Cournot oligopoly with randomly arriving producers

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November, 30, 2017

Abstract

Cournot's model of oligopoly appears as a central model of strategic interaction between competing firms both from a theoretical and an applied perspective (e.g antitrust). As such it is an essential tool in the economics toolbox and always a stimulus. Although there is a huge and deep literature on it and as far as we know, we think that there is a niche which has not yet been investigated: Cournot oligopoly with randomly arriving producers. In a companion paper [Bernhard and Deschamps, 2017] we have proposed a rather general model of a discrete dynamic decision process where producers arrive as a Bernoulli random process and we have given some examples relating to oligopoly theory (Cournot, Stackelberg, cartel). In this paper we study Cournot oligopoly with random entry in *discrete* (Bernoulli) and *continuous* (Poisson) time, whether the time horizon is finite or infinite. Moreover we consider here both constant and variable probabilities of entry or density of arrivals. In this framework, we are able to provide algorithms answering four classical questions: 1/ what is the expected profit for a firm as a function of its rank of arrival on the market?, 2/ How do individual quantities evolve?, 3/ How do market quantities evolve?, and 4/ How does market price evolve?

Keywords Cournot market structure, Bernoulli process of entry, Poisson density of arrivals, Dynamic Programming.
JEL code: C72, C61, L13
MSC: 91A25, 91A06, 91A23, 91A50, 91A60.

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1 Introduction

While it was ignored for many years it seems almost impossible today to think about competition in economics without considering the Cournot oligopoly model. As H. Demsetz said in his *Economics of Business firms* book in 1995 it is one of the "safe harbors" of economic analysis, a statement also shared by A. Daughety who considers that the Cournot oligopoly model "over the recent decades has come to be an essential tool in many economist's toolbox, and is likely to continue as such" in his *New Palgrave Dictionary* notice on Cournot competition.

In its classical form, Cournot's model is static, each producer's strategy is the quantity of output she will produce in the market for a specific homogeneous good and when the number of identical producers goes to infinity the market price converges toward the marginal cost. Along the years, economists have extended this classical form in many directions, including asymmetric producers, differentiated goods and dynamics. On this last topic economists have notably considered the Cournot model with such characteristics as several periods of production ([Saloner, 1987], [Pal, 1991]), game with free entry ([Mankiw and Whinston, 1986], [Amir et al., 2014]), or in the theory of repeated games ([Abreu, 1986]), or stochastic games ([Kebriaei and Rahimi-Kian, 2011]), or as a Poisson game ([Myerson, 1998], [Myerson, 2000]) and, recently, a mean field game ([Chan and Sircar, 2015]), in continuous-time ([Snyder et al., 2013]) or with intertemporal capacity constraints ([van den Berg et al., 2012]). But as far as we know, despite this huge and deep literature, there is a niche which has not yet been investigated: a Cournot oligopoly model with randomly arriving producers.

To begin the investigation of this question we consider a model where there is at the initial step a fixed number of symmetric producers of an homogeneous good playing according to a complete information Cournot oligopoly with the common knowledge hypothesis that, at each future step, and irrespective of past entries, an identical producer may enter the game and, if it does so, stay for ever. We will use control theoretic and dynamic game theoretic methods as in [Harris et al., 2010] and [Ludkovski and Sircar, 2012].

The paper is organized as follows: in the next section we present the structure of the general dynamic game model with randomly arriving players as we developed in [Bernhard and Deschamps, 2017]. Section 3 is devoted to the discrete time problem while section 4 tackles the continuous time problem, considered here as the limit of the former as the step size vanishes. In both cases, we consider both constant and variable entry probabilities, and finite and infinite time horizon. We also provide numerical results.

Section 5 ends the paper by discussing conclusions and limits of our analysis.

2 General model

In a companion paper [Bernhard and Deschamps, 2017], we investigated a rather general model of a discrete dynamic decision process where players arrive as a Bernoulli random process. We summarize here the results obtained there, simplified to fit our need in this article.

Time t is an integer, or, in section 4, a (continuous) real variable. At time t_1 one player is present, then players arrive as a Bernoulli process with a unit probability p. We will first consider the case of a constant p; then we will let it depend on the rank of the arriving player, or equivalently, on the number of producers already on the market. Of particular interest is the case where p is a known function of the expected future return at the arrival rank. Player number m arrives at time t_m, a random variable. The game is played over an horizon T which may be finite or infinite. A sequence of (usually positive decreasing) numbers $\{\pi_m\}$ is given, denoting the reward of each player during one time period if there are m players present. Typically, π_m may be taken as the individual profit in a static m-player Cournot oligopoly. We let $m(t)$ be the number of players actually present at time t, a random variable. Therefore, at each period of time t, all players get a reward $\pi_{m(t)}$. Let finally $r \in (0,1)$ be a discount factor. The reward of the n-th player arrived is

$$\Pi_n = \sum_{t=t_n}^{T} r^{t-t_n} \pi_{m(t)} \,,$$

and we sought to evaluate its expectation Π_n^e. Figure 1 illustrates that problem.

Concerning the sequence $\{\pi_m\}$, we will use the following definitions:

Definition 1 *The sequence $\{\pi_m\}$ is said to be*

- bounded by π *if there exists a positive number π such that*

$$\forall m \in \mathbb{N}\,, \quad |\pi_m| \le \pi\,,$$

- exponentially bounded by π *if there exists a positive number π such that*

$$\forall m \in \mathbb{N}\,, \quad |\pi_m| \le \pi^m\,.$$

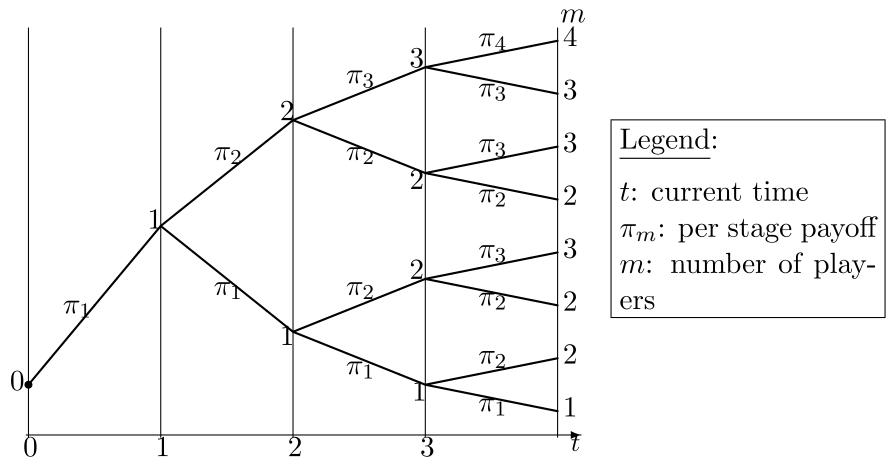

Figure 1: The events tree

Notice that if the sequence $\{\pi_m\}$ is bounded by π, it is exponentially bounded by $\max\{1, \pi\}$, while if it is exponentially by $\pi \leq 1$, it is bounded by π.

We also need the following notation for a domain of the discrete plane, for any positive integer (a time interval) ν:

$$\mathcal{D}_\nu = \{(k, \ell) \in \mathbb{N}^2 \mid 0 \leq \ell \leq k \leq \nu\}\,. \tag{1}$$

The theorems proved in [Bernhard and Deschamps, 2017] can be simplified here, with the use of the combinatorial coefficients

$$\forall k \geq \ell \in \mathbb{N}\,, \qquad \begin{pmatrix} k \\ \ell \end{pmatrix} = \frac{k!}{\ell!(k-\ell)!}$$

Theorem 2 (Bernhard and Deschamps [2016]) *If $T < \infty$, or if $T = \infty$ and the sequence $\{\pi_m\}$ is bounded or exponentially bounded by $\varpi < 1/r$, the expected payoff of the n-th arrived player is*

$$\Pi_n^e = \begin{cases} \displaystyle\sum_{(k,\ell)\in\mathcal{D}_{T-t_n}} [(1-p)r]^k \left(\frac{p}{1-p}\right)^\ell \begin{pmatrix} k \\ \ell \end{pmatrix} \pi_{n+\ell} & \text{if} \quad p < 1\,, \\ \displaystyle\sum_{k=0}^{T-t_n} r^k \pi_{n+k} & \text{if} \quad p = 1\,. \end{cases} \tag{2}$$

3 Discrete time

3.1 Constant entry probability

3.1.1 Algorithm

We offer an alternative approach to theorem 2 to evaluate Π_n^e, recovering an algorithm that can easily be derived from formula (2). Given any natural integer (time interval) k, let $q_\ell(k)$ be the probability that ℓ players arrive during that time interval. In a Bernoulli process, only one player may arrive at each instant of time. Thus, there are only two incompatible ways to achieve exactly ℓ arrivals at time k: either there were $\ell-1$ arrivals at time $k-1$ and one arrived at time k, or there were already ℓ arrivals at time $k-1$ and none arrived at time k. Hence

$$q_\ell(k) = pq_{\ell-1}(k-1) + (1-p)q_\ell(k-1)\,. \tag{3}$$

Now, it holds that

$$\Pi_n^e = \sum_{t=t_n}^{T} r^{t-t_n}\mathbb{E}(\pi_{m(t)} \mid t_n)\,, \quad \text{and} \quad \mathbb{E}(\pi_{m(t)} \mid t_n) = \sum_{\ell=0}^{t-t_n} q_\ell(t-t_n)\pi_{n+\ell}\,.$$

Therefore, using $t - t_n = k$,

$$\Pi_n^e = \sum_{k=0}^{T-t_n}\sum_{\ell=0}^{k} r^k q_\ell(k)\pi_{n+\ell}$$

We define $w_\ell(k) = r^k q_\ell(k)$ to obtain

$$\Pi_n^e = \sum_{k=0}^{T-t_n}\sum_{\ell=0}^{k} w_\ell(k)\pi_{n+\ell} = \sum_{(k,\ell)\in\mathcal{D}_{T-t_n}} w_\ell(k)\pi_{n+\ell}\,, \tag{4}$$

and also the recursive formula, useful for numerical computations:

$$\Pi_n^e(T) = \Pi_n^e(T-1) + \sum_{\ell=0}^{T-t_n} w_\ell(T-t_n)\pi_{n+\ell}\,.$$

The $w_\ell(k)$ can be computed according to the following recursion. (The first two lines may be seen as initialization tricks, while the third one directly derives from equation (3))

$$\begin{aligned} w_0(0) &= 1\,, \\ \forall k\in\mathbb{N}\,, \quad w_{-1}(k) &= w_k(k-1) = 0\,, \\ \forall \ell \le k\,, \quad w_\ell(k) &= rpw_{\ell-1}(k-1) + r(1-p)w_\ell(k-1)\,, \end{aligned}$$

We know from appendix A that the π_m of interest here are uniformly bounded, and therefore, from theorem 2, that this can be extended to the case where $T = \infty$.

Indeed, that algorithm may be derived from formula (2) identifying

$$w_\ell(k) = r^k (1-p)^{k-\ell} p^\ell \begin{pmatrix} k \\ \ell \end{pmatrix}$$

and using the classical formula of "Pascal's triangle":

$$\begin{pmatrix} k \\ \ell \end{pmatrix} = \begin{pmatrix} k-1 \\ \ell-1 \end{pmatrix} + \begin{pmatrix} k-1 \\ \ell \end{pmatrix} .$$

3.1.2 Numerical results

Figure 2 provides a plot of $\Pi_1^e(T)$ assuming $t_1 = 0$, for T varying from 0 to 20, and for $p \in \{0, .1, .4, .7, 1\}$. In this plot, we chose $r = .9$ and $\pi_n = 1/(n+1)^2$. All the computations in this article were done with Scilab.

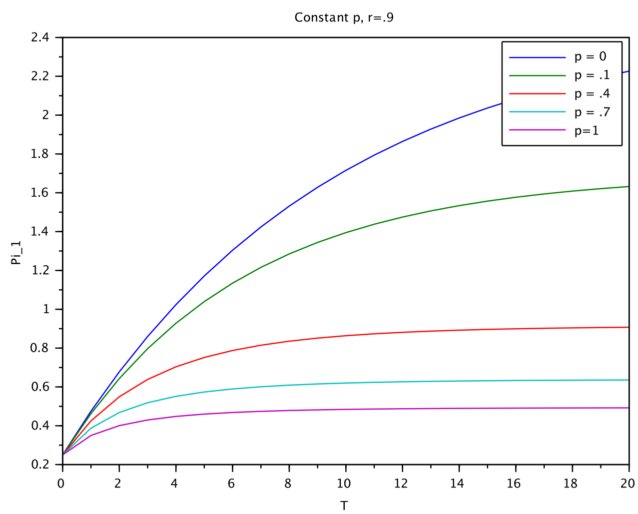

Figure 2: Plots of $\Pi_1^e(T)$ against T for various values of p, with $r = .9$.

3.2 Variable entry probability

We have seen in the introduction that it may be desirable to let the probability of entry depend on the rank of the entrant. The method of paragraph 3.1.1 can easily be extended to such a case.

3.2.1 Algorithm

Let therefore p_m be the probability of entry of the competitor of rank m.

We modify slightly our previous algorithm, by introducing now the probability $q_{n,m}(k)$ of having m players present at time $t_n + k$ knowing that they were n at time t_n, and $w_{n,m}(k) = r^k q_{n,m}(k)$. (Notice that if p_n happens to be constant equal to p, then $w_{n,m}(k) = w_{m-n}(k)$.) Equation (3) is now replaced by

$$q_{n,m}(k) = p_m q_{n,m-1}(k-1) + (1 - p_{m+1}) q_{n,m}(k-1) \,,$$

and we get

$$\begin{aligned} \forall n \in \mathbb{N} \,, \quad & w_{n,n}(0) = 1 \,, \\ \forall n, k \,, \quad & w_{n,n-1}(k) = w_{n,n+k}(k-1) = 0 \,, \\ \forall n \le m \le n+k \,, \quad & w_{n,m}(k) = r p_m w_{n,m-1}(k-1) + r(1 - p_{m+1}) w_{n,m}(k-1) \,. \end{aligned}$$

Formula (4) generalizes into

$$\Pi_n^e = \sum_{k=0}^{T-t_n} \sum_{\ell=0}^{k} w_{n,n+\ell}(k) \pi_{n+\ell} = \sum_{(k,\ell) \in \mathcal{D}_{T-t_n}} w_{n,n+\ell}(k) \pi_{n+\ell} \,. \tag{5}$$

3.2.2 A backward algorithm

While the algorithm of the previous paragraph is well adapted to an infinite horizon (neglecting terms of high order thanks to the discount factor), it does not fit our aim to let p_m depend on Π_m^e. To reach this goal, we need to compute the latter before using the former. This is provided by the following algorithm.

Let F be a fixed entry cost. We take advantage of the fact that we assume that no entry will occur once $\Pi_n^e < F$. The last entrant, say of rank N has an expected profit

$$\Pi_N^e = \frac{\pi_N}{1 - r}$$

and is defined by the fact that

$$\pi_{N+1} < (1 - r) F \le \pi_N \,.$$

Therefore, we know N and Π^e_N. From there, we can proceed by backward induction.

We write the profit Π_n as a function of the arrival times t_n and t_{n+1} as the sum of the profits accumulated between these two time instants, plus the profit to be made after time t_{n+1} (all these quantities are random variables) as

$$\Pi_n(t_n) = \sum_{t=t_n}^{t_{n+1}} r^{t-t_n}\pi_n + r^{t_{n+1}-t_n}\Pi_{n+1}(t_{n+1})\,.$$

We may use the fact that, on the one hand

$$\Pi^e_n(t_n) = \mathbb{E}(\Pi_n \mid t_n)$$

and on the other hand

$$\mathbb{E}(\Pi_{n+1} \mid t_n) = \mathbb{E}_{t_n+1}\left[\mathbb{E}(\Pi_{n+1} \mid t_{n+1})\right] = \mathbb{E}_{t_n+1}\left[\Pi^e_{n+1}(t_{n+1})\right]\ ,$$

to get

$$\Pi^e_n = \sum_{t_{n+1}=t_n+1}^{\infty} (1-p_{n+1})^{t_{n+1}-t_n-1} p_{n+1}\left[\sum_{t=t_n}^{t_{n+1}} r^{t-t_n}\pi_n + r^{t_{n+1}-t_n}\Pi^e_{n+1}(t_{n+1})\right]. \tag{6}$$

It takes some calculations given in appendix B to conclude

$$\Pi^e_n = \frac{1}{1-(1-p_{n+1})r}\left(\pi_n + p_{n+1}r\Pi^e_{n+1}\right). \tag{7}$$

This formula can be used backward from Π^e_N.

3.2.3 Numerical results

Figure 3 provides a plot of the expected profit in the infinite horizon game as a function of the rank of entry, computed with the backward algorithm. In this computation, we chose $r = .9$, $\pi_n = 100/(n+1)^2$. We let somewhat arbitrarily $F = 2$ and

$$p_n = 1 - \frac{F}{\Pi^e_n}$$

in order to have an entry probability increasing with the expected payoff (i.e decreasing with the rank of entry) and equal to zero as Π^e_n drops below the cost of entry. For these parameters, only 21 players enter the game.

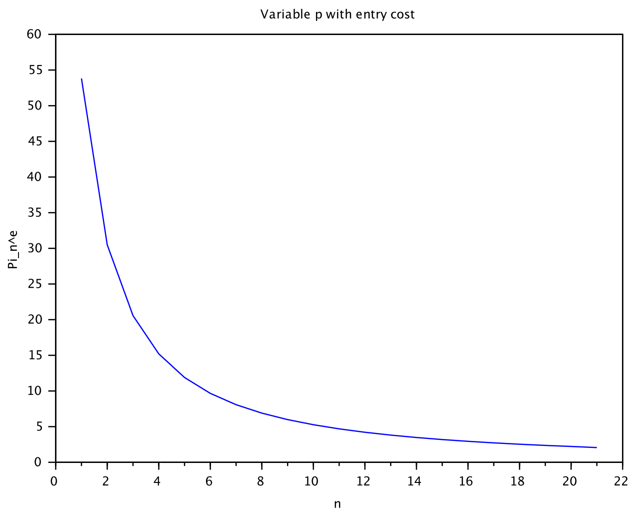

Figure 3: Expected payoff for the infinite horizon game as a function of the rank of entry for $p_n = 1 - \frac{F}{\Pi_n^e}$, with $r = .9$.

4 Continuous time

We aim to derive the continuous time limit formulas and algorithms as the step size vanishes. The profit when n players are present is now a *rate of profit per unit time*, denoted by ϖ_n. Accordingly, q, Q, and $C(q)$ are now rates of production, respectively expense, per unit time (although we keep the same notation as in the discrete time case). Let h (instead of 1) be the step size, an integer submultiple of the horizon T when in finite horizon. Of course, the number of steps to reach a fixed time goes to infinity as h goes to zero, but the per step discount factor goes to one. Let an upperindex (h) denote the relevant quantities when the step size is h. Specifically, we set

$$r^{(h)} = \mathrm{e}^{-\delta h}$$

for a fixed continuous discount factor δ.

In the limit, the individual profit of the player of rank n arrived at time t_n is

$$\Pi_n(t_n) = \int_{t_n}^{T} \mathrm{e}^{-\delta(t-t_n)} \varpi_{m(t)} \, \mathrm{d}t \, .$$

4.1 Constant Poisson density of arrivals

4.1.1 Continuous formula

Formula (2) must now be slightly modified into

$$\Pi_n^e = \sum_{k=0}^{\frac{T-t_n}{h}} [r^{(h)}(1-p^{(h)})]^k \sum_{\ell=0}^{k} \left(\frac{p^{(h)}}{1-p^{(h)}}\right)^{\ell} \binom{k}{\ell} \pi_{n+\ell}^{(h)} . \tag{8}$$

The limit as $h \to 0$ in the above formula leads to the following one, which has been derived directly with a Poisson process in [Bernhard and Hamelin, 2016]:

Theorem 3 *In the limit as the step size goes to zero, the expected payoff is given by formula*

$$\Pi_n^e = \int_0^{T-t_n} \mathrm{e}^{-(\lambda+\delta)t} \sum_{\ell=0}^{\infty} \frac{(\lambda t)^{\ell}}{\ell!} \varpi_{n+\ell} \, \mathrm{d}t . \tag{9}$$

If the sequence $\{\varpi_m\}$ is bounded or bounded by the powers $\varpi^m \geq |\varpi_m|$ of a number $\varpi \leq 1 + \delta/\lambda$, this formula can be extended to $T = \infty$.

Proof The calculation hereafter is directly inspired by the classical analysis of the continuous limit of a Bernoulli process, which is known to be a Poisson process. Let the discrete quantities be expressed, up to second order in h, in terms of the continuous ones as follows, where δ, λ and the sequence $\{\varpi_m\}$ are the continuous data:

$$r^{(h)} = \mathrm{e}^{-\delta h} , \qquad p^{(h)} = \lambda h , \qquad \pi_n^{(h)} = \varpi_n h . \tag{10}$$

In formula (8), we let simultaneously h go to zero and each k go to infinity keeping $kh = t$ constant. It is a classic fact that

$$(1-\lambda h)^k = \mathrm{e}^{k \ln(1-\lambda h)} \simeq \mathrm{e}^{k(-\lambda h)} = \mathrm{e}^{-\lambda t} .$$

Therefore,

$$[r^{(h)}(1-p^{(h)})]^k \simeq \mathrm{e}^{-(\lambda+\delta)t} .$$

Furthermore

$$\left(\frac{\lambda h}{1-\lambda h}\right)^{\ell} \binom{k}{\ell} \varpi_{n+\ell} h = \frac{\prod_{i=0}^{\ell-1} \lambda h (k-i)}{(1-\lambda h)^{\ell} \ell!} \varpi_{n+\ell} h .$$

When $h \to 0$ and $k \to \infty$ with $kh = t$, $\lambda h(k - i) \to \lambda t$. Also, $(1 - \lambda h)^\ell \to 1$. We therefore have

$$\Pi_n^e \simeq \sum_{k=0}^{\frac{T-t_n}{h}} \mathrm{e}^{-(\lambda+\delta)kh} \sum_{\ell=0}^{k} \lambda^\ell \frac{(kh)^\ell}{\ell!} \varpi_{n+\ell} h$$

which converges to (9). Finally, if $\varpi_n \leq \varpi^n$, (9) yields

$$\Pi_n^e \leq \int_0^{T-t_n} \mathrm{e}^{-(\lambda+\delta)t} \varpi^n \sum_{\ell=0}^{\infty} \frac{(\lambda \varpi t)^\ell}{\ell!} \,\mathrm{d}t = \int_0^{T-t_n} \mathrm{e}^{(-\lambda-\delta+\lambda\varpi)t} \,\mathrm{d}t$$

which converges when $T \to \infty$ provided that $-\lambda - \delta + \lambda\varpi < 0$, i.e. $\varpi < 1 + \delta/\lambda$. (Notice however that in our application to Cournot equilibrium, the sequence $\{\varpi_n\}$ is decreasing, therefore bounded, thus we do not need that bound which concerns increasing returns.) ∎

One may notice that $\mathrm{e}^{-\lambda t}(\lambda t)^\ell/\ell!$ is just the probability that, in a Poisson process of intensity λ, exactly ℓ positive events (here player's arrivals) occur during a time period of length t. Therefore, this can also be written, as expected

$$\Pi_n^e = \int_{t_n}^{T} \mathrm{e}^{-\delta(t-t_n)} \mathbb{E}(\varpi_{m(t)} \mid t_n) \,\mathrm{d}t = \mathbb{E}\left[\int_{t_n}^{T} \mathrm{e}^{-\delta(t-t_n)} \varpi_{m(t)} \,\mathrm{d}t \,\middle|\, t_n\right].$$

where the expectation is taken under a Poisson law of intensity λ.

4.1.2 Algorithm

We introduce the notation

$$w_{n,n+\ell} = \int_0^{T-t_n} \mathrm{e}^{-(\lambda+\delta)t} \frac{(\lambda t)^\ell}{\ell!} \,\mathrm{d}t\,. \tag{11}$$

with which we re-write formula (9):

$$\Pi_n^e = \sum_{m=n}^{\infty} w_{n,m} \varpi_m = \sum_{\ell=0}^{\infty} w_{n,n+\ell} \varpi_{n+\ell}\,.$$

Successive integrations by parts easily yield (see appendix C)

$$w_{n,n+\ell} = \frac{\lambda^\ell}{(\lambda+\delta)^{\ell+1}} \left[1 - \mathrm{e}^{-(\lambda+\delta)(T-t_n)} \sum_{k=0}^{\ell} \frac{(\lambda+\delta)^k (T-t_n)^k}{k!}\right]. \tag{12}$$

Introduce the notation

$$v_{n,\ell} = \mathrm{e}^{-(\lambda+\delta)(T-t_n)} \frac{\lambda^{\ell-1}(T-t_n)^\ell}{\ell!} .$$

We propose the following algorithm:

$$\begin{aligned}
\forall n \in \mathbb{N}\,, \quad v_{n,n} &= \frac{1}{\lambda}\mathrm{e}^{-(\lambda+\delta)(T-t_n)}\,,\\
\forall (n,\ell) \in \mathbb{N}^2\,, \quad v_{n,n+\ell} &= \frac{\lambda(T-t_n)}{\ell} v_{n,n+\ell-1}\,,\\
\forall n \in \mathbb{N}\,, \quad w_{n,n} &= \frac{1}{\lambda+\delta}(1-\lambda v_{n,n})\,,\\
\forall (n,\ell) \in \mathbb{N}^2\,, \quad w_{n,n+\ell} &= \frac{\lambda}{\lambda+\delta}(w_{n,n+\ell-1} - v_{n,n+\ell})\,.
\end{aligned}$$

Moreover, in the case of an infinite horizon: $T - t_n = \infty$, all the $v_{n,n+\ell}$ are equal to zero, and the $w_{n,n+\ell} = \lambda^\ell/(\lambda+\delta)^{\ell+1}$ are independent from n.

Two simple cases in infinite horizon Two simple cases are as follows:

1. If $\varpi_m = \varpi_0 r^m$ for some positive ϖ_0 and r. Then formula (9) integrates in closed form, giving

$$\Pi_n^e = \frac{\varpi_0 r^n}{(1-r)\lambda+\delta}[1-\mathrm{e}^{-[(1-r)\lambda+\delta](T-t_n)}]\,,$$

   which simplifies to

$$\Pi_n^e = \frac{\varpi_0 r^n}{(1-r)\lambda+\delta}$$

   in infinite horizon.

2. If the players just share equally a fixed flux ϖ_1 of resource, i.e. $\varpi_m = \varpi_1/m$, then the last remark above yields

$$\Pi_1^e = \frac{\varpi_1}{\lambda}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\lambda}{\lambda+\delta}\right)^k = \frac{\varpi_1}{\lambda}\ln\left(1+\frac{\lambda}{\delta}\right).$$

4.1.3 Numerical results

The above two cases are not Cournot payoffs. We give in Figure 4 numerical results for the case where $\varpi_m = 1/(m+1)^2$, infinite horizon, $\delta = .1$, and $\lambda \in \{0, .1, .4, .7, 1\}$.

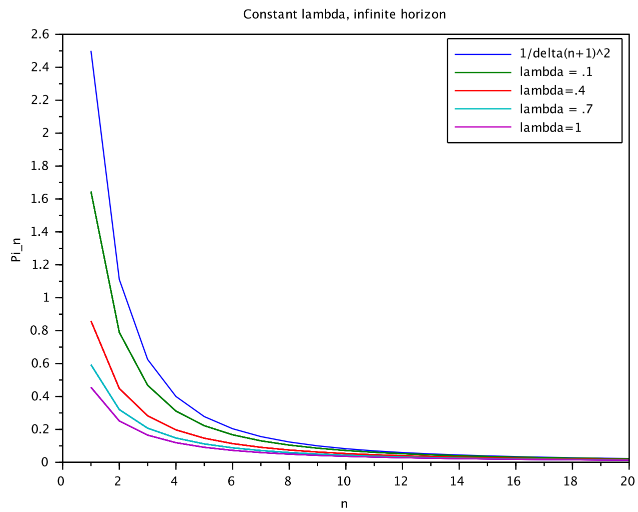

Figure 4: Curves Π_n^e against n for the infinite horizon game, with $\delta = .1$ and for various intensities λ.

4.2 Variable Poisson density of arrivals

4.2.1 General formula

We wish now to let the density of arrivals λ be a function of the rank of the next player to arrive (the number of players already present plus one). We are therefore confronted with a sequence of inter-arrival time intervals which are independent random variables each with an exponential law of coefficient, or intensity, λ_m. We can use an approach similar to that of subsection 3.2.1.

Let λ_m be the arrival density for the m-th player. Let also $q_{n,m}(t)$ be the probability that m players be present at time t, knowing that they were n at time t_n.([1]) We claim the following:

Theorem 4 *The sequence $q_{n,m}(t)$ is the unique solution of the following set of differential equations:*

$$\begin{aligned} \forall n \in \mathbb{N}\,, \quad q_{n,n}(t_n) = 1\,, \quad \dot{q}_{n,n} &= -\lambda_{n+1} q_{n,n}\,, \\ \forall m > n\,, \quad q_{n,m}(t_n) = 0\,, \quad \dot{q}_{n,m} &= \lambda_m q_{n,m-1} - \lambda_{m+1} q_{n,m}\,. \end{aligned}$$

[1]We should denote this probability it as $q_{n,m}(t_n, t)$. We omit the explicit dependence on t_n because it is fixed in the analysis of Π_n^e.

Proof The first differential equation is just another, numerically efficient, way to write $q_{n,n}(t) = \exp[-\lambda_{n+1}(t - t_n)]$, which is the probability that no event occurs during the time interval (t_n, t) for a random variable with an exponential law of density λ_{n+1}.

Consider $q_{n,m}(t)$ for $m > n$. Notice first that by hypothesis $q_{n,m}(0) = 0$. Let h be a time step destined to vanish. Let also $p_n^{(h)}$ be the probability of arrival of the n-th player during a step of length h. Hence $p_n^{(h)} = h\lambda_n + 0(h)$ (where $0(h)/h \to 0$ as $h \to 0$). Also, the probability of arrival of several players during a step of length h is of the order $0(h)$. The event of being m players present at time t is either that there were m players at time $t - h$ and none arrived, or there were $m - 1$ players at time $t - h$, and one arrived, or that there were less than $m - 1$ players at time $t - h$ and several arrived during the interval $[t - h, t]$. Hence we have

$$\begin{aligned} q_{n,m}(t) &= (1 - p_{m+1}^{(h)})q_{n,m}(t - h) + p_m^{(h)} q_{n,m-1}(t - h) + 0(h) \\ &= (1 - h\lambda_{m+1})q_{n,m}(t - h) + h\lambda_m q_{n,m-1}(t - h) + 0(h)\,. \end{aligned}$$

Hence

$$\frac{q_{n,m}(t) - q_{n,m}(t - h)}{h} = -\lambda_{m+1} q_{n,m}(t - h) + \lambda_m q_{n,m-1}(t - h) + \epsilon(h)$$

where $\epsilon(h) \to 0$ with h. It suffices to take the limit as $h \to 0$ to obtain the result of the theorem. ∎

Knowing these probabilities, we can compute

$$\mathbb{E}\varpi_{m(t)} = \sum_{m=n}^{\infty} q_{n,m}(t)\varpi_m\,,$$

and therefore

$$\Pi_n^e = \int_{t_n}^{T} \mathrm{e}^{-\delta(t-t_n)}\mathbb{E}\varpi_{m(t)}\,\mathrm{d}t = \int_{t_n}^{T} \mathrm{e}^{-\delta(t-t_n)} \sum_{m=n}^{\infty} q_{n,m}(t)\varpi_m\,. \tag{13}$$

4.2.2 Algorithm

Finite horizon We start from the formula (13) which we rewrite as

$$v_{n,m}(t) := \mathrm{e}^{-\delta t} q_{n,m}(t_n + t)\,, \tag{14}$$

$$w_{n,m} := \int_0^{T-t_n} v_{n,m}(t)\,\mathrm{d}t\,, \tag{15}$$

and

$$\Pi_n^e = \sum_{m=n}^{\infty} w_{n,m} \varpi_{n+m} \,. \tag{16}$$

We propose to compute the $v_{n,m}(t)$ via the integration of the following differential equations, directly derived from those for $q_{n,m}(t)$:

$$\forall n \in \mathbb{N}\,, \quad v_{n,n}(0) = 1\,, \quad \dot v_{n,n} = -(\lambda_{n+1} + \delta) v_{n,n}\,, \tag{17}$$

$$\forall m > n\,, \quad v_{n,m}(0) = 0\,, \quad \dot v_{n,m} = \lambda_m v_{n,m-1} - (\lambda_{m+1} + \delta) v_{n,m}\,. \tag{18}$$

Infinite horizon The computation simplifies in the case where $T = \infty$. Indeed, we can write equation (15) taking equation (18) into account, as

$$w_{n,m} = \int_0^\infty \frac{1}{\delta + \lambda_{m+1}} [\lambda_m v_{n,m-1}(t) - \dot v_{n,m}(t)] \, \mathrm{d}t \,.$$

However, it follows from its definition (14) that $v_{n,m}(t) \to 0$ as $t \to \infty$. Also, for $m > n$, $v_{n,m}(0) = 0$. Therefore the integral of $\dot v_{n,m}$ vanishes. And we are left with

$$\begin{aligned} \forall n \in \mathbb{N}\,, \qquad & w_{n,n} = \frac{1}{\lambda_{n+1} + \delta}\,, \\ \forall m > n\,, \qquad & w_{n,m} = \frac{\lambda_m}{\lambda_{m+1} + \delta} w_{n,m-1} \end{aligned} \tag{19}$$

and formula (16).

4.2.3 A finite entry problem

As in the discrete time case, we wish to investigate a problem where the density λ_m is a (decreasing) function of the expected payoff of the m-th player, becoming null when that expected payoff drops below a fixed entry cost F. We therefore need to compute that specific payoff before we can use λ_m in the algorithm. We proceed as in the discrete time case, with the approximations (10) and taking the limit as $h \to 0$. The last entrant's expected payoff is now

$$\Pi_N^e = \frac{\varpi_N}{\delta}\,,$$

and N is the integer such that

$$\varpi_{N+1} < \delta F \leq \varpi_N \,.$$

Formula (7) reads

$$\Pi_n^e = \frac{1}{1 - (1 - \lambda_{n+1} h)(1 - \delta h)} \left[\varpi_n h + \lambda_{n+1} h (1 - \delta h) \Pi_{n+1}^e \right]$$

and therefore, taking the limit as $h \to 0$:

$$\Pi^e_n = \frac{1}{\lambda_{n+1}+\delta}\left(\varpi_n + \lambda_{n+1}\Pi^e_{n+1}\right) \,, \tag{20}$$

$$\Pi^e_N = \frac{\varpi_N}{\delta}\,, \qquad \text{or equivalently } \lambda_{N+1} = 0\,. \tag{21}$$

We can also expand in

$$\Pi^e_n = \frac{1}{\lambda_{n+1}+\delta}\left\{\varpi_n + \frac{\lambda_{n+1}}{\lambda_{n+2}+\delta}\left[\varpi_{n+1} + \frac{\lambda_{n+2}}{\lambda_{n+3}+\delta}\left(\varpi_{n+2}+\lambda_{n+3}\Pi^e_{n+3}\right)\right]\right\}$$

and continue until the last term is

$$\frac{\lambda_N}{\lambda_{N+1}+\delta}(\varpi_N + \lambda_{N+1}\Pi_{N+1}) = \frac{\lambda_N}{\delta}\varpi_N\,.$$

Clearly, we have

$$\Pi^e_n = \sum_{m=n}^{N} w_{n,m}\varpi_m\,,$$

i.e. formula (16), with the same recursion (19). One may notice that

$$w_{n,m} = \frac{1}{\lambda_n}\prod_{k=n}^{m}\frac{\lambda_k}{\lambda_{k+1}+\delta}\,,$$

but the recursion (20)(21) is more useful. It can be applied backward, with a law $\lambda_n = \Lambda_n(\Pi^e_n)$. (Although adjusting the laws Λ_n is made difficult by the fact that we do not know Π^e_1 beforehand. If the ϖ_n are decreasing with n, we only know that $\Pi^e_1 < \varpi_1/\delta$.)

4.2.4 Numerical results

Figure 5 shows the plot of the expected payoff as a function of the rank of entry for the infinite horizon game, computed with the backward algorithm. We chose $\delta = .1$, $\varpi_n = 100/(n+1)^2$, $F = 2$, and $\lambda_n = 1 - F/\Pi^e_n$.

5 Conclusion

The model we investigated is slightly more general than a sequence of Cournot oligopolies, in that we allow for an arbitrary stepwise profit π_m, and not necessarily one of the the Cournot profits listed in A. But the real innovation we claim is in the random arrival of producers, be it in discrete

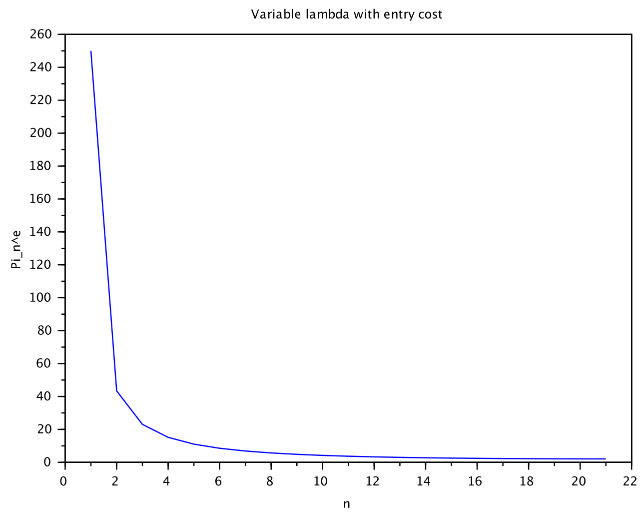

Figure 5: Expected payoff for the infinite horizon game as a function of the rank of entry for $\lambda_n = 1 - \frac{F}{\Pi_n^e}$, with $\delta = .1$.

or continuous time, and whether the game horizon is finite or infinite. Our results are in the same spirit as in the classical Cournot oligopoly but we are able to answer precisely *in this new set up* four questions: what is the expected profit for a firm as a function of its rank of arrival on the market? How do individual quantities evolve? How do market quantities evolve? How does market price evolve?

There are at least five limitations to our analysis. First, we do not have a dynamic equilibrium since there is no intertemporal link between each step (such as capacity constraints, stickiness, etc.). We only have a sequence of static equilibria. Second, there is no exit of players, yet this would be more realistic. In an other paper ([Bernhard and Deschamps, 2016]) we explicitly consider these issues and propose a way to overcome these two limitations.

The third limitation of this paper is that we only consider a Bernoulli or a Poisson process of entry. They are the most simple probability laws concerning random events, and as such the most widely used. But in some contexts they could unfortunately be inappropriate. In each such case, a new analysis needs to be done. Fourth, in our setting market demand is always deterministic (such as linear or isoelastic). Last but not least, we have only considered symmetric producers (incumbents and potential entrants), an hypothesis which leads to symmetric profits for producers who are in the game since the same step. We leave for further developments the case where producers belong to several classes of players and the market demand is random.

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A Classical n-fixed results

We wish to apply our model to Cournot oligopoly, under the combination of two hypotheses, one relative to the demand function —linear or isoelastic—, the other to the production cost of the players —linear or quadratic. We always consider identical players, enjoying complete information about the rules of the game and about the current number of players present in the game at each instant of time.

Let $P(Q)$ be the inverse demand function and $C(q)$ the individual production cost. We have

$$\pi = qP(Q) - C(q)\,,$$

that each player seeks to maximize, assuming other players' production fixed. The method is as follows: write $Q = q + (n-1)q_n$, hence

$$\pi = qP\big(q + (n-1)q_n\big) - C(q)\,,$$

equate the partial derivative with respect to q to 0, and in that equation place $q = q_n^\star$. The results are summarized in Table 1. In table 2, we

	$C = cq$	$C = cq^2$
$P = a - bQ$	$P_n^\star = \frac{a+nc}{n+1}$ $q_n^\star = \frac{a-c}{(n+1)b}$ $\quad Q_n^\star = \frac{n(a-c)}{(n+1)b}$ $\pi_n^\star = \frac{1}{b}\left[\frac{a-c}{n+1}\right]^2$ $\quad \Pi_n^\star = \frac{n}{b}\left[\frac{a-c}{n+1}\right]^2$	$P_n^\star = \frac{a(b+2c)}{(n+1)b+2c}$ $q_n^\star = \frac{a}{(n+1)b+c}$ $\quad Q_n^\star = \frac{na}{(n+1)b+c}$ $\pi_n^\star = \frac{a^2(b+c)}{[(n+1)b+2c]^2}$ $\quad \Pi_n^\star = \frac{na^2(b+c)}{[(n+1)b+2c]^2}$
$P = aQ^{-\frac{1}{\varepsilon}}$	$P_n^\star = \frac{nc}{n-\frac{1}{\varepsilon}}$ $q_n^\star = \left[\frac{a\left(\varepsilon-\frac{1}{n}\right)}{n^{\frac{1}{\varepsilon}}c\varepsilon}\right]^\varepsilon$ $\quad Q_n^\star = \left[\frac{a\left(\varepsilon-\frac{1}{n}\right)}{c\varepsilon}\right]^\varepsilon$ $\pi_n^\star = \frac{cQ_n^\star}{n(n\varepsilon-1)}$ $\quad \Pi_n^\star = \frac{cQ_n^\star}{n\varepsilon-1}$	$P_n^\star = \left(\frac{2a^\varepsilon c}{n-\frac{1}{\varepsilon}}\right)^{\frac{1}{1+\varepsilon}}$ $q_n^\star = \left[\frac{a\left(\varepsilon-\frac{1}{n}\right)}{2n^{\frac{1}{\varepsilon}}c\varepsilon}\right]^{\frac{\varepsilon}{1+\varepsilon}}$ $\quad Q_n^\star = \left[\frac{a\left(n-\frac{1}{\varepsilon}\right)}{2c}\right]^{\frac{\varepsilon}{1+\varepsilon}}$ $\pi_n^\star = \frac{c\left(n+\frac{1}{\varepsilon}\right)}{n^2\left(n-\frac{1}{\varepsilon}\right)}Q_n^{\star 2}$ $\quad \Pi_n^\star = \frac{c\left(n+\frac{1}{\varepsilon}\right)}{n\left(n-\frac{1}{\varepsilon}\right)}Q_n^{\star 2}$

Table 1: Equilibrium values in Cournot oligopoly with n agents

give the asymptotic equivalent as $n \to \infty$. In the case of quadratic costs, the price goes to zero as $n \to \infty$. In the case of linear demand function, total production goes to a finite limit, while individual profits go to zero as n^{-2}, and therefore also total profit as n^{-1}. But the total production and profit behave differently depending on whether the production cost is linear or quadratic: in the former case, gross production goes to a finite limit and total profits go to zero, while in the latter, gross production and total profits go to infinity.

Let us also emphasize that the same theory developed to evaluate the expectation of the cumulative profit of the players applies mutatis mutandis to evaluate the expectation of any other cumulative quantity pertaining to this game, e.g. in the application to the Cournot oligopoly, the finite time cumulative gross production

$$G = \sum_{t=t_1}^{T} Q_n \,.$$

	$C = cq$	$C = cq^2$
$P = a - bQ$	$P_n^\star \sim c + \frac{a-c}{n} \to c$ $q_n^\star \sim \frac{a-c}{b} n^{-1}$ $\quad Q_n^\star \to \frac{a-c}{b}$ $\pi_n^\star \sim (a-c)^2 n^{-2}$ $\quad \Pi_n^\star \sim (a-c)^2 n^{-1}$	$P_n^\star \sim \frac{a(b+2c)}{b} n^{-1} \to 0$ $q_n^\star \sim \frac{a}{b} n^{-1}$ $\quad Q_n^\star \to \frac{a}{b}$ $\pi_n^\star \sim \frac{a^2(b+c)}{b^2} n^{-2}$ $\quad \Pi_n^\star \sim \frac{a^2(b+c)}{b^2} n^{-1}$
$P = aQ^{-\frac{1}{\varepsilon}}$	$P_n^\star \sim c\left(1 + \frac{1}{n\varepsilon}\right) \to c$ $q_n^\star \sim \left(\frac{a}{c}\right)^\varepsilon n^{-1}$ $\quad Q_n^\star \to \left(\frac{a}{c}\right)^\varepsilon$ $\pi_n^\star \sim \frac{a^\varepsilon}{\varepsilon c^{\varepsilon-1}} n^{-2}$ $\quad \Pi_n^\star \sim \frac{a^\varepsilon}{\varepsilon c^{\varepsilon-1}} n^{-1}$	$P_n^\star \sim (2a^\varepsilon c)^{\frac{1}{1+\varepsilon}} n^{-\frac{1}{1+\varepsilon}} \to 0$ $q_n^\star \sim \left(\frac{a}{2c}\right)^{\frac{\varepsilon}{1+\varepsilon}} n^{-\frac{1}{1+\varepsilon}}$ $\quad Q_n^\star \sim \left(\frac{an}{2c}\right)^{\frac{\varepsilon}{1+\varepsilon}}$ $\pi_n^\star \sim \alpha n^{-\frac{2}{1+\varepsilon}}$ $\quad \Pi_n^\star \sim \alpha n^{\frac{\varepsilon-1}{\varepsilon+1}}$ $\alpha = \left(\frac{a}{2}\right)^{\frac{2\varepsilon}{1+\varepsilon}} c^{\frac{1-\varepsilon}{1+\varepsilon}}$

Table 2: Asymptotics as $n \to \infty$

B Derivation of the backward algorithm formula

We start from equation (6)

$$\Pi_n^e = \sum_{t_{n+1}=t_n+1}^{\infty} (1 - p_{n+1})^{t_{n+1}-t_n-1} p_{n+1} \left[\sum_{t=t_n}^{t_{n+1}} r^{t-t_n} \pi_n + r^{t_{n+1}-t_n} \Pi_{n+1}^e(t_{n+1}) \right].$$

The inner sum over t can be expressed in closed form, to get

$$\Pi_n^e = \sum_{t_{n+1}=t_n+1}^{\infty} (1 - p_{n+1})^{t_{n+1}-t_n-1} p_{n+1} \left[\frac{1 - r^{t_{n+1}-t_n}}{1-r} \pi_n + r^{t_{n+1}-t_n} \Pi_{n+1}^e(t_{n+1}) \right].$$

substitute t for the dummy summation index t_{n+1}, and expand in

$$\Pi_n^e = \frac{p_{n+1}}{1-p_{n+1}} \left[\left(\sum_{t=t_n+1}^{\infty} (1-p_{n+1})^{t-t_n} - \sum_{t=t_n+1}^{\infty} [(1-p_{n+1})r]^{t-t_n} \right) \frac{\pi_n}{1-r} + \sum_{t=t_n+1}^{\infty} [(1-p_{n+1})r]^{t-t_n} \Pi_{n+1}^e \right].$$

Use again the closed form of the sums of powers:

$$\Pi_n^e = \frac{p_{n+1}}{1-p_{n+1}} \left[\left(\frac{1-p_{n+1}}{p_{n+1}} - \frac{(1-p_{n+1})r}{1-(1-p_{n+1})r} \right) \frac{\pi_n}{1-r} + \frac{(1-p_{n+1})r}{1-(1-p_{n+1})r} \Pi_{n+1}^e \right].$$

or

$$\Pi_n^e = \left(1 - \frac{p_{n+1}r}{1-(1-p_{n+1})r}\right)\frac{\pi_n}{1-r} + \frac{p_{n+1}r}{1-(1-p_{n+1})r}\Pi_{n+1}^e.$$

or, finally, formula (7):

$$\Pi_n^e = \frac{1}{1-(1-p_{n+1})r}\left(\pi_n + p_{n+1}r\Pi_{n+1}^e\right).$$

C Evaluating the $w_{n,n+\ell}$

We aim to derive formula (12) from formula (11). To simplify the calculation, we let $\lambda+\delta=\gamma$ and $T-t_n=S$. We aim to evaluate

$$w_{n,n+\ell} = \int_0^S \mathrm{e}^{-\gamma t}\frac{(\lambda t)^\ell}{\ell!}\,\mathrm{d}t\,.$$

Performing an integration by parts, we obtain

$$\begin{aligned} w_{n,n+\ell} = \int_0^S \mathrm{e}^{-\gamma t}\frac{(\lambda t)^\ell}{\ell!}\,\mathrm{d}t &= \lambda^\ell\int_0^S \frac{t^\ell}{\ell!}\mathrm{d}\left(-\frac{1}{\gamma}\mathrm{e}^{-\gamma t}\right) \\ &= \frac{\lambda^\ell}{\gamma}\left(\left[-\mathrm{e}^{-\gamma t}\frac{t^\ell}{\ell!}\right]_0^S + \int_0^S \mathrm{e}^{-\gamma t}\frac{t^{\ell-1}}{(\ell-1)!}\,\mathrm{d}t\right) \\ &= \frac{\lambda^\ell}{\gamma}\left(-\mathrm{e}^{-\gamma t}\frac{S^\ell}{\ell!} + \int_0^S \mathrm{e}^{-\gamma t}\frac{t^{\ell-1}}{(\ell-1)!}\,\mathrm{d}t\right) \end{aligned}$$

Apply the same integration by parts to the last integral, and repeat once:

$$\begin{aligned} w_{n,n+\ell} = \frac{\lambda^\ell}{-\gamma}\Bigg\{&\mathrm{e}^{-\gamma S}\frac{-S^\ell}{\ell!} + \frac{1}{\gamma}\Bigg[-\mathrm{e}^{-\gamma S}\frac{S^{\ell-1}}{(\ell-1)!} \\ &+ \frac{1}{\gamma}\left(-\mathrm{e}^{-\gamma S}\frac{S^{\ell-2}}{(\ell-2)!} + \int_0^S \mathrm{e}^{-\gamma t}\frac{t^{\ell-3}}{(\ell-3)!}\mathrm{d}t\right)\Bigg]\Bigg\} \\ = \frac{\lambda^\ell}{\gamma^4}&\left[-\mathrm{e}^{-\gamma S}\left(\frac{\gamma^3 S^\ell}{\ell!} + \frac{\gamma^2 S^{\ell-1}}{(\ell-1)!} + \frac{\gamma S^{\ell-2}}{(\ell-2)!}\right) + \gamma\int_0^S \mathrm{e}^{-\gamma t}\frac{t^{\ell-3}}{(\ell-3)!}\,\mathrm{d}t\right]. \end{aligned}$$

Performing the same substitution ℓ times, we end up with

$$\begin{aligned} w_{n,n+\ell} &= \frac{\lambda^\ell}{\gamma^{\ell+1}}\left[-\mathrm{e}^{-\gamma S}\left(\frac{\gamma^\ell S^\ell}{\ell!} + \frac{\gamma^{\ell-1}S^{\ell-1}}{(\ell-1)!} + \ldots + \gamma S\right) + \gamma\int_0^S \mathrm{e}^{-\gamma t}\mathrm{d}t\right] \\ &= \frac{\lambda^\ell}{\gamma^{\ell+1}}\left[1 - \mathrm{e}^{-\gamma S}\left(\frac{\gamma^\ell S^\ell}{\ell!} + \frac{\gamma^{\ell-1}S^{\ell-1}}{(\ell-1)!} + \ldots + \gamma S + 1\right)\right], \end{aligned}$$

which is formula (12).