \left[-f - \frac{nh}{(n+1)b} + \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b}\right)P\right]x + \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b}\right)p - \frac{nh\alpha}{(n+1)b}\,.$$

Therefore, x remains positive provided that α be smaller than a limit value easy to get. (Actually, if $\alpha < 0$ but $x + \alpha > 0$, although $p < 0$, we still have $u^\star > 0$.)

A.2.3 Analysis, infinite horizon

If the horizon is taken as infinite, and if the algebraic Riccati equation has real roots, then all coefficients P, p and π are constant at their limit values given by the algebraic equations obtained by setting the derivatives equal to zero. Closed form solutions are easy to write, useful for an efficient numerical implementation, but otherwise of little interest.

Existence of the positive limit value $\bar{p}$ requires that $1-(n^2+1)h\bar{P} > 0$, a condition we have already investigated, and

$$f + \rho + \frac{(n^2+1)h}{(n+1)^2 b} > \left((2n-1)\frac{g^2}{r} + \frac{4n^2h^2}{(n+1)^2 b}\right)\bar{P}\,,$$

again a condition satisfied provided that n be large.

The environment quality dynamics is stable provided that

$$f + \frac{nh}{(n+1)b} > \left(\frac{ng^2}{r} + \frac{2nh^2}{(n+1)b}\right)\bar{P}\,,$$

again a condition satisfied provided that n be large enough. It also ensures that $\mathrm{e}^{-\rho t}x^2$ is asymptotically stable, and that this stability is preserved if some of the players play $u_i = 0$, both necessary conditions for the Nash equilibrium in infinite horizon to exist. (See [Mageirou, 1976]).

A.3 Closing remarks

With this little example, our main objective has been to show

1. that REK's standard LQ theory can easily be extended in various directions, and
2. that some dynamic population ecology models may lead to such problems.

The infinite horizon problem in a one player version and with $h = 0$, was amenable to the pre-kalmanian theory of [Newton Jr et al., 1957]. Indeed, in that case, the v_i have no dynamic effect and may just be chosen so as to maximize the integrand. Thus the problem becomes monovariable. The game version was more problematic[26]. The case with $h \neq 0$ was solvable in principle, provided that one be able to perform a spectral factorization for the two-input multivariable system. But the simplest way to achieve that is via the Riccati equation resurrected by R.E.K., as shown by [Willems, 1971]. The finite horizon, seasonal problems were completely beyond the reach of that theory.

[26] See, however, the surprising reference [Roos, 1925] where the author obtains the Cournot-Nash equilibrium of a linear quadratic differential game by standard variational methods of the calculus of variations.