Kalman on dynamics and control

Linear System Theory, Optimal Control, and Filter

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Abstract

Rudolf Emil Kalman ("R.E.K.") passed away on July, 2nd, 2016. Among contemporary economists Kalman is mainly remembered for his filter, an algorithm that allows recursive estimation of unobserved time varying variables in a system. However, he has also a key part on the whole of recursive macroeconomic theory as is notably expressed by Lars Ljungqvist's and Thomas Sargent's book [Ljunqvist and Sargent, 2012]. Our paper is a contribution to show the links between Kalman's works on filtering, linear quadratic optimal control, and system theory. We also provide a model on cooperative advertising to show that Kalman's works on dynamics and control can be useful in macroeconomics as in microeconomics, a domain where his contributions seem to be unfortunately less used.

Keywords: Dynamic Programming, System Theory, Linear Quadratic Optimal Control, Kalman Filter, Riccati Equation.

JEL codes: C61, C60, C02

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1 Introduction

Rudolf Emil Kalman ("R.E.K.") passed away on July, 2nd, 2016. His name will remain tied to the most famous *Kalman filter*, which, by the end of his life, he would himself call "the KF". (We will follow his lead here.)

This is a mathematical method, which can be implemented as a set of algorithms, to estimate time varying variables from noisy and incomplete measurements, using a (noisy) model of the underlying dynamics.[1] Because it lets one estimate a variable which is not directly measured, it has sometimes been "commercialized" by computer scientists as a "software sensor". It, and its extensions, are now widely used in a great variety of domains, industrial and technological of course, but also in social, biological, and earth sciences, health systems, etc. It is a fact that even persons who have never heard of it have used it extensively, be it only in their GPS receivers.

But the point we want to make here is that the filter article [Kalman, 1960b] is only one of the three major papers of R.E.K. in 1960. The other two are the great article about Linear Quadratic optimal control [Kalman, 1960a] —a 1959 conference communication—, and the system theory article [Kalman, 1960c], to be followed by [Kalman, 1962] —dated January 1961 in the proceedings— and [Kalman, 1963]. To control theoreticians, these were earth-shaking contributions, definitively transforming control theory and linear system theory, well beyond filtering and prediction.[2]

Bolstered by the advent of the digital computer, these theories were extensively put to use via the "Automatic Synthesis Program" in the Apollo lunar landing program which started at the same time. Seldom has a brand new piece of theory been embraced so quickly by the practitioners in such an important endeavour.[3] But, as we will see, it was also quickly used in economic and management sciences.

The paper is organized as follows : in the next section we present the system theory and control before and after Kalman works. In Section 3, we present the linear quadratic gaussian theory by introducing to the linear quadratic optimal control and its duality with the Kalman filter. Then, in Section 4, we briefly discuss the applications of linear quadratic gaussian theory in macroeconomics and develop a microeconomic model on cooperative advertising. Section 5 ends the paper.

2 System theory

2.1 A paradigmatic change

2.1.1 State of the theory before 1960

To appreciate the importance of R.E.K.'s contribution to systems and control, one must have a crude idea of what was the state of these fields before 1960. The theory was

[1] Dont say that it belongs to "estimation theory", as R.E.K. strongly objected to that qualification, arguing that this phrase had come to name a body of non-science. He later denied its being based upon probabilities, and refused its definition as (only) an algorithm, because it is complete.

[2] Another pair of important articles on stability of linear dynamical systems also appeared in 1960: [Kalman and Bertram, 1960a] [Kalman and Bertram, 1960b], complementing the other control articles.

[3] Another example, though, may be the KF's predecessor: the Wiener filter, largely conceived in 1940 for the anti-aircraft radar, and embargoed until 1949 for its military sensitivity.

confined to linear time-invariant (LTI) systems, and essentially to so-called "monovariable" ones, i.e. where the signals considered were scalar.

A (linear) dynamical system is a device which is excited by a time varying signal, the *input*, and simultaneously produces a time varying signal, the *ouput*. To the mathematicians, a linear system is therefore a linear operator, transforming a time function called input into another time function called output. Because this happens in real time, some properties are required: causality (the current output does not depend on future inputs), and some others (relating notably to behaviour "at infinity"), and linearity and time invariance to be amenable to the theory then available.

Mathematicians had developed an extremely elegant and powerful way of manipulating such transformations, through the use of an esoteric mathematical transformation of the signals : the *Laplace transform*, which led to the representation of the system via a "transfer function" (a ratio of polynomials of a complex variable). An important consideration is that it is straightforward to derive the transfer function from a linear differential equation describing the system. It had, inter alia, the property of transforming cascade of systems (the output of the first one becoming the input of the second one) into simple products of their transfer functions, making possible the analysis of *feedback* systems, where the output is re-introduced as a component of the input of the same system, a necessary ingredient of any servomechanism. It also led to the Wiener filter, then the standard tool in signal processing.

These methods were intimately tied to the *frequency response* of the system: its behaviour if excited by a sinusoidal signal. Engineers had developed both powerful analytical and graphical tools and a deep understanding of the frequency content of signals and its meaning for the analysis and control of LTI systems.

A clever trick of Wiener filtering was to consider a noisy signal to be "filtered" (freed from its noisy content) as the output of a linear system excited by a noise with adequate statistical properties. This was used by R.E.K. in the KF, so that he once told one of the authors of this article: "Take the Kalman filter, which, as everybody knows, was invented by Wiener…". This author would not have dared that quip.

2.1.2 Inovations of 1960

R.E.K. chooses to represent the transformation of inputs into outputs by the mediation of an internal *state* of the system: in his case a vector of several real variables that also vary with time according to a forced first order differential or difference equation. Hence the name of an *internal* description of the system (the classical one becoming *external*.) In that representation, the input acts on the dynamics of the state, and the state instantly produces an output. Since the state itself is a vector, and all relations considered are linear, matrices and linear algebra are at play. And this mathematical apparatus lends naturally itself to the consideration of vector-valued inputs and outputs. Moreover, if the matrices defining the system are time varying, the system is no longer time invariant. If part of realization theory, the heart of linear system theory, still concerns LTI systems, both filtering and what we want to stress here: control, can naturally be extended to non-time invariant systems.

The notion of state of a system was known inprecisely as "*a set of numbers from which the entire future behavior of the plant may be determined provided that the future* *inputs of the plant are known*" (quoted from [Kalman and Koepcke, 1958])[4]. Typically, positions and velocities in a mechanical system, intensities in inductors and charges of capacitors in an electrical system. The intimate link with first order differential equations was obviously recognized. But their almost exclusive use was in deriving from them the transfer function of the (LTI) system, a straightforward process. The intimate link with first order differential equations was obviously recognized. But their almost exclusive use was in deriving from them the transfer function of the (LTI) system, a straightforward process. The direct use of the differential (or difference) equations in optimization was becoming more frequent in the late fifties under the influence of Bellman's Dynamic Programming [Bellman, 1957]. R.E.K.'s bold move was to make it the core definition of a linear system and to invesigate in depth its properties.

With the internal description, the extension to non-time invariant systems is possible because the tools developed dispense with the Laplace transform (the frequency analysis). They manipulate the signals as time functions, hence also the name of *time domain* analysis, as opposed to the *frequency domain*. For control theory, they delve into calculus of variations, and for filtering into Markov processes.

The advent of the digital computer and of direct digital control also led R.E.K. and later researchers to systematically develop a discrete-time theory along the continuous-time one, at first as a theory of *sampled data systems* i.e. looking at a continuous-time system at discrete instants of time. It turned out that the parallel was particularly natural and elegant in the new theory.[5]

The transfer function of a time invariant system in internal form is obtained via a simple algebraic formula. The converse: finding the internal representation of a system given in external form, is a deeper question, involving a detailed analysis of the mathematical nature of a linear system in internal form, the topic of realization theory.

2.2 Realization theory

The general theory of control systems was initiated by R.E.K. in [Kalman, 1960c]. But the real founding article, which we shall follow here, is [Kalman, 1962], actually written in 1960 (or the journal article [Kalman, 1963]).

Definition 1 *A* realization *of a linear (or affine) input-output transformation is a representation in internal form as* (1)(2) *or* (3)(4) *below.*

Let $x \in \mathbb{R}^n$ be the state of the system (n is called the dimension of the realization), $u \in \mathbb{R}^m$ be the input, or control, and $y \in \mathbb{R}^p$ the output. To avoid some trivialities, we assume that $m \leq n$ and $p \leq n$. We use Newton's notation for time derivatives:

[4][Kalman, 1960c] adds the precision that it is the *smallest* such set. This was to be made precise later via the (earlier) concept of Nerode equivalence class of formal languages and automata theory.

[5]It should be mentioned, though, that with an admirable prescience, Kolmogorov had developed in 1941, independently from Wiener, a discrete-time version of the Wiener —or Wiener-Kolmogorov— filter.

$\dot{x} = \mathrm{d}x/\mathrm{d}t$. A continuous-time system is of the form[6]

$$\dot{x}(t) = Fx(t) + Gu(t)\,, \tag{1}$$
$$y(t) = Hx(t)\,, \tag{2}$$

and in discrete time

$$x(t+1) = Fx(t) + Gu(t)\,, \tag{3}$$
$$y(t) = Hx(t)\,. \tag{4}$$

Notice that some would like to extend the output equation adding a term $+Ju(t)$. R.E.K. himself argued against in general system theory, with good arguments.[7]

For the sake of completeness, let us mention that if all three matrices F, G, and H are constant, the system is LTI, and its transfer function is

$$\mathcal{H}(s) = H(sI - F)^{-1}G\,.$$

It follows from Cramer's theorem that this is a matrix of (strictly proper) rational fractions of s, with the characteristic polynomial of F as their common denominator, and thus for poles its eigenvalues.

Even without referring to the transfer function, it is clear that the transformation from input to output induced by these equations is not altered if we make a change of basis in the state space, or equivalently if we use for new state $\xi = Tx$ with T an invertible matrix. The continuous-time sytem becomes

$$\dot{\xi} = TFT^{-1}\xi + TGu\,,$$
$$y = HT^{-1}\xi\,.$$

Hence changing (H, F, G) into (HT^{-1}, TFT^{-1}, TG) is unessential, representing the same system, with the same transfer function if it is time-invariant. This points to a weakness of the new representation: it is non unique for the same input-output system. And other non-uniqueness may show up, as the following trivial one, where we introduce a higher dimensional vector, say z made of x and a vector ξ of arbitrary dimension:

$$z = \begin{pmatrix} x \\ \xi \end{pmatrix}, \qquad \dot{z} = \begin{pmatrix} F & 0 \\ A & B \end{pmatrix} z + \begin{pmatrix} G \\ C \end{pmatrix} u$$
$$y = (H \quad 0)z\,.$$

A, B and C are arbitrary matrices. They play no role, since ξ does not influence y, neither directly nor via x. Therefore, realizations of different dimensions may represent the same input-oupt system. In the case above, it is trivial, but assume that a change of basis such as the previous one mixes x and ξ, and the excess dimension may be more difficult to detect. Moreover, other cases may appear.

[6]Essentially all the current literature replaces (F, G, H) by (A, B, C). We keep R.E.K.'s notation as he really meant it ! He noted a system as (H, F, G).

[7]Such a "feed-through" term is not dynamic, and consequently spoils the elegance of the algebraic theory.

The solution of this problem involves two fundamental concepts. The first is *controllability*. A state is controllable if there exists a control function $u(\cdot)$ that drives the system from this state as initial state to the origin. The system is said to be *completely controllable* if every state is controllable. We will here cheat somewhat with history by using instead *reachability*. A state is reachable if there exists a control function that drives the system from the origin to that state. A system is *completely reachable* if every state is reachable. The two concepts are equivalent for continuous-time systems, but not for discrete-time systems, unless the matrices $F(t)$ are invertible for all t. Hence a system is said completely reachable if the application $u([t_0, t_1]) \mapsto x(t_1)$, which is linear if $x(t_0) = 0$, is onto for some $t_1 > t_0$.

The second fundamental concept is that of *observability*. A state $x_0 \neq 0$ is *unobservable* if the ouput of the "free" system, i.e. with $u(\cdot) = 0$, intialized at that state, is $y([t_0, t_1]) = 0$ for any $t_1 \geq t_0$. The system is *completely observable* if no state is unobservable. Hence the system is said completely observable if the application $x(t_0) \mapsto y([t_0, t_1])$, which is linear if $u(\cdot) = 0$, is one to one for some $t_1 > t_0$.

The article [Kalman, 1960a] also gives efficient criteria to check these properties. In the case of time-invariant systems the Kalman criteria are in terms of the rank of composite matrices:

Theorem 1

$$(F\ G)\ \textit{completely reachable} \Leftrightarrow \mathrm{rank}[G\ FG\ F^2G \ldots F^{n-1}G] = n\,,$$

$$(H\ F)\ \textit{completely observable} \Leftrightarrow \mathrm{rank}\begin{bmatrix} H \\ HF \\ HF^2 \\ \vdots \\ HF^{n-1} \end{bmatrix} = n$$

The article also gives simple criteria for non time-invariant systems. They are less algebraic, more analytic, but they share the following striking property (stated in two equivalent forms below), called *duality*: (we use prime for transposed)

$$(F\ G)\ \text{completely reachable} \Leftrightarrow (G'\ F')\ \text{completely observable}$$

$$(H\ F)\ \text{completely observable} \Leftrightarrow (F'\ H')\ \text{completely reachable}$$

We will see that this duality reaches into optimal control and filtering. R.E.K. himself gave a detailed analysis of the duality between the Wiener filter and the linear quadratic regulator [Kalman, 1960c][8]. But the whole extent of duality in the LQG theory to be seen below remains difficult to explain, and even more in modern $\mathcal{H}_\infty$-optimal control (see [Başar and Bernhard, 1995]).

We use these concepts in realization theory with the help of the following formal definition:

[8]Defining the observability of *costates* in the dual space of the state space. Duality clearly has to do with the fact that a linear operator between linear spaces is onto if and only if its adjoint operator is one to one, and it is one to one if and only if its adjoint operator is onto. As pointed out in [Kalman, 1960c], it is also related to the known duality between the differential equations $\dot{x} = Fx$ and $\dot{p} = -F'p$ or the difference equations $x(t+1) = Fx(t)$ and $p(t) = F'p(t+1)$ which leave the inner product $p'x$ invariant.

Definition 2 *A realization completely reachable and completely observable is called* canonical.

The main theorem is as follows:

Theorem 2 *A realization is minimal (has a state space of minimum dimension) if and only if it is canonical. And then it is unique up to a change of basis in the state space.*

The article [Kalman, 1962] further shows that the state space of any linear system in internal form, even not time invariant (i.e. a system such as (1),(2) with matrices H, F, and G depending continuously on time) can be decomposed canonically as the direct sum of four (variable if the system is not LTI) subspaces as in the left diagram of figure 1 borrowed from [Kalman, 1962], or more classically in block diagram form as in the right one borrowed from [Kalman, 1963] and reproduced in all textbooks since. A, B, C, and D are the subspaces corresponding respectively to states reachable but unobservable, reachable and observable, unreachable and unobservable, unreachable but observable. Using a basis adapted to that decomposition yields a canonical decom-

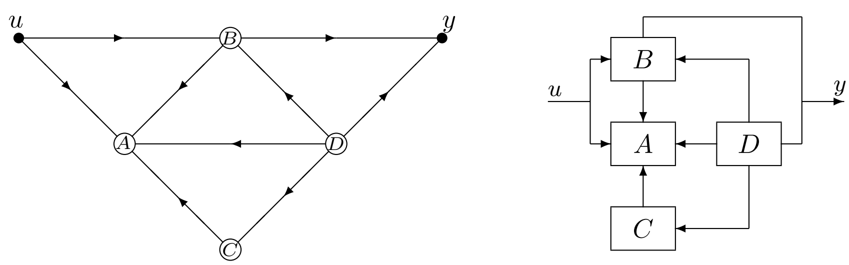

Figure 1: The canonical decomposition of a linear system in internal form

position of the system matrices that exhibits these properties as follows:

$$F = \begin{bmatrix} F_{AA} & F_{AB} & F_{AC} & F_{AD} \\ 0 & F_{BB} & 0 & F_{BD} \\ 0 & 0 & F_{CC} & F_{CD} \\ 0 & 0 & 0 & F_{DD} \end{bmatrix}, \quad G = \begin{bmatrix} G_A \\ G_B \\ 0 \\ 0 \end{bmatrix},$$
$$H = \begin{bmatrix} 0 & H_B & 0 & H_D \end{bmatrix}.$$

The subspaces B, C, and D are not uniquely defined, but the decomposition of the system matrices is, up to changes of basis within each of the four subspaces.

2.3 Compensator design: pole placement approach

The KF was the first *observer*[9], originally in discrete time, i.e. a system of either form

$$\hat{x}(t+1) = F\hat{x}(t) + Gu(t) + K[y(t) - H\hat{x}(t)]\,, \qquad (5)$$
$$\dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + K[y(t) - H\hat{x}(t)]\,, \qquad (6)$$

[9]Called "observing system" in [Kalman, 1960c] which proposes an "optimal" one in terms of the number of time steps necessary to exactly recover the state in a discrete time system. The term "observer" was coined by [Luenberger, 1964], which extends the concept, in a less explicit form that lacks the simplicity displayed here and the rather crucial stability argument invoked in paragraph A.2.

providing an estimate $\hat{x}(t)$ of the state, optimal in the sense that it minimizes the expected squared L^2 norm of the *error signal* $\tilde{x}(t) = x(t) - \hat{x}(t)$. The natural idea, proposed as early as [Kalman, 1960c] for monovariable discrete-time systems, is to associate such an observer with a control law

$$u(t) = -C\hat{x}(t)\,.$$

There remains to choose the gains K and C. This can be done via the following results.

Writing the overall system in terms of $(x, \tilde{x})$ instead of $(x, \hat{x})$, it is a simple matter to prove the *principle of separation of the dynamics* ([Luenberger, 1964]):

Theorem 3 (Separation of dynamics) *: The eigenvalues of the dynamic matrix of the closed loop observer-controller are the union of the eigenvalues of $F - GC$ —the controller— and those of $F - KH$ —the observer.*

A later result [Wonham, 1967][10] is the following extension to multi-input systems of the *pole shifting theorem*, known in 1960 (and used in [Luenberger, 1964]) for single input systems:

Theorem 4 *If the pair (F, G) is completely reachable, then given any monic n-th degree polynomial $p(z)$, there exists a matrix C such that the characteristic polynomial of $F - GC$ be p. Dually, if the pair (H, F) is completely observable, the characteristic polynomial of $F - KH$ can be assigned to any desired one by the choice of K.*

Hence a purely system theoretic argument in favor of the proposed control structure, and a means of choosing C and K (see paragraph A.2 below).

A further important remark is that in the discrete-time case, hence also in the sampled data problem of any digital control, the observer is a one step predictor: It gives the estimate $\hat{x}(t+1)$ of $x(t+1)$ with the data $y(\tau)$, $\tau \leq t$. Hence one has one time step to compute the contol $u(t+1) = -C\hat{x}(t+1)$, the gain C being pre-computed off line.

3 Linear Quadratic Gaussian (LQG) theory

3.1 Linear Quadratic (LQ) optimal control

The topic covered here is partially investigated in [Kalman, 1960c], but the definitive article is [Kalman, 1960a]. We will essentially adopt its notation.

The introduction of [Kalman, 1960a] states, we quote: *"This problem dates back, in its modern form, to Wiener and Hall at about 1943."* It also quotes, although in rather denigrating terms, [Newton Jr et al., 1957] as the state of the art at that time. Therefore in its infinite horizon (optimal regulator) form, it was not new. But the solutions offered were in terms comparable to those of the Wiener filter, i.e. frequency domain analysis and spectral factorization, and did not easily lead to efficient algorithms, particularily so for "multivariable" problems.[11]

[10] Other proofs were quickly given as comments of this article by [Heymann, 1968] and [Davison, 1968].

[11] Actually, a finite-horizon nonhogeneous scalar-control linear-quadratic optimization problem is solved in [Merriam III, 1959], with the correct Riccati equation and the linear equations for the non homogeneous terms, although difficult to recognize.

3.1.1 Finite horizon problem

The new approach started with the investigation of a finite horizon optimal control problem, i.e. not time invariant. It involves quadratic forms that we denote as follows: for any positive definite or semi-definite $\ell \times \ell$ matrix M and z a ℓ-vector,

$$\langle z, Mz \rangle = z' M z = \|z\|_M^2 \,.$$

The problem investigated is as follows[12]:

Linear quadratic optimal control problem Given the system (1)(2), with all system matrices possibly (piecewise) continuous functions of time, and $x(t_0) = x_0$, and given the symmetric $n \times n$ matrix $A \geq 0$, and the symmetric (piecewise) continuous respectively $n \times n$ and $m \times m$ matrix funtions, $Q(t) \geq 0$ and $R(t) > 0$, find, if it exists, the control law that minimizes the performance index

$$V(x_0, t_0, t_1; u(\cdot)) = \|x(t_1)\|_A^2 + \int_{t_0}^{t_1} \left[\|y(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2 \right] \mathrm{d}t \,. \tag{7}$$

Kalman's approach to the problem followed Carathéodory's.[13] The solution is as follows. Define a symmetric matrix function $P(t)$ as the solution, if it exists, of the matrix Riccati equation (where all matrices are time dependent)

$$-\dot{P} = PF + F'P - PGR^{-1}G'P + H'QH \,, \qquad P(t_1) = A \,. \tag{8}$$

(By Cauchy's theorem, there exists a solution on some open time interval (t_2, t_1). However existence of a solution over the time interval $[t_0, t_1]$ is by no means guaranteed a priori, as the solution might diverge to infinity before reaching down t_0.)

The full theorem is as follows:

Theorem 5

1. *The Riccati equation* (8) *has a solution* $P(t) \geq 0$ *over* $[t_0, t_1]$ *for every* $t_0 < t_1$.
2. *The solution of the linear quadratic optimal control problem is given in state feedback form by*

$$u(t) = -C(t)x(t) \,, \qquad C(t) = R(t)^{-1}G'(t)P(t) \,, \tag{9}$$

3. *and the optimal value of the performance index is*

$$V^0(x_0, t_0, t_1) = \|x_0\|_{P(t_0)}^2 \,.$$

Important remark: The Riccati equation (8) and the optimal feedback gain (9) are the duals of the KF's Riccati equation and gain. (See section 3.2)

[12] In [Kalman, 1960a], the problem is first posed and investigated in a more general form.

[13] While [Kalman and Koepcke, 1958] and [Kalman, 1960c] explicitely use [Bellman, 1957] and (discrete) dynamic programming, a strong incentive for using the state space representation, [Kalman, 1960a] does not quote it for the continuous-time problem, using instead Carathéodory's theory. Symmetrically, [Bellman, 1957] does not refer to Carathéodory. Yet, continuous dynamic programming is essentially a rediscovery of Carathéodory's theory, Bellman's return function being the control equivalent of Carathéodory's principal function.

3.1.2 Optimal regulator (infinite horizon) problem

The challenge in this theory is to investigate the infinite horizon problem. For the sake of simplicity, we only give here its LTI version, the only one amenable to the previous, "classical" at that time, theory, and the most widely used. But [Kalman, 1960a] also gives the solution for a non time-invariant problem.

Optimal regulator problem Given the time invariant linear system (1)(2) with initial state $x(0) = x_0$, positive definite $p \times p$ respectively $m \times m$ matrices Q and R, find, if it exists, the control that minimizes the performance index

$$V(x_0; u(\cdot)) = \int_0^\infty \left[\|y(t)\|_Q^2 + \|u(t)\|_R^2\right] \mathrm{d}t\,.$$

This investigation requires the introduction of both controllability and observability. Indeed, in its introduction, R.E.K. states "*The principal contribution of the paper lies in the introduction and exploitation of the concepts of controllability and observability*". In retrospective, he might also have quoted the Riccati equation (8). We shall use here its algebraic version, where all matrices are now constant:

$$PF + F'P - PGR^{-1}G'P + H'QH = 0\,. \tag{10}$$

The full theorem is as follows:

Theorem 6

1. *If the pair (F, G) is completely controllable,*[14] *then*
   (a) *the solution $P(t)$ of the Riccati equation* (8) *has a limit $\bar{P}$ as $t \to -\infty$, which solves the algebraic Riccati equation* (10)*,*
   (b) *the solution of the optimal regulator problem in state feedback form is*
   $$u(t) = -Cx(t)\,, \qquad C = R^{-1}G'\bar{P}\,, \tag{11}$$
   (c) *and the optimal value of the performance index is*
   $$V^0(x_0) = \|x_0\|_{\bar{P}}^2\,.$$
2. *If furthermore the pair (H, F) is completely observable,*[15] *$\bar{P}$ is positive definite and the system governed by the law* (11) *is asymptotically stable.*

Of course, the duality pointed out in the finite horizon problem holds here, making the optimal regulator dual to the stationary KF, i.e. to a realization of the Wiener filter.

[14] *Stabilizable* i.e. $\exists D : F - GD$ stable, suffices.
[15] For the stability result, *detectable*, i.e. $\exists L : F - LH$ stable, suffices.

3.1.3 Discrete-time case

Articles [Kalman, 1960b] and [Kalman, 1960c] provide the equivalent discrete-time results[16]. The system is (3)(4), and the performance index is:

$$V(x_0,t_0,t_1;\{u(\cdot)\}) = \|x(t_1)\|_A^2 + \sum_{t=t_0}^{t_1-1}\left[\|y(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2\right]. \qquad (12)$$

The Riccati differential equation is replaced by the so-called discrete Riccati equation (the system matrices may all be time dependent) or its "algebraic" version (with all system matrices constant) where $P(t) = P(t+1) = \bar{P}$:

$$\begin{aligned}&P(t) = F'P(t{+}1)F - F'P(t{+}1)G(G'P(t{+}1)G+R)^{-1}G'P(t{+}1)F + H'QH,\\ &P(t_1) = A\,.\end{aligned} \qquad (13)$$

The optimal feedback control is

$$u(t) = -C(t)x(t)\,, \qquad C(t) = (G'P(t+1)G+R)^{-1}G'P(t+1)F\,. \qquad (14)$$

Both the finite and infinite horizon results follow exactly as for the continuous-time case, with the same controllability and observability conditions.

3.2 The Kalman Filter

For the sake of completeness, and to stress duality, we quickly review the famous KF, [Kalman, 1960b] and [Kalman and Bucy, 1961]. Existence and stability properties for both the finite horizon and infinite horizon cases are directly derived from those of the dual LQ control problem.

3.2.1 Discrete-time

We start with the discrete-time problem, after [Kalman, 1960b].[17] Let us consider a discrete time linear system excited by *white noise* and a known control $u(\cdot)$:

$$\begin{aligned}x(t+1) &= F(t)x(t) + G(t)u(t) + D(t)v(t)\,, \quad x(t_0) = x_0\,,\\ y(t) &= H(t)x(t) + w(t)\,,\end{aligned}$$

where $(v(t),w(t))$ is a gaussian random variable with zero mean and known covariance, independent from all the $(v(\tau),w(\tau))$ for $\tau \neq t$. In the simplest case, $v(t)$ is also independent from $w(t)$, but this is not necessary for the theory to hold. A possible non-zero cross correlation between them is dual to the presence of a cross term $x'Su$ in the quadratic performance index of LQ control, which we will not introduce here. The noise is therefore characterized by its covariance matrix (with $\delta_{t,\tau}$ the Kronecker symbol):

$$\mathbb{E}\begin{pmatrix} v(t)\\ w(t)\end{pmatrix}(v'(\tau)\ w'(\tau)) = \begin{pmatrix} V(t) & 0\\ 0 & W(t)\end{pmatrix}\delta_{t,\tau}\,.$$

[16] First approached in [Kalman and Koepcke, 1958, Appendix], dealing with sampled data control of a continuous-time system. But the treatment there is not completely satisfactory.

[17] [Kalman, 1960b] has no added noise in the measurement equation, nor control.

The initial state is also given as a gaussian random variable of known distribution:

$$\mathbb{E}(x(t_0)) = \hat{x}_0 \,, \qquad \mathbb{E}(x(t_0) - \hat{x}_0)(x(t_0) - \hat{x}_0)' = \Sigma_0 \,.$$

The problem is to compute the conditional mathematical expectation:

$$\hat{x}(t) = \mathbb{E}(x(t)|y(\tau), \tau < t) \,. \tag{15}$$

The solution is of the form (5) initialized at $\hat{x}(t_0) = \hat{x}_0$, where the gain K is given via the error covariance matrix $\Sigma(t) = \mathbb{E}(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))'$, solution of the discrete Riccati equation (16) dual of (13), and by the formula (17) dual of (14):

$$\Sigma(t+1) = F\Sigma(t)F' - F\Sigma(t)H'(H\Sigma(t)H' + W)^{-1}H\Sigma(t)F' + DVD' \,, \tag{16}$$

$$K(t) = F\Sigma(t)H'(H\Sigma(t)H' + W)^{-1} \,. \tag{17}$$

The time invariant, infinite horizon case is the internal form of the Kolmogorov filter.

3.2.2 Continuous-time

Given a continuous-time system in internal form excited by both dynamic and measurement "white noises" as above, but in continuous-time, with (in terms of the Dirac δ as in [Kalman and Bucy, 1961], probabilists now have a different way of stating things)

$$\mathbb{E}\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} (v'(\tau)\ w'(\tau)) = \begin{pmatrix} V(t & 0 \\ 0 & W(t) \end{pmatrix} \delta(t-\tau) \,,$$

with W invertible[18], the conditional expectation (15) sought is the solution of the continuous-time observer (6) with the dual formulas from LQ control:

$$\dot{\Sigma} = F\Sigma + \Sigma F' - \Sigma H'W^{-1}H\Sigma + DVD' \,, \qquad \Sigma(t_0) = \Sigma_0 \,.$$

and

$$K(t) = \Sigma(t)H'W^{-1} \,.$$

The time invariant, infinite horizon case coincides with the internal representation of the Wiener filter which, as the Kolmogorov filter, was given in external form.

3.3 The separation theorem

The control laws (9) or (11) or (14) assume that the state $x(t)$ is exactly measured. But the uderlying assumption of this whole theory is that only the output $y(t)$ is measured. The natural idea, then, is to associate a KF with the optimal LQ in an "optimum" observer controller $u(t) = -C\hat{x}(t)$. This idea was proposed in the two 1960 papers [Kalman, 1960c] and [Kalman, 1960a]. In the latter, it is hinted that the duality principle makes it legitimate to associate the KF estimate and the optimal LQ gain, anticipating the separation theorem. But this was not quite sorted out at that time. What follows somehow ties the loose ends of 1960 with later results.

[18]Otherwise, a linear combination of the state variables is exactly observed. The problem solution then involves a Luenberger reduced observer

One may recover optimality through the *separation theorem* initially proved in [Joseph and Tou, 1961] in discrete-time[19] [20], and for a more general (non quadratic) performance index and continous time by [Wonham, 1968][21]. The continuous-time problem is much more difficult. It turns out that part of the problem is… to precisely state the problem. This involves continuous Brownian motions, Ito calculus, filtrations[22] and measurability. We will not attempt to state it in modern rigourous terms, but be content with the engineering form of the early sixties. The aim is to have firm grounds to devise a feed-back dynamic compensator. (See paragraph A.3 below.)

Linear Quadratic Gaussian (LQG) stochastic optimal control problem. Given a linear system in internal form with additive gaussian white random disturbances in the dynamics and output equations, find, if it exists, a control law where $u(t)$ only depends on past outputs $y(\tau)$, $\tau < t$, that minimizes the mathematical expectation of a quadratic performance index among all such control laws.

The answer is the *separation and certainty equivalence theorem*, true for the discrete-time and continuous-time, finite horizon and stationary infinite horizon problems:

Theorem 7 *The solution of the LQG stochastic optimal control problem exists and is obtained by replacing the state $x(t)$ by the KF estimate $\hat{x}(t)$ in the LQ deterministic optimal control state feedback law.*

4 Applications in economics

4.1 LQG theory in macroeconomics

Kalman is most often quoted in the economic science literature for the KF. The LQ control theory is often quoted in macroeconomics, and in discrete time. It is also the basis of more advanced (robust) methods used in recent macroeconomic literature such as [Hansen and Sargent, 2008]. The continuous time theory has also been used in macroeconomics for a long time, although less often. Typical examples can be found in [Petit, 1990] which writes, we quote: "*quadratic objective functions have been definitively adopted in economic policy analysis following Theils's important contributions […] (see [Theil, 1958], [Theil, 1964])*". As an example, the author solves a 20-dimensional LQ optimal control problem for a model of the Italian economy with three or four policy instruments. The procedure followed is typically quadratic synthesis as described in more detail in the appendix.

As a matter of fact, macroeconomic theory can easily be thought of as a control problem where the emphasis is on devising "good" policies, with satisfactory qualitative behaviour of the generated trajectories, much more than on maximizing or minimizing a specific performance index. This is the natural realm of quadratic synthesis,

[19] With no observation noise, as in [Kalman, 1960b], and, dually, no control cost.

[20] Early "certainty equivalence" results, in some particular cases with perfect state information, and without the system theoretic formulation, appeared in the economic literature: [Simon, 1956] and [Theil, 1957].

[21] An early proof, more specific to the LQG case, was also due to [Faurre, 1968].

[22] We quote R.E.K. in a conference on applications of the KF in Hydrogeology, Hydraulics, and Water Resources (1978): "There are three types of filters: (i) those which keep tea leaves from falling into the tea cup, (ii) those we are talking about today, (iii) those which are so fancy that only topologists use them."

and explains the popularity that R.E.K.'s linear quadratic control theory has enjoyed in that field.

Because these models can be very large (close to one hundred variables), computational issues become important. For continuous time models, specific algorithms have been devised for the infinite horizon, stationary Riccati equation. Yet, it can remain a numerically heavy burden, and moreover macroeconomic models are often meant to give finite horizon specifications. As a consequence, authors are often content with open-loop computations of isolated trajectories, while closed loop control is intimately linked to fields of optimal trajectories as understood ever since Weierstrass. The standard tool is then Euler's equation, an ordinary differential equation bearing upon the $2n$-dimensional vector (x, λ), where in fact $\lambda = V_x$, usually introduced via Pontryagin's Maximum Principle [Pontryagin et al., 1962]. The resulting two point boundary value problem requires an iterative procedure with several integrations of this linear ODE in $\mathbb{R}^{2n}$, sparing us the complexities of the nonlinear Riccati equations with its big matrix products and various numerical difficulties.[23]

As far as we know, the corresponding discrete time formulas are (almost ?) never used. Yet, if the dynamic matrix F can be inverted once for all, then there is a very simple parallel theory again entirely in terms of linear equations. This could be worthwile when the number of decision variables (policy instruments) is also very large, since the discrete Riccati equation involves an inversion of a matrix of corresonding size at each time step.

4.2 LQG in microeconomics

It seems that R.E.K.'s optimal LQ theory is less well known in microeconomics, probably because the apparent rigidity of the LQ formalism does not fit very well its models, and where the relevent criterion may fail to be quadratic. Moreover, because the criterion used is an economic payoff, one is confronted with a game problem rather than a simple control problem. Yet, R.E.K.'s LQ control theory may be usefull as we shall try to show in the next subsection.

In that subsection, we provide an example in microeconomics that extends the basic LQ model of our subsection 3.1 on four counts:

1. A more general performace index with no x^2 term and with cross terms state times decision in the payoff. This only adds some terms in the Riccati equation.

2. Non homogeneous (first degree) terms in the payoff. This is dealt with by allowing for a non homogeneous Bellman function, with terms of degree 1 and 0 in x.

[23] An alternative way of considering these equations is Carathéodory's *canonical equations*, that represent the field of optimal trajectories as the projection on $\mathbb{R}^n$ of a n-dimensional linear subspace of the (infinite dimensional) space of functions $\mathbb{R} \to \mathbb{R}^{2n}$. With this tool, n integrations of a $2n$-dimensional linear ODE let one recover any required optimal trajectory as a linear combination of these n trajectories. (See [Bernhard, 1982] or [Weber, 2011].) This was taught by REK at Stanford University as early at least as the academic year 1968–1969.

3. Exponentially discounted payoff.[24] This is dealt with by taking the Bellman function as $V(t,x) = \exp(-\rho t)W(t,x)$.

4. More significantly, seeking a Nash equilibrium rather than a mere maximizing control. This is done by considering each player's problem as an optimal control problem in the presence of the other players' equilibrium strategies, thus allowing for a Bellman (or rather Isaacs) function for each one.

Our example belongs to cooperative advertising, but unlike in the classical literature on that topic (see [Jørgesen and Zaccour, 2014]), the advertising expense is not shared by a producer and a retailer, but by several competing producers who agree on campaigning separately on a joint moto to promote their similar products. (Typically a protected designation of origin such as "Comté" or "Champagne".)

We will see that although the formula obtained look complicated, they are easy to implement and allow for a numerical investigation of the role of the model's parameters.

4.3 A model in microeconomics

4.3.1 The "complete" model

The model Advertising is assumed to be aimed at increasing the *goodwill* for the product, measured as a real positive time varying variable $x(t)$ (which should be thought of as an additive to the market price). There are n producers. The sum devoted by producer i at time t to advertising on the joint trade mark is S_i. But this expense has a decreasing marginal efficiency, represented here by $s_i = \sqrt{S_i}$. Moreover, a larger circulation of the product also has a small positive effect on the goodwill (the market size) through the buzz generated by the consumers who are de facto advertising agents: a "must have" effect. (We will investigate in more detail the simpler case where this effect is neglected.) In the absence of any advertising, the goodwill decays at a constant relative rate $f > 0$. The goodwill dynamics are therefore

$$\dot{x}(t) = -fx(t) + g\sum_{k=1}^{n} s_k(t) + h\sum_{k=1}^{n} q_k(t)\,.$$

Producer i produces at a rate $q_i(t)$ at time t, at a production cost $c_i q_i(t)$. We assume that the market price p is linearly growing with x and decreasing with total production as in a linear model:

$$p(t) = a + x(t) - b\sum_{k=1}^{n} q_k(t)\,. \tag{18}$$

Let

$$a - c_i = \alpha_i\,, \qquad n\alpha_i - \sum_{j\neq i}\alpha_j = \beta_i\,, \qquad \sum_{i=1}^{n}\alpha_i = \sum_{i=1}^{n}\beta_i = n\bar{\alpha}\,.$$

[24] As is done in e.g. [Kalman, 1960c], but not in the continuous time [Kalman, 1960a].

We use a discount factor ρ. Hence player i's profit over an horizon T is

$$\Pi_i = \int_0^T \mathrm{e}^{-\rho t} \left[q_i \left(\alpha_i + x(t) - b \sum_{k=1}^n q_k(t) \right) - s_i^2 \right] \mathrm{d}t \,.$$

The Hamilton-Jacobi-Carathéodory-Isaacs-Bellman equation We look for a Bellman (or rather Isaacs Value) function of the form $V_i(t,x) = \exp(-\rho t) W_i(t,x)$ with W_i a nonhomogeneous quadratic function of x. But we will find that the coefficient of x^2 is the same for all:

$$W_i(t,x) = P(t)x^2 + 2L_i(t)x + M_i(t)\,,$$

Thus, the HJCIB equation reads

$$\begin{aligned} -\dot{W}_i + \rho W_i = \max_{q_i, s_i} \Bigg\{ & 2(Kx + L_i) \left[-fx(t) + g \sum_{k=1}^n s_k(t) + h \sum_{k=1}^n q_k \right] \\ & + q_i \left(\alpha_i + x(t) - b \sum_{k=1}^n q_k(t) \right) - s_i^2 \Bigg\}. \end{aligned}$$

We easily derive the equations for the maximizing policies $s_i^\star$ and $q_i^\star$ as

$$\begin{aligned} s_i^\star &= g(Kx + L_i)\,, \\ bq_i^\star &= (Kx + L_i)h + \frac{1}{2}\Big(x + \alpha_i - b \sum_{j \neq i} q_j^\star \Big)\,. \end{aligned} \tag{19}$$

As is expected from R.E.K.'s theory, this yields feedback strategies explicitely using the goodwill x. Whether this elusive quantity may be directly measured is debatable. But one may assume that the total output $\sum_k q_k$ of the producers may be observed. Then the goodwill x may be recovered from equation (18) as $x = p - a + \sum_k q_k$. A first conclusion is that the advertising efforts of the producers increase with the goodwill, to offset an increasing goodwill decay $-ax$. Let also

$$\sum_{k=1}^n L_k = \mathcal{L}\,, \quad nL_i - \sum_{j \neq i} L_j = (n+1)L_i - \mathcal{L} = \Lambda_i\,.$$

The equations for $q_i^\star$ yield

$$q_i^\star = \frac{1}{(n+1)b} [(2hK + 1)x + 2h\Lambda_i + \beta_i]\,.$$

Not surprisedly, equilibrium productions increase with the goodwill.

Placing these back in the Hamilton-Jacobi-Carathéodory-Isaacs equation and equating the coefficients of the various powers of x, we obtain

$$-\dot{K}+\rho K=\left[(2n-1)g^2+\frac{4n^2h^2}{(n+1)^2b}\right]K^2+2\left[-f+\frac{(n^2+1)h}{(n+1)^2b}\right]K+\frac{1}{(n+1)^2b}.$$

$$-\dot{L}_i+\rho L_i=\left[-f+(n-1)g^2K+\frac{h}{b}\right]L_i+\left[\left(g^2+\frac{4nh^2}{(n+1)^2b}\right)K-\frac{2h}{(n+1)^2b}\right]\mathcal{L}$$
$$+\frac{hK\alpha_i}{b}-\frac{2nhP-1}{(n+1)^2b}\beta_i\,.$$

$$-\dot{M}_i+\rho M_i=\left[g^2(2\mathcal{L}-L_i)+\frac{2h\alpha_i}{b}\right]L_i+\frac{(\beta_i-2h\mathcal{L})^2}{(n+1)^2b}\,.$$

These differential equations are to be integrated backward from zero terminal conditions at T, an easy task to carry out numerically even for a large n. Observe that the equation for K is of Riccati type, but decoupled from the following ones. Once K is computed, the equations for the L_i are linear, and $\exp(-\rho t)M_i(t)$ may be obtained as a mere integral.

If the horizon is taken as infinite, then all coefficients K, L_i and M_i are constant. The differential equations are replaced by the algebraic equations obtained by setting the derivatives equal to zero. In particular, we have

$$\left[(2n-1)g^2+\frac{4n^2h}{(n+1)^2b}\right]K^2-2\left[f+\frac{\rho}{2}-\frac{(n^2+1)h}{(n+1)^2b}\right]K+\frac{1}{(n+1)^2b}=0\,. \tag{20}$$

Two positive solutions exist provided that ρ be large enough, the more so that h is larger, because a larger h clearly allows the producers to sell more for the same price, i.e. increase their return, ending in an infinite payoff if the discount factor is not large enough. The smallest root is the solution sought. And L_i can also be obtained in closed form, via the use of $\mathcal{L}$ obtained by summing the equations for L_i:

$$\left[f+\rho-(2n-1)g^2K-h\frac{4n^2hK+n^2+1}{(n+1)^2b}\right]\mathcal{L}=\frac{(n^2+1)hK+1}{(n+1)^2b}n\bar{\alpha}\,,$$

and

$$\left[f+\rho-(n-1)g^2K-\frac{h}{b}\right]L_i=$$
$$\left[\left(g^2+\frac{4nh^2}{(n+1)^2b}\right)K-\frac{2h}{(n+1)^2b}\right]\mathcal{L}+\frac{hK\alpha_i}{b}-\frac{2nhK-1}{(n+1)^2b}\beta_i, \tag{21}$$

$$\tag{22}$$

and

$$\rho M_i=\left[g^2(2\mathcal{L}-L_i)+2\frac{h}{b}\alpha_i\right]L_i+\frac{(\beta_i-2h\mathcal{L})^2}{(n+1)^2b}\,.$$

These formulas are unappealing. They are nevertheless very easy to implement numerically, and thus make possible a numerical comparative statics as illustrated in the next subsection. Actually, a complete proof requires that we check the stability of the dynamics when one producer deviates from its equilibrium strategy. We give more details in a simpler case in subsection 4.3.3.

4.3.2 Qualitative lessons

Notice first that, in investigating the infinite horizon problem, we must assume that

$$\frac{h}{b} < f\,.$$

(Both have the dimension of a frequency, i.e. the inverse of a time.) It can be seen that otherwise, low production costs producers can make an infinite profit by producing large quantities; the price rise due to the goodwill effect overcoming the price decrease due to the inverse demand equation.

Influence of production costs on equilibrium advertising We investigate first the effect of a production cost differential on the advertising strategies of the producers. To that effect, we consider the case of two players. Observe that, according to equation (19), the difference between the advertising strategies of the producers only comes from L_i. It follows from equation (21) that L_i is increasing with $\alpha_i = a - c_i$ hence decreasing with c_i. (Indeed, our numerical experiments show that $2nK$ is always smaller than one, so that the term in $\beta_i = n\alpha_i - \sum_{j\neq i}\alpha_j$ acts in the same direction as the term in α_i.) The players with lower production costs will invest more in collective advertising, as they are set to profit more from larger sales. We provide in figure 2 a graph of $s_2^\star/s_1^\star$ as a function of c_2/c_1 in the case $f = 4$, $g = 1$, $h = 1$, $a = 2$, $b = 1$, $\rho = .1$.

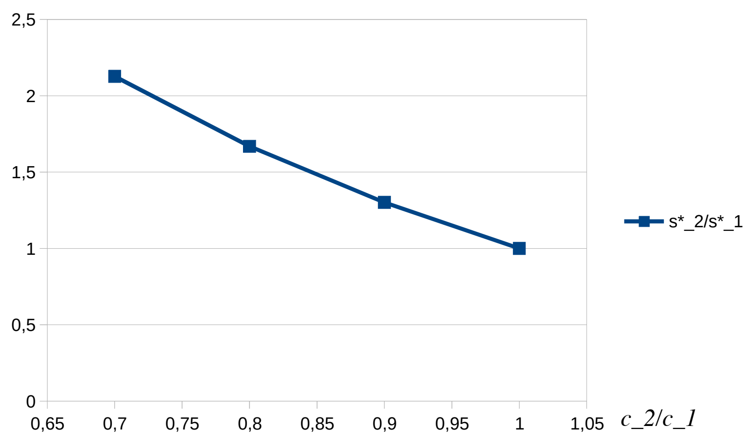

Figure 2: $s_2^\star/s_1^\star$ as a function of c_2/c_1 for $f = 4$, $g = 1$, $h = 1$, $a = 2$, $b = 1$, $\rho = .1$.

Influence of goodwill rate of decay We provide in figure 3 a graph of the influence of the rate of goodwill decay on equilibrium advertising in the case of 8 identical players. We observe a rapid decrease of the advertising effort as the decay rate f increases, "explained" by a correlative decay of x. The true explanation is probably that, when the decay rate f is too large, the effort of advertising is essentially lost by the decay. A large advertising effort would be akin to pouring water in the Danaids' proverbial leaking bucket.

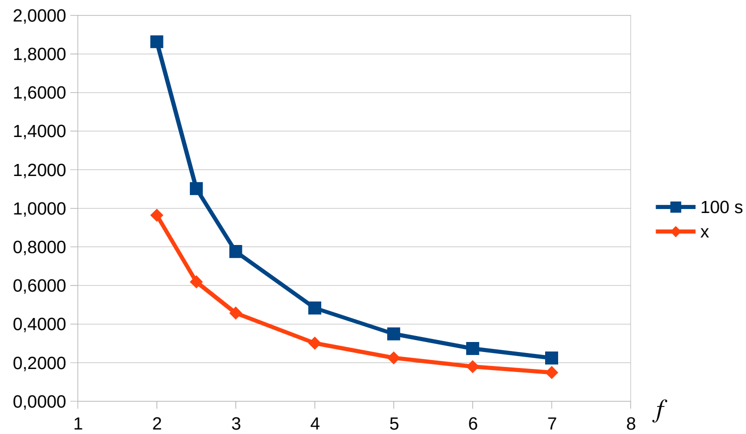

Figure 3: $s^\star$ and x as functions of f for $n = 8$, $g = 1$, $h = 1$, $a = 2$, $b = 1$, $\rho = .1$.

Asymptotic behaviour for large n An analytic consequence to be drawn from our formulas is the infinite horizon behaviour as the number of producers gets large.

The smallest root of equation (20) behaves asymptotically as

$$K \sim \frac{1}{2n^2 b(f + \frac{\rho}{2} - \frac{h}{b})} \,.$$

Also,

$$L_i \sim \frac{1}{n^2 b(f + \rho - \frac{h}{b})} \left[\frac{\frac{h}{b}}{2f + \rho - 2\frac{h}{b}} \alpha_i + \beta_i \right] ,$$

which is also positive, and although more complicated, M_i can also be seen as going to zero as $1/n^2$. Therefore, as could be expected for a Cournot-like oligopoly, the individual profit decreases as $1/n^2$, and therefore the collective profit as $1/n$ when the number of producers goes to infinity.

More significantly, while the total production goes to a finite value

$$\sum_{k=0}^{n} q_k^\star \to \frac{f}{bf - h} \bar{\alpha} \,,$$

the total brand advertising goes to zero as

$$\sum_{k=1}^{n} s_k^\star \sim \frac{gf}{nb(f - \frac{h}{b})(f + \rho - \frac{h}{b})} \bar{\alpha} \,,$$

and the equilibrium goodwill

$$x \to \frac{h}{bf - h} \bar{\alpha} \,,$$

is kept nonzero by the "must have" effect only, since the advertising effort vanishes.

4.3.3 A simplified model

We consider now the simplest possible case, and first of all that all players are identical, and an infinite horizon. Moreover, we neglect the positive effect of the circulation of the good on the market size, i.e. we set $h = 0$. Then the q_i have no dynamic effect. They can be chosen so as to maximize the running profit:

$$q_i^\star = q^\star = \frac{x+\alpha}{(n+1)b}\,, \qquad \Pi_i = \int_0^\infty \mathrm{e}^{-\rho t}\left[\frac{(x+\alpha)^2}{(n+1)^2 b} - s_i^2\right]\mathrm{d}t.$$

Since all players are identical, all have the same Value function $V(x) = Kx^2 + 2Lx + M$. Hence equation (20) simplifies as

$$(2n-1)g^2K^2 - (2f+\rho)K + \frac{1}{(n+1)^2 b} = 0 \tag{23}$$

which has a solution if and only if

$$\rho \geq \frac{2g}{n+1}\sqrt{\frac{2n-1}{b}} - 2f\,.$$

Otherwise, the infinite time problem has no finite solution. The producers may obtain an arbitrarily large payoff. When they exist, the two roots are then positive. It follows from the analysis in [Kalman, 1960a] that we are interested in the limit as $t \to -\infty$ of the solution of the Riccati differential equation integrated from $K(0) = 0$. Hence it is the smallest root of (23), i.e.:

$$K = \frac{1}{2(2n-1)g^2}\left(2f+\rho - \sqrt{(2f+\rho)^2 - \frac{4(2n-1)g^2}{(n+1)^2 b}}\right).$$

Determinig the equilibrium policies also requires that L be known (but not M):

$$L = \frac{\alpha}{(n+1)^2 b}\left[f+\rho-(n-1)g^2K\right]^{-1}\,.$$

To have the solution entirely proved, we must also check that $\exp(-\rho t)x(t)^2$ decreases exponentially as $t \to \infty$, and that this is also true if some of the players play $s_i = 0$ instead of the equilibrium policy. Hence we must check first that

$$-f + ng^2K < \frac{\rho}{2}\,,$$

hence that

$$K < \frac{1}{ng^2}\left(f + \frac{\rho}{2}\right)\,.$$

However, we know that K is the smallest root of the polynomial (23), whose half sum of roots is $(f+\rho/2)/(2n-1)g^2 < (f+\rho/2)/ng^2$. Hence the above inequality is proved. Now, if some of the players play $s_i = 0$, the coefficient of x in the ensuing dynamics is even smaller, hence the equilibrium property is proved.

4.3.4 Closing remarks

With this little example, our only objective has been to show

1. that REK's standard LQ theory can easily be extended in various directions, and
2. that some microeconomic models may rather naturally lead to such problems.

The simplified problem above, in a one player version, was amenable to the pre-kalmanian theory of [Newton Jr et al., 1957], although not easily. The game version was already more problematic[25]. The case with $h \neq 0$ was in principle, provided that one be able to perform a spectral factorisation for the two-input multivariable system. But the simplest way to achieve that is via the Riccati equation resurrected by R.E.K., as shown by [Willems, 1971]. The finite horizon problems were completely beyond the reach of that theory.

5 Going on

R.E.K.'s contributions of 1960-1961 were a powerful stimulus for system theory and control research. Many researchers followed suit, both to get further theoretical advances (such as those hinted at in sections 3.3, 4.2.2 and much more), and to develop algorithms concretely implementing those theoretical results.

Algebraic sytem theory attracted many researchers such as [Wonham, 1967] and remained R.E.K.'s main research area, using sophisticated algebraic tools and ideas from automata theory. (See, e.g. [Kalman, 1965, Kalman et al., 1969, Kalman, 1972]). The linear quadratic theory of control and observation was deeply renewed by the theory of $\mathcal{H}_\infty$-optimal conrol, initiated by [Zames, 1981] in the frequency domain "external" description, and later transfered in a Kalman-like time domain formulation (see [Başar and Bernhard, 1995]), with a minimax, probability-free treatment of uncertainties, where the same duality shows up, in a more complex setup, and a bit mysteriously.

The basic theory quickly found its way into all engineering control textbooks, and more recently in economics textbooks such as [Weber, 2011]. The algorithms were coded into publicly available software packages, and they have been used in a wide range of application domains, well beyond the industrial and transportation systems of the early times, encompassing all branches of engineering as well as natural and biomedical sciences. We have shortly discussed above its uses in economics, with a small example in microeconomics exhibiting some of the easy extensions of the theory.

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A Applications in engineering

Control theory was developped as a science for engineers, the first applications were obviously in that field, and these are still the main domain of application.

A.1 Linear and linearized control systems

Already in "pre-kalmanian" times, the LTI theory was put to use in a variety of physical systems. Some had reasonable linear physical models, and when considered in steady state, were time invariant. But most industrial systems such as transportation systems, energy systems,… have a natural nonlinear model. The engineering practice, then, is to define a desired or *nominal* output trajectory, and take as the output of the control system the *error signal*, i.e. the difference between the actual and the desired outputs. The objective of the control system is then to keep this error signal close to zero via a *dynamic compensator* : a dynamic system whose input is the measured error signal, and the output the control input of the to-be-controlled system. (Hence a feedback system, as understood by Wiener.)

In order to achieve this goal, one builds a linear model as the linearization of the nonlinear model for small deviations around the nominal trajectory. This can be done either from an analytic nonlinear model linearized by a first order expansion, or from experiments using further parts of the theory (such as the consieration of cross correlations between input and output pseudo-random small deviations).

This being done, the aim of the control system is too keep the error variables, approximated as the variables in the control model, close to zero in spite of disturbances in the dynamics, measurement errors, lack of direct measurement of some important variables, not to mention modelization errors and biasses.[26]

A.2 Observer-controller: algebraic approach

Keeping a steady state variable close to zero is achieved by forcing the system to be sufficiently stable. In that process, consideration of the poles of the transfer function, therefore the eigenvalues of the internal description of the overall system, is of the essence, since, for continuous-time systems, their real parts, which must be negative to insure stability, give a measure of the degree of stability while their imaginary parts

[26]These do not have the statistical characteristics of "noises", and were at the inception of "robust control" theories, and most noticeably for our purpose, $\mathcal{H}_\infty$-optimal control by [Zames, 1981].

yield a measure of the oscillatory character of the response of the system. (In discrete time, their modulus must be less than one to insure stability.)

As mentioned in subsection 2.1.1, engineers had developed more sophisticated tools than just the inspection of the poles of the transfer function. But these remain of paramount importance. Hence the use of the separation of dynamics and pole shifting theorems, choosing separately the poles of the observer and those of the controller. A rule of thumb being that the observer must be an order of magnitude faster than the controller. Trial and errors with the localization of the poles using simulation models (both linearized and nonlinear if available) would allow one to construct an efficient control device.

A.3 Quadratic synthesis

However, with the advancement of modelization science, largely driven by the advent of the computational power of digital computers, the dimension of the models used increased to a point where simple methods based on poles location were not practical.[27]

Moreover, some problems such as automatically landing an airplane, controlling an industrial baking cycle by heating and cooling an oven, etc. are intrinsically finite horizon, non LTI problems, with sometimes a great emphasis on terminal error control. These problems are beyond the reach of algebraic methods.

Engineers may have a fairly precise idea of the origin and sizes of the disturbances in the dynamics and error sources in the measurements, let alone modelization biasses. This provides a sound basis for computing an observer gain via the KF.

Concerning the controller, one computes a control gain via a quadratic performance index and the Kalman optimal gain. The process of trial and errors in tuning it is performed on the weighting matrices of the performance index, and is made easy and efficient through the interpretation of the gain as minimizing this performance index, so that one knows how the different state and control variables will respond to a change in the weighting matrices. This process, known as *quadratic synthesis*, is what made the new theory so popular among engineers, to the point of being the main tool used in designing the Apollo control systems.

[27] The rigid body dynamics of a landing airplane are decsribed by a 12th order system. Adding egines and control surfaces dynamics, and in modern airliners flexible modes, leads way beyond that figure.