Dynamic equilibrium in games with randomly arriving players

Pierre Bernhard[*] Marc Deschamps[†]

October 10, 2016

Abstract

This note follows our previous works on games with randomly arriving players [3] and [5]. Contrary to these two articles, here we seek a dynamic equilibrium, using the tools of piecewise deterministic control systems The resulting discrete Isaacs equation obtained is rather involved. As usual, it yields an explicit algorithm in the finite horizon, linear-quadratic case via a kind of discrete Riccati equation. The infinite horizon problem is briefly considered. It seems to be manageable only if one limits the number of players present in the game. In that case, the linear quadratic problem seems solvable via essentially the same algorithm, although we have no convergence proof, but only very convincing numerical evidence.

We extend the solution to more general entry processes, and more importantly, to cases where the players may leave the game, investigating several stochastic exit mechanisms.

We then consider the continuous time case, with a Poisson arrival process. While the general Isaacs equation is as involved as in the discrete time case, the linear quadratic case is simpler, and, provided again that we bound the maximum number of players allowed in the game, it yields an explicit algorithm with a convergence proof to the solution of the infinite horizon case, subject to a condition reminiscent of that found in [20].

As in the discrete time case, we examine the case where players may leave the game, investigating several possible stochastic exit mechanisms.

MSC: 91A25, 91A06, 91A20, 91A23, 91A50, 91A60, 49N10, 93E03.

Foreword *This report is a version of the article [2] where players minimize instead of maximizing, and the linear-quadratic examples are somewhat different.*

[*]BIOCORE team, Université de la Côte d'Azur-INRIA, France.

[†]CRESE, Université de Bourgogne-Franche Comté, CNRS-BETA, CNRS-GREDEG, and OFCE Sciences Po.

1 Introduction

1.1 The problem investigated

In [14], we showed how the diet selection behaviour of a forager on a patch of resources of diverse qualities is affected byt the potential arrival, at a random time, of a competing conspecific. Only one possible arrival was considered, leaving open the question of what happens if the number of possible later arrivals is larger, and even not a priori bounded as in a Poisson process. In [5], we consider such a Poisson arrival process, but where the behaviour of the foragers is given, specified by their functional response. We gave examples with an exhaustable/renewable resource and several types of functional responses.

In the current report, we investigate the case where the behavour of the randomly arriving players is not given a priori. In contrast, we let them adjust their strategies. We know their rate of payoff accumulation as a function of their number and their strategies at each instant of time, and seek their equilibrium strategies in the dynamic game thus defined.

We formulate this problem as a rather general dynamic game with random entry of identical players, in finite or infinite horizon. We also consider different possible schemes of random exit from the game. And we pay a particular attention to the linear quadratic case, which, unsuprisingly, leads to a more explicit solution via a set of Riccati-like equations. We investigate all these questions in both a discrete time and a continuous time versions, using respectively dynamic programming and Piecewise Deterministic Markov Decision Processes (PDMDP) (see [26, 24, 28, 15].

This piece of theory may have applications elsewhere than in foraging theory; most noticeably in economic science where, *e.g.*, competitors on a given market may show-up at unpredictible times. This is developped in the "economics version" of this paper [4], which is essentially the same article as this report, except that

- the notation are somewhat different,
- the introduction and conclusion of the economics version stress applications in the field of economic sciences,
- here the players minimize a cost (to be in the spirit of the mathematical engineering literature and offer the other, minimizing, version of the game) while in the economics version the players maximize,
- the Linear Quadratic (L.Q.) example has both terms in the cost of the same sign, in the tradition of the mathematical engineering literature, while in the

economics version, the underlying model induced us into choosing terms of different signs, leading to a somewhat different theory,

- in the economics version, the L.Q. case is followed by a further example in Cournot oligopoly theory.

Yet, the two versions have so much in common that this technical report is not offered for publication elsewhere than as an INRIA Technical Report.

1.2 Previous literature

We provided in [3] a rather complete review of the literature on games with a random number of players, and of some related topics. The general diagnostic is that where the number of players is random, there is no time involved, and therefore no notion of entry. Typical example are auction theory, see[19] or Poisson games, see [21, 7] . And where there is a time structure in the game, there is a fixed number of players, such as in stochastic games, see [22], or in generalized secretary problems, see [11]. And in the literature on entry equilibrium, such as [25, 6], the players are the would-be entrants, the number of which is known.

One notable exception is the article [16] which explicitly deals with a dynamic game with random entry. In that article, the authors describe a problem more complicated than ours on three counts at least:

1. There are two types of players: a major one, an incumbent, who has an infinite horizon, and identical minor ones that enter at random and leave after a fixed time T (although the authors mention that they can also deal with the case where T is random).

2. Each player has its own state and dynamics. Yet, the criteria of the players only depend on a mean value of these states, simplifying the analysis, and opening the way for a simplified analysis in terms of mean field in the large number of minor players case.

3. All the dynamics are noisy.

It is simpler than ours in that it does not attempt to foray away from the discrete time, linear dynamics, quadratic payoff case. Admittedly, our results in the nonlinear case are rather theoretical and remain difficult to use beyond the L.Q. case. But we do deal with the continuous time case also.

Due to the added complexity, the solution proposed is much less explicit than what we offer in the linear quadratic problem. Typically, the authors solve the two maximization problems with opponents' strategies fixed and state that if the set of strategies is "consistent", i.e. solve the fixed point problem inherent in a Nash equilibrium, then it is the required equilibrium. The algorithm proposed to solve the fixed point problem is the natural Picard iteration. A convergence proof is only available in a very restrictive case.

2 Discrete time

2.1 The problem

2.1.1 Players, dynamics and cost

Time t is an integer. An horizon $T \in \mathbb{N}$ is given, and we will write $\{1, 2, \dots, T\} = \mathbb{T}$, thus $t \in \mathbb{T}$. A state space X is given. A dynamic system in X may be controlled by an arbitrary number of agents. The number m of agents varies with time. We let $m(t)$ be that number at time t. The agents arrive as a Bernoulli process with variable probability; i.e. at each time step there may arrive only one player, and this happens with a probability p^m when m players are present, independently of previous arrivals. We call t_n the arrival time of the n-th player, $u_n \in \mathsf{U}$ its decision (or control).

We distinguish the *finite case* where X and U are finite sets, from the *infinite case* where they are infinite. In that case, they are supposed to be topologic spaces, U compact.

Note concerning the notation *We use lower indices to denote players, and upper indices to denote quantities pertaining to that number of agents in the game. An exception is* U^m *which is the cartesian power set* $\mathsf{U} \times \mathsf{U} \times \cdots \times \mathsf{U}$ m *times.* We use the notation:

$$u^m = (u_1^m, u_2^m, \dots, u_m^m) \in \mathsf{U}^m\,, \quad v^{\times m} = (\overbrace{v, v, \dots, v}^{m\,\text{times}})\,,$$
$$u^{m\setminus n} = (u_1, \dots, u_{n-1}, u_{n+1}, \dots, u_m)\,,$$
$$\{u^{m\setminus n}, u\} = \{u, u^{m\setminus n}\} = (u_1, \dots, u_{n-1}, u, u_{n+1}, \dots, u_m)$$

The dynamics are ruled by the state equation in X:

$$x(t+1) = f^{m(t)}(t, x(t), u^{m(t)}(t))\,, \quad x(0) = x_0\,. \tag{1}$$

A double family of stepwise costs, for $n \le m \in \mathbb{T}$ is given: $L_n^m : \mathbb{T} \times \mathsf{X} \times \mathsf{U}^m \to \mathbb{R} : (t, x, u^m) \mapsto L_n^m(t, x, u^m)$, as well as a discount factor $r \le 1$. The overall cost of player n, which it seeks to minimize, is

$$J_n(t_n, x(t_n), \{u^m\}_{m \ge n}) = \mathbb{E} \sum_{t=t_n}^{T} r^{t-t_n} L_n^{m(t)}(t, x(t), u^{m(t)}(t))\,. \tag{2}$$

	Player				
$m(t)$	1	2	$\cdots$	t	
1	$u_1^1(t)$				$u^1(t)$
2	$u_1^2(t)$	$u_2^2(t)$			$u^2(t)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$		$\vdots$
t	$u_1^t(t)$	$u_2^t(t)$	$\dots$	$u_t^t(t)$	$u^t(t)$
	$U_1(t)$	$U_2(t)$	$\dots$	$U_t(t)$	

Table 1: Representation of $U^t(t)$, the section at time t of an open-loop profile of strategies $U(\cdot)$. In the rightmost column: the names of the lines, in the last line: the names of the columns.

Moreover, all players are assumed to be identical. Specifically, we assume that

1. The functions f^m are invariant by a permutation of the u_n,
2. the functions L_n^m enjoy the properties of a game with identical players as described in Appendix A. That is: a permutation of the u_n produces an identical permutation of the L_n^m.

Finally, in the infinite case, the functions f^m and L^m are all assumed continuous.

2.1.2 Pure strategies and equilibria

We have assumed that the current number of players in the game at each step is common knowledge. We therefore need to introduce $m(t)$-dependent controls: denote by $U_n \in \mathsf{U}_n = \mathsf{U}^{T-n+1}$ a complete n-th player's decision, i.e. an application $\{n, \dots, T\} \to \mathsf{U} : m \mapsto u_n^m$. We recall the notation for a strategy profile: $u^m = (u_1^m, u_2^m, \dots, u_m^m) \in \mathsf{U}^m$. We also denote by U^m a decision profile: $U^m = (U_1, U_2, \dots, U_m)$. It can also be seen as a family $U^m = (u^1, u^2, \dots, u^m)$. The set of elementary controls in U^t is best represented by Table 1 where $u_n^m(t)$ is the control used by player n at time t if there are m players in the game at that time. A partial strategy profile $(U_1, \dots, U_{n-1}, U_{n+1}, \dots, U_m)$ where U_n is missing, will be denoted $U^{m\setminus n}$. An open-loop profile of strategies is characterized by a sequence $U(\cdot) : \mathbb{T} \ni t \mapsto U^t(t)$. A partial open-loop strategy profile where $U_n(\cdot)$ is missing will be denoted $U^{\setminus n}(\cdot)$.

The cost $J_n(t_n, x(t_n), U(\cdot))$ is a mathematical expectation conditioned on the pair $(t_n, x(t_n))$, which is a random variable independent from $U_n(\cdot)$.

Definition 2.1 *An open loop dynamic pure Nash equilibrium is a family history* $\widehat{U}(\cdot)$ *such that*

$$\forall n \in \mathbb{T}\,, \forall (t_n, x(t_n)) \in \mathbb{T} \times \mathsf{X}\,, \forall U_n(\cdot) \in \mathsf{U}_n\,,\ J_n(\{\widehat{U}^{\setminus n}(\cdot), U_n(\cdot)\}) \geq J_n(\widehat{U}(\cdot))\,. \tag{3}$$

Definition 2.2 *A Nash equilibrium will be called* uniform *if at all times, all players present in the game use the same decision, i.e., with our notations, if, for all* t*, for all* m*,* $\hat{u}^m(t) = \hat{u}(t)^{\times m}$ *for some sequence* $\hat{u}(\cdot)$*.*

Remark 2.1 *A game with identical players may have non uniform pure equilibria, and even have pure equilibria but none uniform. However, if it has a unique equilibrium, it is a uniform equilibrium (see appendix A).*

However, we will be interested in *closed loop* strategies, and more specifically *state feedback* strategies; i.e. we assume that each player is allowed to base its control at each time step t on the current time, the current state $x(t)$ and the current number $m(t)$ of players in the game. We therefore allow families of state feedbacks indexed by the number m of players:

$$\varphi^m = (\varphi_1^m, \varphi_2^m, \dots, \varphi_m^m) : \mathbb{T} \times \mathsf{X} \to \mathsf{U}^m$$

and typically let

$$u_n^m(t) = \varphi_n^m(t, x(t))\,.$$

We denote by $\Phi_n \in \mathcal{F}_n$ a whole family $(\varphi_n^m(\cdot,\cdot)\,, m \in \{n, \dots, T\})$ (the complete strategy choice of a player n), Φ a complete strategy profile, $\Phi^{\setminus n}$ a partial strategy profile specifying their strategy Φ_ℓ for all players except player n. A closed loop strategy profile Φ generates through the dynamics and the entry process a random open-loop strategy profile $U(\cdot) = \Gamma(\Phi)$. With a transparent abuse of notation, we write $J_n(\Phi)$ for $J_n(\Gamma(\Phi))$.

Definition 2.3 *A closed loop dynamic pure Nash equilibrium is a profile* $\widehat{\Phi}$ *such that*

$$\forall n \in \mathbb{T}\,, \forall (t_n, x(t_n)) \in \mathbb{T} \times \mathsf{X}\,, \forall \Phi_n \in \mathcal{F}_n\,, \quad J_n(\{\widehat{\Phi}^{\setminus n}, \Phi_n\}) \geq J_n(\widehat{\Phi})\,. \tag{4}$$

It will be called uniform *if it holds that* $\hat{\varphi}^m = \hat{\varphi}^{\times m}$.

We further notice that using state feedback strategies (and dynamic programming) will naturally yield time consistent and subgame perfect strategies.

2.1.3 Mixed strategies and disturbances

For the sake of simplicity, we will emphasize pure strategies hereafter. But of course, a pure Nash equilibrium may not exist. In the discrete time case investigated here, we can derive existence results if we allow mixed strategies.

Let $\mathcal{U}$ be the set of probability distributions over U. Replacing U by $\mathcal{U}$ in the definitions of open-loop and closed-loop strategies above yields equivalent open-loop and closed-loop behavioral mixed strategies. By behavioral, we mean that we use sequences of random choices of controls and not random choices of sequences of controls. See [1] for a more detailed analysis of the relationship between various concepts of mixed strategies for dynamic games.

In case the strategies are interpreted as mixed strategies, $u^{m(t)}(t)$ in equations (1) and (2) are random variables, and the pair $(m(\cdot), x(\cdot))$ is a (controlled) markov chain. But since anyhow, $m(\cdot)$ is already a markov chain even with pure strategies, the rest of the analysis is unchanged.

We might go one step further and introduce disturbances in the dynamics and the cost. Let $\{w(\cdot)\}$ be a sequence of independent random variables in $\mathbb{R}^\ell$, and add the argument $w(t)$ in both f^m and L_n^m. All results hereafter in the discrete time problem remain unchanged (except for formula (9) where one term must be added). We keep with the undisturbed case for the sake of simplicity of notation, and because in the continuous time case, to be seen later, it spares us the Ito terms in the equations.

2.2 Isaacs' equation

2.2.1 Finite horizon

We use dynamic programming, and therefore Isaacs' equation in terms of a family of Value functions $V_n^m : \mathbb{T} \times \mathsf{X} \to \mathbb{R}$. It will be convenient to associate to any such family the family W_n^m defined as

$$W_n^m(t, x) = (1 - p^m)V_n^m(t, x) + p^m V_n^{m+1}(t, x)\,, \tag{5}$$

and the Hamiltonian functions

$$H_n^m(t, x, u^m) := L_n^m(t, x, u^m) + rW_n^m(t + 1, f_n^m(t, x, u^m))\,. \tag{6}$$

We write Isaacs' equation for the general case of a non uniform equilibrium, but the uniform case will be of particular interest to us.

Theorem 2.1 *: An subgame perfect equilibrium $\widehat{\Phi} = \{\hat{\varphi}_n^m\}$ exists, if and only if there is a family of functions V_n^m satisfying the following Isaacs equation, which* *makes use of the notation* (5), (6)*:*

$$\forall n \le m \in \mathbb{T}\,,\ \forall (t,x) \in \{0,\dots,T\} \times \mathsf{X}\,,\ \forall u \in \mathsf{U}\,,$$

$$V_n^m(t,x) = H_n^m(t,x,\hat{\varphi}^{\times m}(t,x)) \le H_n^m(t,x,\{\hat{\varphi}^{m\backslash n}(t,x),u\})\,,$$

$$\forall m \in \mathbb{T}\,,\ \forall x \in \mathsf{X}\,,\ V_n^m(T+1,x) = 0\,.$$

And then, the equilibrium cost of player n *joining the game at time* t_n *at state* x_n *is* $V_n^n(t_n,x_n)$*. If the equilibrium is uniform, i.e. for all* $n \le m$*,* $\hat{\varphi}_n^m = \hat{\varphi}_1^m$*, then* $V_n^m = V_1^m$ *for all* m,n *(and we may call it* V^m*).*

Proof This is a classical dynamic programming argument. We notice first that the above system can be written in terms of conditional expectations given (m,x) as

$$\begin{aligned}
&\forall n \le m \in \mathbb{T}\,,\ \forall (t,x) \in \{0,\dots,T\} \times \mathsf{X}\,,\ \forall u \in \mathsf{U}\,,\\
&V_n^m(t,x) \;=\; \mathbb{E}^{m,x}\Big[L_n^m(t,x,\hat{\varphi}^m(t,x))\\
&\qquad\qquad + rV_n^{m(t+1)}\big(t+1,f^m(t,x,\hat{\varphi}^m(t,x))\big)\Big]\\
&\qquad\quad \le\; \mathbb{E}^{m,x}\Big[L_n^m(t,x,\{\hat{\varphi}^{m\backslash n}(t,x),u\})\\
&\qquad\qquad + rV_n^{m(t+1)}\big(t+1,f^m(t,x,\{\hat{\varphi}^{m\backslash n}(t,x),u\})\big)\Big]\\
&\forall m \in \mathbb{T}\,,\ \forall x \in \mathsf{X}\,,\ V_n^m(T+1,x) = 0\,.
\end{aligned}$$

Assume first that all players use the strategy $\hat{\varphi}$. Fix an initial time t_n (which may or may not be the arrival time of the n-th player) an state x_n and an initial m. Assume all players use their control $\hat{\varphi}_n(t,x(t))$, and consider the random process $(m(t),x(t))$ thus generated. For brevity, write $\hat{u}^m(t) := \hat{\varphi}^m(t,x(t))$. Write the equality in theorem 2.1 at all steps of the stochastic process $(m(t),x(t),\hat{u}^{m(t)}(t))$:

$$V^m(t,x(t)) = \mathbb{E}^{m(t),x(t)}\left[L_n^{m(t)}\big(t,x(t),\hat{u}^{m(t)}(t)\big) + rV_n^{m(t+1)}\big(t+1,x(t+1)\big)\right].$$

Multiply by r^{t-t_n}, take the a priori expectation of both sides and use the theorem of embedded conditional expectations, to obtain

$$\begin{aligned}
\mathbb{E}\Big[&-r^{t-t_n}V_n^{m(t)}(t,x(t)) + r^{t-t_n}L_n^{m(t)}\big(t,x(t),\hat{u}^{m(t)}(t)\big)\\
&+ r^{t+1-t_n}V_n^{m(t+1)}(t+1,x(t+1))\Big] = 0\,.
\end{aligned}$$

Sum these equalities from t_n to T and use $V_n^m(T+1,x) = 0$ to obtain

$$-V_n^m(t_n,x_n) + \mathbb{E}\left[\sum_{t=t_n}^{T} r^{t-t_n}L_n^{m(t)}(t,x(t),\hat{u}^{m(t)})\right] = 0\,,$$

hence the claim that the cost of all players from (t_n, x_n, m) is just $V_n^m(t_n, x_n)$, and in particular the cost of player n as in the theorem.

Assume now that player n deviates from $\hat\varphi_n$ according to any sequence $u_n(\cdot)$. Exactly the same reasoning, but using the inequality in the theorem, will lead to $V_n(t_n, x_n) \le J_n$. We have therefore shown that the conditions of the theorem are sufficient for the existence of a subgame perfect equilibrium.

Finally, assume that the subgame perfect equilibrium exists. Let $V_n^m(t,x)$ be defined as the cost to player n in the subgame starting with m players at (t,x). The equality in the theorem directly derives from the linearity (here, additivity) of the mathematical expectation. And if at one (m,t,x) the inequality were violated, for the subgame starting from that situation, a control $u_n(t) = u$ would yield a lower expectation for player n, which is in contradiction with the fact that $\widehat{\Phi}$ generates an equilibrium for all subgames.

Concerning a uniform equilibrium, observe first that (for all equilibria), for all m, n, for all $x \in \mathsf{X}$, $V_n^m(T+1, x) = 0$. Assume that $V_n^m(t+1,x) = V_1^m(t+1,x)$. Observe that then, in the right hand side of Isaacs' equation, only L_n^m depends on n. let π be a permutation that exchanges n and 1. By hypothesis, $L_n^m(t,x,\hat\varphi^{\pi[m]}(t,x)) = L_1^m(t,x,\hat\varphi^m)$. But for a uniform equilibrium, it also holds that $\hat\varphi^{\pi[m]}(t,x) = \hat\varphi^m(t,x)$. Hence $V_n^m(t,x) = V_1^m(t,x)$. ∎

Isaacs' equation in the theorem involves a sequence of Nash equilibria of the Hamiltonian. In general, stringent conditions are necessary to ensure existence of a pure equilibrium. However, our hypotheses ensure existence of a mixed equilibrium (see, e.g. [8] and [1]). And since the equation is constructive via backward induction, we infer

Corollary 2.1 *A dynamic subgame perfect Nash equilibrium in behavioural strategies exists in the finite horizon discrete time game..*

A natural approach to using the theorem is via Dynamic Programming (backward induction). Assume that we have discretized the set of reachable states in N_t points at each time t. (Or $x \in \mathsf{X}$, a finite set) The theorem brings the determination of a subgame perfect equilibrium set of strategies to the computation of $\sum_t t \times N_t$ Nash equilibria (one for each value of m at each (t,x)). A daunting task in general. However, the search for a *uniform* equilibrium may be much simpler. On the one hand, there is now a one-parameter family of functions $V^m(t,x)$, and, in the infinite case, if all functions are differentiable (concerning W_n^m this is *not* guaranteed by regularity hypotheses on f^m and L_n^m) and if the equilibrium is interior, the search for each static Nash equilibrium is brought back to solving an equation of the form (34):

$$\partial_{u_1} L_1^m(t,x,u^{\times m}) + r\partial_x W^m(t+1, f^m(t,x,u^{\times m}))\,\partial_{u_1} f^m(t,x,u^{\times m}) = 0\,.$$

We will see that in the linear quadratic case that we will consider, this can be done.

2.2.2 Infinite horizon

We consider the same problem as above, with both f^m and L_n^m independent from time t. We assume that the L_n^m are uniformly bounded by some number L, and we let the cost of the n-th player in a (sub)game starting with n players at time t_n and state $x(t_n) = x_n$ be

$$J_n(t_n, x_n; U(\cdot)) = \mathbb{E} \sum_{t=t_n}^{\infty} r^{t-t_n} L_n^{m(t)}(x(t); u^{m(t)}(t)) \,. \tag{7}$$

We look for a subgame perfect equilibrium set of strategies $\hat{\varphi}_n^m(x)$. Isaacs equation becomes an implicit equation for a bounded infinite family of functions $V_n^m(x)$. Using the time invariant form of equations (5) and (6), we get:

Theorem 2.2 *Let $r < 1$. Then, a subgame perfect equilibrium $\widehat{\Phi}$ of the infinite horizon game exists if and only if there is a two-parameter infinite family of uniformly bounded functions $V_n^m(x)$ satisfying the following Isaacs equation:*

$$\forall n \le m \in \mathbb{N}\,,\ \forall x \in \mathsf{X}\,,\ \forall u \in \mathsf{U}\,,$$

$$V_n^m(x) = H_n^m(x, \hat{\varphi}^m(x)) \le H_n^m(x, \{\hat{\varphi}^{m\setminus n}(x), u\}) \,.$$

Then, the equilibrium cost of player n joining the game at state x_n is $V_n^n(x_n)$. If the equilibrium is uniform, $V_n^m = V_1^m$ for all n, m.

Proof The proof proceeds along the same lines as in the finite horizon case. In the summation of the sufficiency proof, there remains a term $r^{T-t_n}V^m(x(T))$ that goes to zero as T goes to infinity, because the functions V^m have been assumed to be bounded. And this is indeed necessary since the bound assumed on the L_n^m implies that the Value functions are bounded by $L/(1-r)$. ∎

We restrict our attention to uniform equilibria, so that we have a one-parameter family of Value functions V^m. But it is infinite. To get a feasible algorithm, we make the following assumption:

Hypothesis 2.1 *There is a finite $M \in \mathbb{N}$ such that $p^M = 0$.*

Thanks to that hypothesis, there is a finite number M of Value functions to consider. There remains to find an algorithm to solve for the fixed points bearing on the family $\{V^m(x)\}_m$ for all $x \in \mathsf{X}$. We offer the *conjecture* that the mapping from the family $\{V^m(t+1,\cdot)\}_m$ to the family $\{V^m(t,\cdot\}_m$ in the finite horizon Isaacs equation is a contraction in an appropriate distance. If so, then it provides an algorithm of "iteration on the Value" to compute the $V^m(x)$ of the infinite horizon problem. (We will offer a different conjecture in the linear quadratic case.)

Remark 2.2 *Hypothesis 2.1 is natural in case the payoff is decreasing with the number of players and there is a fixed entry cost. Otherwise, it may seem artificial and somewhat unfortunate. Yet, we may notice that for any numerical implementation, we are obliged to consider only a bounded (since finite) set of x. We are accustomed to doing so, relying upon the assumption that very large values of x will be reached very seldom, and play essentially no role in the computation. In a similar fashion, we may think that very large values of $m(t)$ will be reached for very large t, which, due to the discount factor, will play a negligible role in the numerical results. This is an unavoidable feature of numerical computations, not really worse in our problem than in classical dynamic programming.*

2.3 Entering and leaving

2.3.1 Methodology

It would be desirable to extend the theory to a framework where players may also leave the game at random. However, we must notice that although our players are identical, the game is not anonymous. As a matter of fact, players are labelled by their rank of arrival, and their payoffs depend on that rank. We must therefore propose exit mechanisms able to take into account *who* leaves the game. Before doing so, we agree on the fact that once a player has left the game, it does not reenter. (Or if it does, this new participation is considered as that of another player.) Let T_n be the exit time of the player of rank n, a random variable. We now have

$$J_n(t_n, x(t_n), U(\cdot)) = \mathbb{E} \sum_{t=t_n}^{T_n} r^{t-t_n} L_n^{m(t)}(t, x(t), u^{m(t)}(t)) \,.$$

In defining the controls of the players, we may no longer have $n \leq m \leq t$ as previously, and Table 1 must be modified accordingly. Let $N(m)$ be the maximum possible rank of players present when there are m of them, and $M(n)$ the maximum possible number of players present when player n is. Then $u^m(t) = \{u_n^m\}_{n \leq N(m)}$ and $U_n(t) = \{u_n^m(t)\}_{m \leq M(n)}$. And of course, a choice of $u_n^m(t)$ means the decision that player of rank n chooses at time t if there are m players present at that time, *including himself.*

We also insist that the probabilities of entry (or exit) are functions such as p^m of the current number of players present, and not of the rank of entry.

When a player leaves the game, from the next time step on it will not get any cost. Thus, we may just consider that for it, the Value functions $V_n^m(t+1,x)$ are null. To take this into account we determine the probabilities $\mathbb{P}^{m,k}$ that there be k players at the next time step *and that the focal player has not left*, knowing that there are m players present at the current step. And then, Theorem 2.1 above and its proof remain unchanged upon substituting

$$W_n^m = \sum_k \mathbb{P}^{m,k} V_n^k$$

to equation (5). (In the Bernoulli entry-only version of the problem, we may set $\mathbb{P}^{m,m+1} = p$ and $\mathbb{P}^{m,m} = (1-p)$.)

We propose several entry and exit mechanisms as examples.

2.3.2 A joint scheme

In this scheme, there is a probability q^m that one player leaves the game at the end of a step where there are m players present. (And of course, $q^0 = 0$.) Moreover, we add the dictum that should one player actually leave, which one leaves is chosen at random with uniform probability among the players present. As a consequence, each player present has a probability q^m/m to leave the game at (the end of) each time step. Let $m(t) = m$, then the probabilities that a given player among the m present at step t be still present at time $t+1$ and that $m(t+1)$ take different values is given by the following table:

$m(t+1)$	probability
$m+1$	$\mathbb{P}^{m,m+1} = p^m(1-q^m)\,,$
m	$\mathbb{P}^{m,m} = p^m q^m \dfrac{m-1}{m} + (1-p^m)(1-q^m)$
$m-1$	$\mathbb{P}^{m,m-1} = (1-p^m) q^m \dfrac{m-1}{m}\,.$

2.3.3 Individual schemes

The previous scheme is consistent with our entry scheme. But it might not be the most realistic. We propose two other schemes.

In the first, each player, once it has joined the game, has a probability q of leaving the game at each time step, independently of the other players and of the past and current arrivals sequence. We need powers of p and q. So, to keep the sequel readable, we take them constant, and upper indices in the table below are powers. It is only a matter of notation to take them dependent on m. In computing the probability that a given number of players has left, we must remember that those must be chosen among the other $m-1$ players, and that the focal player must have remained. The corresponding table of probabilities is now

$m(t+1)$	probability
$m+1$	$\mathbb{P}^{m,m+1} = p(1-q)^m$,
$1 < k \le m$	$\mathbb{P}^{m,k} = \frac{(m-1)!}{(m-k)!(k-2)!} q^{m-k}(1-q)^{k-1}\left[\frac{(1-p)(1-q)}{k-1} + \frac{pq}{m-k+1}\right]$,
1	$\mathbb{P}^{m,1} = (1-p)(1-q)q^{m-1}$.

A more coherent scheme, but that drives us away from the main stream of this article, is one where there is a finite pool of M agents who are eligible to enter the game. At each time step, each of them has a probability p of actually entering. Once into the game, each has a probability q of leaving at each time step, and if so, it re-enters the pool. In that case, we set

$$\mathcal{L}^{m,k} = \{\ell \in \mathbb{N} | \ell \ge 0\,,\ \ell \ge m-k\,,\ \ell \le m-1\,,\ \ell \le M-k\}$$

and we have, for all m, k less or equal to M:

$$\mathbb{P}^{m,k} = \sum_{\ell \in \mathcal{L}^{m,k}} \begin{pmatrix} m-1 \\ \ell \end{pmatrix} \begin{pmatrix} M-m \\ k-m+\ell \end{pmatrix} p^{k-m+\ell}(1-p)^{M-k+\ell} q^{\ell}(1-q)^{m-\ell}\,.$$

2.3.4 Beyond the Bernoulli process

At this stage, it is not difficult to generalize our model to one where several players may join the game at each instant of time, provided that it remains a finite Markov chain. Introduce probabilities p_ℓ^m that ℓ players join the game when m players are already there. In a similar fashion, in the so called "joint scheme" above, we might have probabilities q_ℓ^m that ℓ players leave at the same time.

Set $p_j^m = 0$ for any $j < 0$. We then have

$$\mathbb{P}^{m,k} = \sum_{\ell=0}^{m-1} \frac{m-\ell}{m} q_\ell^m p_{k-m-\ell}^m\,. \tag{8}$$

2.4 Linear quadratic problem

2.4.1 The problem

We consider an academic example as follows: the state space is $\mathsf{X} = \mathbb{R}^d$, the control set $\mathsf{U} = \mathbb{R}^a$. the dynamics are defined by a sequence of square $d \times d$ matrices $A(t)$

and a sequence of $d \times a$ matrices $B(t)$ (both could be m-dependent) and

$$x(t+1) = A(t)x(t) + B(t)\sum_{n=1}^{m(t)} u_n \,.$$

There is a discount factor r (presumably no larger than one, super indices to it are to be understood as powers), and the cost of player n is given in terms of two sequences of (families of) square matrices $Q^m(t)$ and $R^m(t)$, the first nonnegative definite, the second positive definite, as

$$J_n = \mathbb{E}\sum_{t=t_n}^{T} r^{t-t_n}\left[\|x(t)\|^2_{Q^{m(t)}(t)} + \|u_n(t)\|^2_{R^{m(t)}(t)}\right].$$

2.4.2 Solution via the Riccati equation

As usual, we seek a solution with a quadratic Value function. We look for a uniform equilibrium, and a one-parameter family of Value functions of the form

$$V_n^m(t,x) = r^{t-t_n}V^m(t,x)\,, \qquad V^m(t,x) = \|x\|^2_{P^m(t)}\,. \tag{9}$$

Notice first that, for all $m \le T$, $P^m(T) = Q^m(T)$ (and clearly, the equilibrium control, if we want to define it at time T, is $u = 0$.) Assume, as a recursion hypothesis, that $V^m(t+1,x)$ is, for all m, a quadratic form in x, i.e. that there exist symmetric matrices $P^m(t+1)$ such that

$$V^m(t+1,x) = \|x\|^2_{P^m(t+1)}\,.$$

Since the costs are always nonnegative, so are the $P^m(t+1)$. (Indeed they are even positive definite as one can easily figure out.) Isaacs equation is now

$$\begin{aligned} V^m(t,x) = \min_u\Big\{ & \|x\|^2_{Q^m(t)} + \|u\|^2_{R^m(t)} \\ & + r\Big[(1-p^m)\|A(t)x + (m-1)B(t)\hat{u} + B(t)u\|^2_{P^m(t+1)} \\ & \qquad + p^m\|A(t)x + (m-1)B(t)\hat{u} + B(t)u\|^2_{P^{m+1}(t+1)}\Big]\Big\}, \end{aligned}$$

the minimum in u being reached at $u = \hat{u}$. Let

$$S^m(t+1) = r\left[(1-p^m)P^m(t+1) + p^m P^{m+1}(t+1)\right]. \tag{10}$$

These are symmetric non-negative definite matrices. Isaacs' equation can be written

$$V^m(t,x) = \min_u\Big\{\|x\|^2_{Q^m(t)} + \|u\|^2_{R^m(t)} + \|A(t)x + (m-1)B(t)\hat{u} + B(t)u\|^2_{S^m(t+1)}\Big\}.$$

The right hand side is a (non homogeneous) quadratic form in u, with a quadratic term coefficient $R(t) + B'(t)S^m(t+1)B(t)$, which is positive definite. Hence there exists a unique minimum in u. Equating the derivative with respect to u to zero, and equating all controls, yields

$$[R^m(t) + B'(t)S^m(t+1)B(t)]\hat{u} + B'(t)S^m(t+1)[A(t)x + (m-1)B(t)\hat{u}] = 0\,.$$

Hence

$$\hat{u} = -F^m(t)x =: \hat{\varphi}_1^m(t,x) \tag{11}$$

with

$$F^m(t) = \left[R^m(t) + mB'(t)S^m(t+1)B(t)\right]^{-1} B'(t)S^m(t+1)A(t)\,. \tag{12}$$

Finally, placing this value of u in the right hand side, we find that $V^m(t,x)$ is indeed a quadratic form in x. Thus we have proven that (9) holds, with

$$\begin{aligned} P^m(t) &= Q^m(t) + F^{m\prime}(t)R^m(t)F^m(t) \\ &\quad + [A'(t) - mF^{m\prime}(t)B'(t)]S^m(t+1)[A(t) - mB(t)F^m(t)]\,. \end{aligned}$$

and after substituting $F^m(t)$ and reordering:

$$\begin{aligned} P^m(t) = {}& Q^m(t) + A'(t)S^m(t+1)A(t) - \\ & A'(t)S^m(t+1)B(t)[R^m(t) + mB'(t)S^m(t+1)B(t)]^{-1} \\ & \quad [(2m-1)R^m + m^2B'(t)S^m(t+1)B(t)] \\ & [R^m(t) + mB'(t)S^m(t+1)B(t)]^{-1}B'(t)S^m(t+1)A(t)\,, \end{aligned} \tag{13}$$

$$\forall m \in \mathbb{T}\,, \quad P^m(T) = Q^m(T)\,. \tag{14}$$

Equation (13) may be rearranged in a slightly more appealing form, where the first three terms of the right hand side are reminiscent of a classic discrete time Riccati equation, and the fourth and last one cancels for $m = 1$ (and omitting the explicit dependences on t of the system matrices)

$$\begin{aligned} P^m(t) = {}& Q^m + A'S^m(t+1)A \\ & - A'S^m(t+1)B[\tfrac{1}{m}R^m + B'S^m(t+1)B]^{-1}B'S^m(t+1)A \\ & - \tfrac{m-1}{m}A'S^m(t+1)B[\tfrac{1}{m}R^m + B'S^m(t+1)B]^{-1} \\ & \qquad \tfrac{1}{m}R^m[\tfrac{1}{m}R^m + B'S^m(t+1)B]^{-1}B'S^m(t+1)A\,. \end{aligned}$$

Recall that each matrix S^m involves P^{m+1}. But there cannot be more than T players at any time in the game (and T of them only at final time !) Therefore, starting with $P^T(T) = Q^T$ and computing the $P^m(t)$ backward, this is a constructive algorithm. We therefore end up with the following:

Theorem 2.3 *The finite horizon, linear quadratic problem admits a unique uniform subgame perfect equilibrium given by equations (9,10,11,12,13,14).*

Entering and leaving It is now easy to get the solution of the same problem with one of our more general entry and leaving mechanisms: according to equation (8), it suffices to replace the definition (10) of S^m by

$$S^m(t+1) = r \sum_{k=1}^{m+1} \mathbb{P}^{m,k} P^k(t+1)$$

with the relevant set of probabilities $\mathbb{P}^{m,k}$.

2.4.3 Infinite horizon

We may want to consider the case where the matrices A, B, Q, and R are constant, and the horizon infinite. For all practical purposes, we need hypothesis 2.1 limiting the maximum number of players in the game to M. Even so, this problem does not completely fit the framework of Theorem 2.2, because the integrand is not bounded over $\mathbb{R}^n$. Hence the added stability condition that we will need. We can nevertheless look for a Nash equilibrium along the lines of Theorem 2.2. For that purpose, we add a constraint for all players, that their controls be of bounded energy, and precisely:

$$\sum_{t=t_n}^{\infty} r^{t-t_n} \|u_n(t)\|^2_{R(t)} < \infty \,. \tag{15}$$

We also assume that $Q^M > 0$.

We state a system of so called algebraic discrete Riccati equations for matrices P^m, and use the notations

$$S^m = r[(1-p^m)P^m + p^m P^m] \,, \qquad F^m = \left[R^m + mB'S^m B(t)\right]^{-1} B'S^m A \,.$$

We state the theorem

Theorem 2.4 *A sufficient condition for the existence of a uniform subgame perfect pure Nash equilibrium to the infinite horizon linear quadratic game with a bounded number of players and $Q^M > 0$ is that there exist a positive definite solution to the system of equations*

$$\begin{aligned} P^m =& Q^m + A'S^m A - \\ & A'S^m B[R^m + mB'S^m B]^{-1} \\ & \qquad [(2m-1)R^m + m^2 B'S^m B] \\ & \qquad\qquad [R^m + mB'S^m B]^{-1} B'S^m A \,, \end{aligned} \tag{16}$$

and that furthermore, the following inequality hold:

$$\frac{1}{r}Q^M > A'P^M B\left[\frac{1}{r}R^M + MB'P^M B\right]^{-1} B'P^M A\,. \tag{17}$$

In that case, a Nash strategy is given by

$$\hat{u}(t) = -F^{m(t)}x(t)\,. \tag{18}$$

Proof The proof goes with the same calculations as for theorem 2.1 and for the above finite horizon case. But because the payoff is not necessarily bounded, we need that $r^t V^{m(t)}(x(t))$ decrease to zero as $t \to \infty$ for the extension of theorem 2.1 to the infinite case to hold. In infinite horizon (and without exit) the number of players present will almost surely reach its maximum possible M. Therefore, the asymptotic behavior of the closed-loop system depends on $A - MBF^M$.

To investigate that condition, we use Lyapunov's stability analysis for linear systems, but with the state $y(t) = r^{t/2}x(t)$. If this state decreases exponentially, then $r^t\|x(t)\|^2_{P^M}$ goes to zero as $t \to \infty$. We pick the Lyapunov function $U(x) = r^t\|x\|^2_{S^M}$. A direct calculation shows that

$$r(A - MF^M)'S^M(A - MF^M) - S^M = rP^M - rF^{M\prime}R^M F^M - rQ^M - S^M\,.$$

We have $S^M = rP^M$. Therefore, and since we have assumed that $Q^M > 0$, this last matrix is negative definite, proving that indeed, if all players use the control (18), the Value function goes to zero as $t \to \infty$.

But we also need that this be true if all players but one use the control (18), the other one using a "finite energy" (according to equation (15)) control. This will be guaranteed if the state $y(t) = r^{t/2}x(t)$ is stabilized by the dynamics driven by $m - 1$ controls (18). We attempt the same Lyapunov analysis, and end up with

$$r[A - (M-1)BF^M]'S^M[A - (M-1)BF^M] - S^M = rP^M - S^M - rQ^M +$$
$$rA'S^M B[R^M + MB'S^M B]^{-1}(R^M + B'S^M B)[R^M + MB'S^M B]^{-1}B'S^M A\,.$$

Therefore, we need that

$$Q^M > A'S^M B[R^M + MB'S^M B]^{-1}(R^M + B'S^M B)[R^M + MB'S^M B]^{-1}B'S^M A$$

A sufficient condition is that

$$Q^M > A'S^M B[R^M + MB'S^M B]^{-1}(R^M + MB'S^M B)[R^M + MB'S^M B]^{-1}B'S^M A$$

which coincides with the condition (17) of the theorem. ∎

Conjecture 2.1 *We conjecture that condition* (17) *is always satisfied for a small enough r.*

It remains to solve for the fixed point equation (16) on $\{P^m\}_m$. A natural approach is to integrate the dynamical Riccati equation backward in hope of reaching a steady state, which is necessarily a solution of the fixed point problem. We have no proof of that convergence at this time, but no counter-example in the many numerical experiments we ran, in dimension up to ten for the state, and three for the control. In every cases, we observed a rapid convergence, in less than twenty steps for all practical purposes. As a typical example, in state dimension 10, with $p^m = .5$, A in companion form, and characteristic polynomial (chosen arbitrarily)

$$\chi_A(z) = z^{10} - 2z^9 - z^8 + 1.5z^7 - 2z^6 - 3.5z^4 + 3.5z^3 - z^2 + 2z - 1\,,$$

and, no less arbitrarily,

$$B' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$r = 1$, $Q = 3I$, and $R = \mathrm{diag}(2, 2, 1)$, the variation $\Delta^m(t) = P^m(t+1) - P^m(t)$ after nineteen steps has a norm $[\mathrm{tr}(\Delta^2)]^{1/2}$ less than 10^{-9} for all $m \in \{1, \ldots, 20\}$. And while we give one typical example here, the conclusion applies to all the experiments we made.

3 Continuous time

3.1 The problem

3.1.1 Players, dynamics and cost

We consider a game with randomly arriving (or arriving and leaving) players as in the previous section, but in continuous-time. The players arrive as a Poisson process of variable intensity: The interval lengths $t_{m+1} - t_m$ between successive arrivals are independent random variables obeying exponential laws with intensity λ_m:

$$\mathbb{P}(t_{m+1} - t_m > \tau) = \mathrm{e}^{-\lambda^m \tau}$$

for a given sequence of positive λ^m. An added difficulty, as compared to the discrete time case, is that the number of possible arrivals is unbounded, even for the finite horizon problem. For that reason, the sequence λ^m is a priori infinite. But we assume that the λ^m are bounded by a fixed Λ. As a matter of fact, for any practical use of the theory, we will have to assume that the λ^m are all zero for m larger than a given integer M, thus limiting the number of players to M. Alternatively, for a finite horizon T, we may notice that for any M, the probability $\mathbb{P}(m(t) > M)$ is less than $(\Lambda T)^M/M!$ and therefore goes to zero as $M \to \infty$, and take argument to neglect very large m's.

The dynamic system is also in continuous time. The state space X is now the Euclidean space $\mathbb{R}^d$, or a subset of it, and the dynamics

$$\dot{x} = f^{m(t)}(t, x, u^{m(t)}), \quad x(0) = x_0 .$$

Standard regularity and growth hypotheses hold on the functions f^m to insure existence of a unique solution in X over $[0, T]$ to the dynamics for every m-tuple of measurable functions $u^m(\cdot) : [0, T] \to \mathsf{U}^m$.

A positive discount factor ρ is given, and the performance indices are given via

$$\mathcal{L}_n(t_n, x(t_n), \{u^m(\cdot)\}_{m\in\mathbb{N}}) = \int_{t_n}^{T} \mathrm{e}^{-\rho(t-t_n)} L_n^{m(t)}(t, x(t), u^{m(t)}(t))\,\mathrm{d}t$$

as

$$J_n(t_n, x(t_n), \{u^m(\cdot)\}_{m\in\mathbb{N}}) = \mathbb{E}\mathcal{L}_n(t_n, x(t_n), \{u^m(\cdot)\}_{m\in\mathbb{N}}) . \tag{19}$$

The functions L_n^m are assumed to be continuous and uniformly bounded.

As in the discrete time case, we consider identical players, i.e. the functions f^m are invariant by a permutation of the u_n, and the functions L_n^m enjoy the properties of a game with identical players as detailed in the appendix A.

3.1.2 Strategies and equilibrium

We seek a state feedback equilibrium. Let $\mathcal{A}^m$ be the set of *admissible feedbacks* when m players are present. A control law $\varphi : [0, T] \times \mathsf{X} \to \mathsf{U}$ will be in $\mathcal{A}^m$ if, on the one hand, the differential equation

$$\dot{x} = f^m(t, x, \varphi(t, x)^{\times m})$$

has a unique solution for any initial data $(t_n, x_n) \in [0, T] \times \mathsf{X}$, and on the other hand, for every measurable $u(\cdot) : [0, T] \to \mathsf{U}$, the differential equation

$$\dot{x} = f^m(t, x(t), \{u(t), \varphi(t, x(t))^{\times m \setminus 1}\})$$

has a unique solution over $[0, T]$ for any initial data $(t_n, x_n) \in [0, T] \times \mathsf{X}$.

We define a state feedback pure equilibrium as in the previous section, namely via definition 2.3. Moreover, we shall be concerned only with uniform such equilibrium strategies.

3.1.3 Mixed strategies and disturbances

We have rather avoid the complexity of mixed strategies in continuous time (see, however, [9]), as experience teaches us that they are often unnecessary.

Adding disturbances to the dynamics and payoff as in the discrete time problem is not difficult. But the notation need to be changed to that of diffusions, and we would get extra second order terms in Isaacs equation, due to Ito calculus. All results carry over with the necessary adaptations. We keep with the deterministic set up for the sake of simplicity.

3.2 Isaacs equation

3.2.1 Finite horizon

The Isaacs equation naturally associated with a uniform equilibrium in this problem is as follows, where $\hat{u}$ stands for the argument of the minimum (we write V_t and V_x for the partial derivatives of V):

$$\forall (t,x) \in [0,T] \times \mathsf{X}, \quad (\rho + \lambda^m) V^m(t,x) - \lambda^m V^{m+1}(t,x) - V_t^m(t,x) - \min_{u \in \mathsf{U}} \Big[V_x^m(t,x) f^m(t,x,\{u, \hat{u}^{\times m \setminus 1}\}) + L_1^m(t,x,\{u, \hat{u}^{\times m \setminus 1}\}) \Big] = 0\,, \tag{20}$$

$$\forall x \in \mathsf{X}, \quad V^m(T,x) = 0\,.$$

As already mentioned, even for a finite horizon, the number of players that may join the game is unbounded. Therefore, equation (20) is an infinite system of partial differential equations for an infinite family of functions $V^m(t,x)$. We will therefore make use of the hypothesis similar to 2.1:

Hypothesis 3.1 *There exists an integer M such that $\lambda^M = 0$.*

As hypothesis 2.1 of the discrete time case, this is a natural hypothesis in case of a decreasing payoff and fixed finite entry cost, and akin to classical approximations of the Isaacs equation in dynamic programming algorithms.

Under that hypothesis, using the tools of piecewise deterministic Markov decision Processes, we have the following easy extension of [12]:

Theorem 3.1 *A uniform subgame perfect equilibrium exists if and only if there exists a family of admissible feedbacks $\varphi^m \in \mathcal{A}^m$ and a family of bounded uniformly continuous functions $V^m(t,x)$ that are, for all $m \le M$, viscosity solutions of the partial differential equation* (20)*. Then, $u_n(t) = \hat{\varphi}^{m(t)}(t,x(t))$ is a uniform subgame perfect equilibrium, and the equilibrium cost of player n joining the game at time t_n and state x_n is $V^n(t_n,x_n)$.*

A sketch of the proof is given in appendix B.1.

The question naturally arises of what can be said of the problem without the hypothesis 3.1. To investigate this problem, we consider an "original problem" defined by its infinite sequence $\{\lambda^m\}_{m\in\mathbb{N}}$, assumed bounded :

$$\exists\Lambda > 0 : \quad \forall m \in \mathbb{N}\,,\ \lambda^m \leq \Lambda\,.$$

and a family of "modified problems" depending on an integer M, where we modify the sequence $\{\lambda^m\}$ at λ^M that we set equal to zero. (And therefore all λ^m for $m > M$ are irrelevant: there will never be more than M players.) The theorem above holds for all modified problems, whatever the M chosen. We call $V^{m|M}$ (a finite family) the solution of the corresponding equation (20). They yield the equilibrium value of the cost J^M in the modified problems.

We propose in appendix B.2 arguments in favor of the following

Conjecture 3.1 *As M goes to infinity, the equilibrium state feedbacks φ^M of the modified problems converge, in L^1 (possibly weighted by a weight $\exp(-\alpha\|x\|)$) toward an equilibrium feedback $\varphi^\star$ of the original problem, and the functions $V^{m|M}$ converge in C^1 toward the equilibrium value V^m. Consequently, theorem 3.1 holds for the original, unmodified problem.*

3.2.2 Infinite horizon

We assume here that the functions f^m and L_n^m are time invariant, and $\rho > 0$. We set

$$J_n(t_n, x(t_n), \{u^m(\cdot)\}_{m\in\mathbb{N}}) = \mathbb{E}\int_{t_n}^{\infty} \mathrm{e}^{-\rho(t-t_n)} L_n^{m(t)}(t, x(t), u^{m(t)})\,\mathrm{d}t\,.$$

As expected, we get

Theorem 3.2 *Under hypothesis 3.1, a uniform subgame perfect equilibrium in infinite horizon exists if and only if there exists a family of admissible feedbacks $\hat\varphi^m \in \mathcal{A}^m$ and a family of bounded uniformly continuous functions $V^m(x)$ that are, for all m, viscosity solutions of the following partial differential equation, where $\hat u$ stands for $\hat\varphi^m(t,x)$ and the minimum is reached precisely at $u = \hat u$:*

$$\forall x \in \mathsf{X}\,, \qquad 0 = (\rho + \lambda^m)V^m(x) - \lambda^m V^{m+1}(x) - \tag{21}$$

$$\min_{u\in\mathsf{U}}\Big[V_x^m(x) f^m(x, \{u, \hat u^{\times m\setminus 1}\}) + L_1^m(x, \{u, \hat u^{\times m\setminus 1}\})\Big] \tag{22}$$

Then, $u_n(t) = \hat\varphi^{m(t)}(x(t))$ is a uniform subgame perfect equilibrium, and the equilibrium cost of player n joining the game at state x_n is $V^n(x_n)$.

The proof involves extending equation (20) to the infinite horizon case, a sketch of which is provided in appendix B.1, relying on the boundedness of the functions V^m to ensure that $\exp(-\rho T)V^m(x(T))$ goes to zero as T increases to infinity. The rest is exactly as in the previous subsubsection.

The original problem without the bounding hypothesis 3.1 requires a different approach from that of the previous subsection, because in infinite horizon, it is no longer true that $\mathbb{P}(\Omega_M)$ is small. Indeed it is equal to one if the hypothesis does not hold and the λ^m have a lower bound.

3.3 Entering and leaving

As in the discrete time case, we may extend the theory to the case where the players mays also leave the game. We consider that once a player has left, it does not re-enter. We let T_n be the exit time of player n. In the joint exit mechanism, the process that one of the m players present may leave is a Poisson process with intensity μ^m, and if one does, it is one of the players present with equal probability. In the individual scheme, each of the m players present has a Poisson exit process with probability μ^m.

We leave it to the reader to check that Isaacs' equation now reads

$$(\rho + \mathbb{P}^{m,m})V^m(t,x) - \mathbb{P}^{m,m+1}V^{m+1}(t,x) - \mathbb{P}^{m,m-1}V^{m-1}(t,x) - V_t^m(t,x)$$
$$- \min_{u \in \mathsf{U}} \Big[V_x^m(t,x) f^m(t,x,\{u,\hat{u}^{m\setminus 1}\}) + L_1^m(t,x,\{u,\hat{u}^{m\setminus 1}\}) \Big] = 0\,,$$

where the coefficients $\mathbb{P}^{m,\ell}$ are given by the following table:

scheme	$\mathbb{P}^{m,m-1}$	$\mathbb{P}^{m,m}$	$\mathbb{P}^{m,m+1}$
joint	$\frac{m-1}{m}\mu^m$	$\lambda^m + \mu^m$	λ^m
individual	μ^m	$\lambda^m + m\mu^m$	λ^m

(23)

3.4 Linear quadratic problem

3.4.1 Finite horizon

We turn tho the standard linear quadratic case, where the dynamics are given by piecewise continuous (or even measurable) time dependent matrices $A(t)$ and $B(t)$ of dimensions, respectively $d \times d$ and $d \times a$ (both could be m-dependent)

$$\dot{x} = A(t)x + B(t)\sum_{n=1}^{m} u_n\,.$$

The cost to be minimized is given by a discount factor ρ, a family of piecewise continuous, nonnegative definite symmetric $d \times d$ matrices $Q^m(t)$ and positive definite $a \times a$ matrices $R(t)$ as

$$J_n = \mathbb{E}\left[\int_{t_n}^{T} \mathrm{e}^{-\rho(t-t_n)}\left(\|x(t)\|^2_{Q^{m(t)}} + \|u_n(t)\|^2_R\right)\,\mathrm{d}t\right].$$

(It is only for notational convenience that we do not let the matrix R depend on m, because we will need its inverse R^{-1}.) We again seek a uniform solution with Value functions

$$V^m(t,x) = \|x\|^2_{P^m(t)}\,. \tag{24}$$

Isaacs equation now reads

$$\rho\|x\|^2_{P^m(t)} = \min_{u\in\mathsf{U}}\Big[\|x\|^2_{\dot P^m(t)} + 2x'P^m(t)(A(t)x + B(t)u + (m-1)B(t)\hat u) + \|x\|^2_{Q^m(t)} + \|u\|^2_{R(t)}\Big] + \lambda^m\big(\|x\|^2_{P^{(m+1)}(t)} - \|x\|^2_{P^m(t)}\big).$$

We drop explicit explicit time dependences of the system matrices for legibility. We obtain

$$\hat u = -R^{-1}B'P^m(t)x \tag{25}$$

and

$$\dot P^m - (\rho+\lambda^m)P^m + P^mA + A'P^m - (2m-1)P^mBR^{-1}B'P^m + Q^m + \lambda^mP^{m+1} = 0 \tag{26}$$

with the terminal condition

$$P^m(T) = 0\,. \tag{27}$$

To get a practically usable theory, we limit the possible number of players according to hypothesis 3.1. Under that hypothesis, we prove the following:

Theorem 3.3 *The finite horizon linear quadratic problem with a bounded maximum number of players has a unique uniform equilibrium given by equations (24,25,26,27).*

Proof There remains to prove that the Riccati equations have a solution over $[0,T]$. Notice first that the equation for P^M stands alone. It is the same as the Riccati equation associated to the simple optimization problem

$$\dot x = A(t)x + B(t)u\,, \quad x(0) = x_0\,,$$

$$J = \int_0^T \mathrm{e}^{-(\rho+\lambda^M)t}\big(\|x\|^2_{Q^M} + (2M-1)\|u\|^2_{R^M}\big)\,\mathrm{d}t\,.$$

Therefore, $P^M(t)$ is known to exist and be non-negative definite over $[0,T]$. (It is non-negative as long as it exists as giving the infimum of a non-negative cost, and bounded by above by he value of the same cost obtained, e.g. with $u = 0$.) Assume as a recursion hypothesis that $P^m(t)$ is defined and non-negative definite over $[0,T]$. Then, the Riccati equation for $P^m(\cdot)$ is the same as that for the same control problem as above with m instead of M, but with Q^M replaced by $\widetilde{Q}^m = Q^m + \lambda^m P^{m+1}$. Therefore we also have existence of a non-negative definite $P^m(t)$, and therefore recursively, for all $m \leq M$. ▮

Entering and leaving We may of course deal with the case where players may leave the game in the same way as before, replacing in the Riccati equation (26) the term

$$\lambda^m\big(P^{m+1}(t) - P^m(t)\big)$$

by

$$\mathbb{P}^{m,m-1}P^{m-1}(t) - \mathbb{P}^{m,m}P^m(t) + \mathbb{P}^{m,m+1}P^{m+1}(t)\,.$$

the $\mathbb{P}^{m,k}$ being given by the table (23). Te same existence theorem applies, with some care. The Riccati equations can no longer be integrated in sequence from $m = M$ down to $m = 1$. But they can still be integrated backward jointly, as a finite dimensional ordinary differential equation. As long as all $P^m(t)$ exist, the interpretation as Value functions of nonnegative costs still guarantees te nonnegativity of all P^m matrices. But this in turn guarantees the existence over $[0,T]$ with the interpretation as Riccati equations of ordinary linear quadratic control problems with positive weighting matrices.

3.4.2 Infinite horizon

We consider now the linear quadratic problem with A, B, Q^m, and R^m constant matrices, and

$$J_n = \mathbb{E}\int_0^\infty \mathrm{e}^{-\rho(t-t_n)}\big(\|x(t)\|^2_{Q^{m(t)}} + \|u(t)\|^2_R\big)\,.$$

We assume that hypothesis 3.1 holds, and that $Q^M > 0$. Furthermore, we impose on all players the constraint that their controls be of “finite energy” in the precise sense that

$$\int_{t_n}^\infty \mathrm{e}^{-\rho t}\|u_n(t)\|^2_R\,\mathrm{d}t < \infty\,. \tag{28}$$

The Riccati equation is replaced by its algebraic counterpart

$$(\rho+\lambda^m)P^m = P^mA + A'P^m - (2m-1)P^mBR^{-1}B'P^m + Q^m + \lambda^mP^{m+1}\,. \tag{29}$$

Under the hypothesis 3.1 of a bounded number of players, we have the following result:

Theorem 3.4 *If the pair (A, B) is stabilizable and $Q^M > 0$, the Riccati equations* (26) *integrated from $P^m(0) = 0$ have positive definite limits $\bar{P}^m$ as $t \to -\infty$ solutions of the algebraic Riccati equations* (29).

Furthermore, for a large enough ρ, it holds that

$$\bar{P}^M BR^{-1}B'\bar{P}^M - Q^M < 0\,, \tag{30}$$

and then the infinite horizon linear quadratic game with a bounded number of players has a uniform subgame perfect pure Nash equilibrium obtained by replacing P^m by $\bar{P}^m$ in formula (25).

Proof The proof of the asymptotic behavior of equations (26) relies on the same identification with a control Riccati equation as for the finite horizon problem. Thus, $P^M(t)$ does have a positive definite limit as $t \to -\infty$. It remains to extend to the variable matrix $\widetilde{Q}^{M-1}(t) := Q^{M-1} + \lambda^M P^M(t)$ the standard theory of the linear quadratic regulator, a simple matter, to conclude that the solution $P^{M-1}(t)$ of the differential equation in P^{M-1} also has a limit as $t \to -\infty$. In this process, we notice that since Q^{M-1} has been assumed nonnegative definite and P^M is positive definite, it follows that $\widetilde{Q}^{M-1}(t)$ is positive definite, and therefore the pair $(\widetilde{Q}^{M-1}(\cdot), A)$ is detectable over $(-\infty, 0)$. And then, the same reasoning applies recursively as m decreases to 1.

Consider $x'\dot{P}^m x$ with a constant x. Applying a standard comparison theorem for ordinary differential equations shows that $P^m(t)$ is decreasing as ρ increases. Moreover, equation (29) divided through by ρ shows that the limit as $\rho \to \infty$ is $\bar{P}^m = 0$. Hence, for a large enough ρ, condition (30) is indeed satisfied.

In infinite time, the maximum number of players will almost surely reach M. Therefore, the asymptotic behavior of the closed-loop system depends on the controls for $m = M$. It is a simple matter to check that if all players use this strategy,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\rho t}\|x\|^2_{\bar{P}^M}\right) = -\mathrm{e}^{-\rho t}x'[\bar{P}^M BR^{-1}B'\bar{P}^M + Q^M]x\,.$$

Using Lyapunov stability theory for linear systems, it follows that the Value function $\exp(-\rho t)\|x(t\|^2_{\bar{P}^M}$ goes to zero as $t \to \infty$, which is needed to apply theorem 3.2.

We also need that this be true if only $(M-1)$ players use their control (25), and the other one any finite energy (in the sense of (28)) control. This will be true if the system with the $M-1$ feedbacks stabilizes $\exp(-\rho t\|x(t)\|^2_{\bar{P}^M}$. Applying the same technique, we find that, in these conditions,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\rho t}\|x\|^2_{\bar{P}^M}\right) = \mathrm{e}^{-\rho t}x'[\bar{P}^M BR^{-1}B'\bar{P}^M - Q^M]x\,.$$

Hence the condition (30). ∎

4 Conclusion

The tools of piecewise deterministic Markov decision processes have been extended to games with random players arrivals and exits. We have chosen some specific problems within this wide class, namely identical players (there might be several classes of players as in, e.g. [27]). We have emphasized a Bernoulli arrival process in the discrete time case, Poisson in the continuous time case, at first with no exit. Yet, we have given a few examples of other schemes, both with more general arrivals, and with exit at random times also.

We have also considered a restricted class of linear quadratic problems as illustration. The continuous time problem lends itself to a rather complete theory where we have existence and limit results similar to what is available for the elementary one-player optimization counterpart. All these might be extended to other cases. The present article shows clearly how to proceed. The question is to find which other cases are both interesting and, if possible, lead to feasible computational algorithms.

In that respect, the unbounded number of players in the infinite horizon discrete time problem, and in all cases in continuous time, poses a problem, mainly computational in the former case, also mathematical in the later, because of the difficulty of dealing with an infinite set of partial differential equations. The computational problem, however, is nothing very different from that of discretizing an infinite state space.

We consider that the main weaknesses of the present theory, beyond its difficulty of dealing with an unbounded number of players, is the lack of a private state for each player. This would allow us to deal with several classes of players as, say, in [27], or more general models such as in [16], and also with other exit mechanisms such as a fixed residence time T for all players.

References

[1] Pierre Bernhard. Information and strategies in dynamic games. *SIAM Journal on Control and Optimization*, 30(1):212–228, 1992.

[2] Pierre Bernhard and Marc Deschamps. Dynamic equilibrium in games with randomly arriving players. Working paper, CRESE, Université de Bourgogne-Franche Comté, France, 2016.

[3] Pierre Bernhard and Marc Deschamps. On dynamic games with randomly arriving players. *Dynamic Games And Applications*, To appear, 2016.

[4] Pierre Bernhard and Marc Deschamps. Dynamic equilibrium in games with randomly arriving players. *?*, To be submitted, and not quoted before publication, May be 2017.

[5] Pierre Bernhard and Frédéric Hamelin. Sharing a resource with randomly arriving foragers. *Mathematical Biosciences*, 273:91–101, 2016.

[6] Michèle Breton, Lucia Sbragia, and Georges Zaccour. A dynamic model for international environmental agreements. *Environmental and Resource Economics*, 45:25–48, 2010.

[7] Francesco De Sinopoli, Claudia Meroni, and Carlos Pimienta. Strategic stability in Poisson games. *Journal of Economic Theory*, 153:46–63, 2014.

[8] Ivar Ekeland. *Théorie des jeux et introduction à l'économie mathématique*. Presses Universitaires de France, 1974.

[9] Robert J. Elliot and Nigel J. Kalton. The existence of value in differential games of pursuit and evasion. *Journal of Differential Equations*, 12:504–523, 1972.

[10] L. C. Evans and P. E. Souganidis. Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations. *Indiana University Mathematical Journal*, 33:773–797, 1984.

[11] A. T. S. Ferguson. Selection by committee. In A Nowak and K. Szajowski, editors, *Advances in Dynamic Games*, number 7 in Annals of the International Society of Dynamic Games, pages 203–209. Birkhäuser, Boston, 2005.

[12] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*. Springer, 1993.

[13] Avner Friedman. Differential games. In Robert Auman and Hart Sergiu, editors, *Handbook of Game Theory, Vol. 2*, pages 781–789. Elsevier, 1994.

[14] Frédéric Hamelin, Pierre Bernhard, A.J. Shaiju, and Éric Wajnberg. Diet selection as a differential foraging game. *SIAM journal on Control and Optimization*, 46:1539–1561, 2007.

[15] A. Haurie and F. Moresino. A stochastic programming approach to manufacturing flow control. *IEEE Transactions on Automatic Control*, 32:907919, 2000.

[16] Ioannis Kordonis and George P. Papavassilopoulos. LQ Nash games with random entrance: An infinite horizon major player and minor players of finite horizon. *IEEE Transactions on Automatic Control*, AC 60:1486–1500, 2015.

[17] Nicolai S. Kukushkin. Best response dynamics in finite games with additive aggregation. *Games and Economic Behavior*, 48:94–110, 1994.

[18] Rida Laraki and Sylvain Sorin. Advances in dynamic games. In Peyton Young and Shmuel Zamir, editors, *Handbook of Game Theory, Vol. 4*, pages 27–93. Elsevier, 2015.

[19] Dan Levin and Emre Ozdenoren. Auctions with uncertain number of bidders. *Journal of Economic Theory*, 118:229–251, 2004.

[20] Evangelos F. Mageirou. Values and strategies for infinite time linear quadratic games. *IEEE Transactions on Automatic Control*, AC-21:547–550, 1976.

[21] Roger B. Myerson. Population uncertainty and Poisson games. *International Journal of Game Theory*, 27:375–392, 1998.

[22] Abraham Neyman and Sylvain Sorin. *Stochastic Games*. NATO ASI series. Kluwer academic publishers, 2003.

[23] Marc Quincampoix. Differential games. In N. Meyers, editor, *Encyclopaedia of Complexity and System Science*, pages 1948–1956. Springer, 2009.

[24] Raymond Rishel. Dynamic programming and minimum principles for systems with jump markov disturbances. *SIAM Journal on Control*, 13:338–371, 1975.

[25] William F. Samuelson. Competitive bidding with entry costs. *Economic Letters*, 17:53–57, 1985.

[26] David D. Sworder. Feedback control of a class of linear systems with jump parameters. *IEEE Transactions on Automatic Control*, AC-14:9–14, 1969.

[27] Hamidou Tembine. Evolutionary networking games. In *Game Theory for Wireless Communications and Networking*, Auerbach Publications, pages 133–158. CRC Press, 2010.

[28] D. Vermes. Optimal control of piecewise deterministic Markov process. *Stochastics*, 14:165–207, 1985.

A Games with identical players

A.1 Model and properties

By assumption, in the game considered here, all players are identical. To reflect this fact in the mathematical model, we need to consider permutations $\pi^m \in \Pi^m$ of the elements of $\{1, 2, \dots, m\}$. We also recall the notation

$$u^{m\backslash n} := (u_1, \dots, u_{n-1}, u_{n+1}, \dots, u_m)\,,$$
$$\{u^{m\backslash n}, u\} := (u_1, \dots, u_{n-1}, u, u_{n+1}, \dots, u_m)$$

Furthermore, we denote

$$u^\pi = u^{\pi[m]} := \left(u_{\pi(1)}, u_{\pi(2)}, \dots, u_{\pi(m)}\right)\,,$$
$$u^{\pi[m]\backslash\pi(n)} := \left(u_{\pi(1)}, \dots, u_{\pi(n-1)}, u_{\pi(n+1)}, \dots, u_{\pi(m)}\right)\,,$$
$$\{u^{\pi[m]\backslash\pi(n)}, u\} := \left(u_{\pi(1)}, \dots, u_{\pi(n-1)}, u, u_{\pi(n+1)}, \dots, u_{\pi(m)}\right)\,,$$
$$u^{\times m} := (u, u, \dots, u) \in \mathsf{U}^m\,.$$

Definition A.1 *A m-person game $\{J_n : \mathsf{U}^m \to \mathbb{R}\}$, $n = 1, \dots, m$ will be called a* game with identical players *if, for any permutation π of the set $\{1, \dots, m\}$, it holds that*

$$\forall n \le m\,, \quad J_n(u_{\pi(1)}, \dots, u_{\pi(m)}) = J_{\pi(n)}(u_1, \dots, u_m)\,. \tag{31}$$

We shall write this equation as $J_n(u^{\pi[m]}) = J_{\pi(n)}(u^m)$.

An alternate definition of a game with identical players is given by the following:

Lemma A.1 *A game with identical player is defined by a function $G : \mathsf{U}\times\mathsf{U}^{m-1} \to \mathbb{R}$ invariant by a permutation of the elements of its second argument, i.e. such that,*

$$\forall u \in \mathsf{U}\,, \forall v^{m-1} \in \mathsf{U}^{m-1}\,, \forall \pi \in \Pi^{m-1}\,, \quad G(u, v^{m-1}) = G(u, v^{\pi[m-1]})\,. \tag{32}$$

And the J_n are defined by

$$J_n(u^m) = G(u_n, u^{m\backslash n}) \tag{33}$$

Proof It is clear that if the J_n are defined by (33) with G satisfying (32), they satisfy (31). Indeed, then

$$J_n(u^{\pi[m]}) = G(u_{\pi(n)}, u^{\pi[m]\backslash\pi(n)}) = G(u_{\pi(n)}, u^{m\backslash\pi(n)}) = J_{\pi(n)}(u^m)\,.$$

Conversely, assume that the J_n satisfy (31). Define

$$G(u_1, u^{m\setminus 1}) = J_1(u^m)\,.$$

Let $\pi_1 \in \Pi^{m-1}$, and π defined by $\pi(1) = 1$, and for all $j \geq 2$, $\pi(j) = \pi_1(j-1)$. (i.e. π is any permutation of Π^m that leaves 1 invariant.) It follows from (31) that

$$G(u_1, u^{m\setminus 1}) = J_1(u^m) = J_{\pi(1)}(u^m) = J_1(\{u_1, u^{\pi_1[m\setminus 1]} = G(u_1, u^{\pi_1[m\setminus 1]})\,.$$

Therefore G is invariant by a permutation of the elements of its second argument. Let now π be a permutation such that $\pi(1) = n$. We have

$$J_n(u^m) = J_{\pi(1)}(u^m) = J_1(u^\pi) = G(u_{\pi(1)}, u^{\pi\setminus\pi(1)}) = G(u_n, u^{m\setminus n})\,,$$

which is equation (33). And this proves the lemma. ▮

The main fact is that the set of pure Nash equilibria is invariant by a permutation of the decisions:

Theorem A.1 *Let $\{J_n : \mathsf{U}^m \to \mathbb{R}\}$, $n = 1, \ldots, m$ be a game with identical players. Then if $\hat{u}^m$ is a Nash equilibrium, so is $\hat{u}^{\pi[m]}$.*

Proof Consider $J_n(\hat{u}^{\pi[m]})$, and then substitute some u to $\hat{u}_{\pi(n)}$ in the argument. Because $J_n(u^{\pi[m]}) = J_{\pi(n)}(u^m)$, it follows that

$$J_n(\{\hat{u}^{\pi[m]\setminus\pi(n)}, u\}) = J_{\pi(n)}(\{\hat{u}^{m\setminus\pi(n)}, u\}) \geq J_{\pi(n)}(\hat{u}^m) = J_n(\hat{u}^{\pi[m]})\,.$$

And this is true for all $n \leq m$, which proves the theorem. ▮

Example An example of the above reasoning is as follows. Let $m = 2$ and by hypothesis, $\forall(u_1, u_2)$, $J_1(u_2, u_1) = J_2(u_1, u_2)$. Let $(\hat{u}_1, \hat{u}_2)$ be a Nash equilibrium. Let us show that $(\hat{u}_2, \hat{u}_1)$ is also an equilibrium:

$$\forall u\,, \quad J_1(u, \hat{u}_1) = J_2(\hat{u}_1, u) \geq J_2(\hat{u}_1, \hat{u}_2) = J_1(\hat{u}_2, \hat{u}_1).$$

Corollary A.1 *A pure Nash equilibrium of a game with identical players can be unique only if it is* uniform*, i.e. with all players using the same control:*

$$\exists \hat{u} \in \mathsf{U} : \forall n \leq m\,, \quad \hat{u}_n^m = \hat{u}\,.$$

Existence of such a Nash equilibrium is not guaranteed, and even if it exists, it might not be the only one. However there is a simple way to look for one. Let us first assert the following fact:

Theorem A.2 *Let $\{J_n : \mathsf{U}^m \to \mathbb{R}\}, n = 1, \ldots, m$ be a game with identical players. If the function $u_1 \mapsto J_1(\{u^{m\setminus 1}, u_1\})$ is convex, so are all the functions $u_n \mapsto J_n(\{u^{m\setminus n}, u_n\})$.*

Proof Let $\tilde{u}^m = (u_n, u_2, \dots, u_{n-1}, u_1, u_{n+1}, \dots, u_m)$, and let $\pi^{1,n}$ be the permutation that exchanges 1 and n. Then, $u^m = \tilde{u}^{\pi^{1,n}}$. Thus,

$$J_n(u^m) = J_n(\tilde{u}^{\pi^{1,n}}) = J_1(\tilde{u}^m) = J_1(u_n, \dots)\,.$$

Now, J_1 is by hypothesis convex in its first argument, here u_n. Therefore J_n is convex in u_n. ▮

Finally, we shall use the corollary of the following theorem[1]:

Theorem A.3 *Let $\{J_n : \mathsf{U}^m \to \mathbb{R}\}, n = 1, \dots, m$ be a game with identical players. Let $u \in \mathsf{U}$ and $u^{\times m} = (u, u, \dots, u) \in \mathsf{U}^m$. Then*

$$\forall n \le m\,, \quad \mathrm{D}_n J_n(u^{\times m}) = \mathrm{D}_1 J_1(u^{\times m})\,.$$

Proof Observe first that obviously,

$$\forall n \le m\,, \quad J_n(u^{\times m}) = J_1(u^{\times m}).$$

Let now $\tilde{u}^m = (u + \delta u, u, \dots, u)$, and as previously $\pi^{1,n}$ be the permutation that exchanges 1 and n. Let perturb the n-th control in $J_n(u^{\times m})$ by δu. We get

$$J_n(u, \dots, u, u + \delta u, u, \dots, u) = J_n(\tilde{u}^{\pi^{1,n}}) = J_1(\tilde{u}^m).$$

Therefore, the differential quotients involved in $\mathrm{D}_n J_n(u^{\times m})$ and $D_1 J_1(u^{\times m})$ are equal, hence the result. ▮

Corollary A.2 *If $u_1 \mapsto J_1(u^m)$ is convex, an interior solution $\hat{u} \in \mathsf{U}$ of the equation*

$$D_1 J_1(u^{\times m}) = 0 \tag{34}$$

yields a uniform Nash equilibrium $\hat{u}^{\times m}$.

A.2 Examples of games with identical players

The best known example of game with identical players is Cournot's duopoly. This, incidentally, is an *aggregative game* according to the definition of [17], which are a sub-class of games with identical players. We propose three 2-player games with identical players with different structures of equilibria. Let $i \in \{1, 2\}$:

$$\Pi_i(u_1, u_2) = (u_1 - u_2)^2 - a u_i^2\,. \tag{35}$$

[1]Where we use Dieudoné's notation $\mathrm{D}_k J$ for the partial derivative of J with respect to its k-th variable

A.2.1 Example 1

In this example, $\mathsf{U} = \mathbb{R}$ and $1 < a < 2$. Then $u_1 \mapsto \Pi_1(u_1, u_2)$ is concave for all u_2. Moreover

$$\mathrm{D}_1\Pi_1(u_1, u_2) = 2(1-a)u_1 - 2u_2$$

so that the unique maximum in u_1 is reached at $u_1 = -u_2/(1-a)$. Therefore a Nash equilibrium requires that

$$\begin{aligned}(a-1)u_1 + u_2 &= 0\,,\\ u_1 + (a-1)u_2 &= 0\,.\end{aligned}$$

The determinant of the matrix of this system is $a(a-2) < 0$. Therefore, the matrix is invertible, the only solution is $u_1 = u_2 = 0$. There is a single (pure) Nash equilibrium, which is uniform.

The question of whether there can exist a mixed Nash equilibrium is investigated as follows: let u_2 be a random variable (a mixed strategy). Clearly, then

$$\mathbb{E}\Pi_1(u_1, u_2) = (1-a)u_1^2 - 2\mathbb{E}(u_2)u_1 + \mathbb{E}(u_2^2)\,.$$

This has a unique maximum at $u_1 = -\mathbb{E}(u_2)/(a-1)$. Therefore Player 1's strategy is necessarily pure, but then Player 2's strategy also.

A.2.2 Example 2

We use the same example as above, but with $a = 2$. Then any pair $(u_1, u_2) = (u, -u)$ solves the Nash necessary condition, and in view of the concavity of the payoffs, is a (pure) Nash equilibrium. Indeed $\Pi_1(-v, v) - \Pi_1(u, v) = (u+v)^2 \geq 0$, and symmetrically $\Pi_2(u, -u) - \Pi_2(u, v) = (u+v)^2 \geq 0$. The set of Nash equilibria is, as predicted, invariant by a permutation $u_1 \leftrightarrow u_2$. (However, $\Pi_i(u, -u) = -2u^2$, so that both players prefer the equilibrium $(0, 0)$.)

No mixed equilibrium is possible for the same reason as above.

A.2.3 example 3

We use now $\mathsf{U} = [0, 1]$ and $a < 1$. Now $u_1 \mapsto \Pi_1(u_1, u_2)$ is convex for all u_2. Therefore a maximum in u_1 can only be reached at $u_1 = 0$ or $u_1 = 1$. Observe that

$$\Pi_1(1, u_2) - \Pi_1(0, u_2) = 1 - a - 2u_2\,.$$

Therefore, for $u_2 < (1-a)/2$, the maximum of Π_1 is reached at $u_1 = 1$, while for $u_2 > (1-a)/2$, it is reached at $u_1 = 0$. We therefore find two pure Nash equilibria: $(1, 0)$ and $(0, 1)$.

Indeed, once it is established that pure Nash equilibria can only be found at $u_i \in \{0, 1\}$, we can investigate the matrix game

$u_1 \backslash u_2$		0		1
		0		$1-a$
0	0		1	
		1		$-a$
1	$1-a$		$-a$	

The two pure Nash equilibria appear naturally. We can also look for a mixed equilibrium, obtained for $\mathbb{P}(u_i = 0) = (1+a)/2$, $\mathbb{P}(u_i = 1) = (1-a)/2$. In that case $\mathbb{E}(u_2) = (1-a)/2$. This an mixed equilibrium of the matrix game, *and also an equilibrium of the game over the unit square*, since the maxima can only be attained at 0 or 1. (The fact that also $\Pi_1(0, (1-a)/2) = \Pi_1(1, (1-a)/2)$ is a coincidence, due to the fact that $\Pi_1(1, u_2) - \Pi_1(0, u_2)$ is affine in u_2.)

B Continuous Isaacs equation

B.1 Modified, bounded m, problem

We first evaluate the following mathematical expectation, given t_m:

$$\mathcal{S}^m = \mathbb{E}\left[\int_{t_m}^{t_{m+1}} \mathrm{e}^{-\rho t} L^m(t, x(t), u^m(t))\,\mathrm{d}t + \mathrm{e}^{-\rho t_{m+1}} V^{m+1}(t_{m+1}, x(t_{m+1})\right].$$

given that both $L^m(t)$ and $V^{m+1}(t)$ are taken equal to zero if $t > T$. We have

$$\mathcal{S}^m = \mathrm{e}^{-\lambda^m(T-t_m)} \int_{t_m}^{T} \mathrm{e}^{-\rho t} L^m(t, x(t), u^m(t))\,\mathrm{d}t +$$
$$\int_{t_m}^{T} \lambda^m \mathrm{e}^{-\lambda^m(\tau - t_m)} \left[\int_{t_m}^{\tau} \mathrm{e}^{-\rho t} L^m(t, x(t), u^m(t))\,\mathrm{d}t + \mathrm{e}^{-\rho\tau} V^{m+1}(\tau, x(\tau))\right] d\tau\,.$$

Exchanging the order of summations in the double integral, changing the name of the integration variable in the second, it comes, after cancellation of the first term with one of those coming from the double integral:

$$\mathcal{S}^m = \int_{t_m}^{T} \mathrm{e}^{-\lambda_m(t-t_m)-\rho t}\big(L^m(t, x(t), u^m(t)) + \lambda^m V^{m+1}(t, x(t))\big)\,\mathrm{d}t\,. \tag{36}$$

We turn to the Isaacs equation (20), and deal with it as if the Value functions V^m were of class C^1. Multiply both sides of the equation by $\exp(-\lambda(t - t_m) - \rho t)$

and rewrite it as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda^m(t-t_m)-\rho t}V^m(t,x(t)\right) + \mathrm{e}^{-\lambda^m(t-t_m)-\rho t}L^m(t,x(t),u^m(t)) \\ + \lambda^m \mathrm{e}^{-\lambda^m(t-t_m)-\rho t}V^{m+1}(t,x(t)) \geq 0\,,$$

being understood that the lagrangian derivative and L^m are evaluated at $u^m(t) = \{u(t), \hat{u}^{(m\setminus 1)}(t)\}$, and that the inequality becomes an equality for $u(t) = \hat{u}(t)$. Integrating from t_m to T, we recognize $\mathcal{S}^m$ and write

$$\mathrm{e}^{-\rho t_m}V^m(t_m,x(t_m)) \leq \mathrm{e}^{-(\lambda^m+\rho)T+\lambda^m t_m}V^m(T,x(T)) \\ + \mathbb{E}\left[\int_{t_m}^{t_{m+1}} L^m(t,x(t),u^m(t)) + \mathrm{e}^{-\rho t_{m+1}}V^{m+1}(t_{m+1},x(t_{m+1}))\right].$$

In the finite horizon version, we have $V^m(T,x) = 0$, so that the first term in the right hand side cancels, and we are left with

$$\mathrm{e}^{-\rho t_m}V^m(t_m,x(t_m)) \leq \\ + \mathbb{E}\left[\int_{t_m}^{t_{m+1}} L^m(t,x(t),u^m(t)) + \mathrm{e}^{-\rho t_{m+1}}V^{m+1}(t_{m+1},x(t_{m+1}))\right]$$

if player one, say, deviates alone from $\hat{u}^m(t)$, and equality if $u^m(t) = \hat{u}^{(m)}(t)$. In the infinite horizon case, use the fact that V^m is bounded to see that the same first term of the r.h.s. cancels in the limit as T goes to infinity.

With this last inequality, we proceed as in discrete dynamic programing: take the a priori expectation of both sides, sum for all $m \leq M$, cancel the terms that appear on both sides of the sum and use $t_1 = 0$ (the first player starts at time 0) to get

$$V^1(0,x_0) \leq \mathbb{E}\int_0^T \mathrm{e}^{-\rho t}L^{m(t)}(t,x(t),u^m(t))\,\mathrm{d}t = J_1(0,x_0,u^m)\,,$$

for $u^m(t) = \{u(t), u^{(m\setminus 1)}(t)\}$, and equality if $u^m(t) = \hat{u}^{(m)}(t)$.

Having restricted our search to state feedback strategies and to a uniform equilibrium of identical players, and ignoring the intrinsic fixed point problem that for each (m,t,x) the minimizing control be precisely $\hat{\phi}^m(t,x)$ used by all other players, the inequality in definition 2.3 defines a unique minimization problem. As a consequence, in the case where the functions V^m are not globally C^1, both the necessary and the sufficiency characters with viscosity solutions are derived from this calculation in the same way as for one-player control problems. But a major difference with that case is that here, existence is far from granted. On the one hand, the fixed point for each (m,t,x) may not exist, and on the other hand, if it always does, it might not define an admissible strategy as characterized in paragraph 3.1.2. The situation is more complex for many player games than for two player games, where one can dispense with state feedback strategies. For these difficult technical matters, see [10, 13, 23, 18].

B.2 Unmodified unbounded m problem

We aim to extend theorem 3.1 to the unmodified problem where the number of players who may join the game before the time T is unbounded, and therefore equation (20) involves an infinite number of functions V^m. We simplify the notations as follows. Given two admissible state feedbacks ϕ and ψ, let

$$G(\phi,\psi) = J_1(\{\phi, \psi^{\times m(t)\setminus 1}\})$$

and the same with upper index M (respectively N) be the corresponding quantity in the modified problem where $\lambda^M = 0$ (resp. $\lambda^N = 0$).

We make the following hypotheses which would need to be converted into hypotheses bearing on the data f^m and L^m of the problem, probably via the hamiltonian

$$H^m(t,x,p,u,v) = \langle p, f(t,x,\{u,v^{\times m\setminus 1}\})\rangle + L(t,x,\{u,v^{\times m\setminus 1}\})\,.$$

We endow the set of state feedbacks with the topology of L^1 and assume:

Hypothesis B.1

1. *The function $\phi \mapsto G(\phi,\psi)$ is, for all ψ quasi convex with a unique minimum and differentiable.*

2. *there exists a positive number β such that,*

$$\begin{aligned}&\forall M\in\mathbb{N}\,,\ \forall\phi,\chi,\psi\in\mathcal{A}\,,\ \forall\mu\in[0,1]\\ &G^M\big((1-\mu)\phi+\mu\chi,\psi)\big)\geq\\ &(1-\mu)G^M(\phi,\psi)+\mu G^M(\chi,\psi)-\frac{\beta}{2}\mu(1-\mu)\|u-v\|^2\,.\end{aligned}$$

   If $\phi\mapsto G^M(\phi,\psi)$ is of class C^2, this is equivalent to

$$\forall\phi,\chi,\psi\in\mathcal{A}\,,\quad |\langle \mathrm{D}_{11}G(\phi,\psi)\chi,\chi\rangle|\leq\beta\|\chi\|^2\,.$$

3. *For all M and ψ, the map $\phi\mapsto D_1G^M(\phi,\phi)$ is locally invertible in a neighborhood of zero with an inverse locally uniformly Lipshitz of modulus γ. If $(\phi,\psi)\mapsto G^M(\phi,\psi)$ is of class C^2, it suffices that the operator $\mathrm{D}_{11}G(\phi,\psi)+\mathrm{D}_{12}G(\phi,\psi)$ be onto, with an inverse uniformly bounded by a positive number γ.*

With this set of hypotheses, too abstract at this stage, we can prove the conjecture 3.1. We first prove a simple lemma.

Let $\mathbb{P}$ be the probability structure induced by the entry process in the original problem, $\mathbb{E}$ the mathematical expectation in that probability law, $\mathbb{P}^M$ the probability law induced by the modified problem with $\lambda^M = 0$, and $\mathbb{E}^M$ the mathematical expectation in that law. We prove the following lemma.

Lemma B.1 *Let $X(\omega)$ be a bounded random variable measurable on the sigma-field generated by the entry process. $\mathbb{E}^M X$ converges to $\mathbb{E}X$ as M goes to infinity.*

Proof In the original problem, let Ω^M be the set of events for which $m(T) < M$ and Ω_M the complement: events such that $m(T) \geq M$. These sets belong to the sigma-field generated by the entry process. We have

$$\mathbb{E}(X) = \int_{\Omega^M} X(\omega)\,\mathrm{d}\mathbb{P}(\omega) + \int_{\Omega_M} X(\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

and similarly for $\mathbb{E}^M X$. Now, both laws coincide over Ω^M. Therefore

$$\begin{aligned}|\mathbb{E}X - \mathbb{E}^M X| &= \left|\int_{\Omega_M} X(\omega)\,\mathrm{d}\big(\mathbb{P}(\omega) - \mathbb{P}^M(\omega)\big)\right| \\ &\leq \sup_{\omega\in\Omega_M} |X(\omega)|\,\big(\mathbb{P}(\Omega_M) + \mathbb{P}^M(\Omega_M)\big)\ .\end{aligned}$$

Notice finally that $\mathbb{P}(\Omega^M) = \mathbb{P}^M(\Omega^M)$, and therefore for their complements:

$$\mathbb{P}(\Omega_M) = \mathbb{P}^M(\Omega_M) = \mathbb{P}(m(T) \geq M) < \frac{(\Lambda T)^M}{M!}$$

which goes to zero with M. As a consequence, $\mathbb{E}^M X$ converges to $\mathbb{E}X$ as M goes to infinity. ∎

Let $M < N$ be two integers. Let φ^M and φ^N be the equilibrium feedbacks of the modified problems G^M and G^N respectively. Using the lemma, we see that given a positive number ε, there exists an integer K such that for any M and N lager than K, and any φ,

$$|G^M(\varphi,\varphi^N) - G^N(\varphi,\varphi^N)| \leq \varepsilon\,.$$

It follows that

$$\forall\varphi\,, G^M(\varphi,\varphi^N) \geq G^N(\varphi,\varphi^N) - \varepsilon \geq G^N(\varphi^N,\varphi^N) - \varepsilon \geq G^M(\varphi^N,\varphi^N) - 2\varepsilon\,.$$

From the fact that $G^M(\varphi^N, \varphi^N)$ is close to the minimum in ϕ of $G^M(\phi, \varphi^N)$ and hypothesis 2, we may derive that

$$\|\mathrm{D}_1 G^M(\varphi^N, \varphi^N)\| \le 2\sqrt{\beta\varepsilon}.$$

On the other hand, $\mathrm{D}_1 G^M(\varphi^M, \varphi^M) = 0$. From hypothesis 3 we conclude that

$$\|\varphi^N - \varphi^M\| \le 2\gamma\sqrt{\beta\varepsilon}\,.$$

Hence the sequence $\{\varphi^M\}$ is Cauchy, and thus converges to some $\varphi^\star$. Hence the $V^{m|M}$ converge, and because all satisfy the P.D.E. they converge in C^1.

The hypotheses B.1 can be made mode concrete with the following approach, only sketched here. We consider again the system of partial differential

$$\begin{aligned}
\rho W^m(t,x) &= W_t^m(t,x) + H^m(t,x,W_x^m(t,x),u,v) \\
&\quad + \lambda^m[W^{m+1}(t,x) - W^m(t,x)]\,, \\
W^m(T,x) &= 0\,.
\end{aligned}$$

When we set furthermore, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $W^{M+1}(t,x) = 0$ in the system above, this uncouples the equation for W^M from the other ones, and allows one to consider the system in decreasing order of m as a finite sequence of P.D.E's. We denote with $W^{m|M}$ such a family of solutions with $m \le M$.

Choose a pair of admissible state feedbacks ϕ and ψ. Consider the above system with, for all (t,x), $u = \phi^{m(t)}(t,x)$ and $v = \psi^{m(t)}(t,x)$. It follows from the analysis of the previous subsection that a viscosity solution exists and that, if $x(t_n) = x_n$

$$W^{m|M}(t_n, x_n) = G_n^M(\phi, \psi)\,.$$

And the $V^{m|M}(t,x)$ are the equilibrium values for a uniform equilibrium, obtained for $\phi(t,x) = \psi(t,x) = \varphi^M(t,x)$. We will use the shorthand notation u^M (respectively u^N) in the equations. As a consequence, for instance

$$V^{m|M}(t_m, x(t_m)) = G_m^M(\varphi^M, \varphi^M) = \min_\phi G_m^M(\phi, \varphi^M)\,.$$

It follows that each $\varphi^{m|M}$ minimizes the criterion $\mathcal{S}^m$ (36), and that, for M and N large enough, $|W^{m|M} - W^{m|N}|$ is smaller than an arbitrarily chosen ε. Let

$$H^m(t,x,p,u,v) = L^m(t,x,\{u, v^{\times m\setminus 1}\}) + \langle p, f^m(t,x,\{u, v^{\times m\setminus 1}\})\rangle\,.$$

The second derivative version of hypothesis B.1.2 derives from the standard second variation theory and the hypothesis that the solution $y(t)$ of the linear differential equations

$$\dot y = \mathrm{D}_2 f^m(t,x,(\varphi^M(t,x))^{\times m})y + \mathrm{D}_{u_1} f^m(t,x,(\varphi^M(t,x))^{\times m})w(t)$$

and the second derivative

$$\begin{pmatrix} \mathrm{D}_{22}H^m & \mathrm{D}_{24}H^m \\ \mathrm{D}_{42}H^m & \mathrm{D}_{44}H^m \end{pmatrix}$$

are uniformly bounded. Then we conclude that the L^∞ norm

$$\|\mathrm{D}_4 H^m(t, x, W_x^{m|M}(t,x), u^N, u^N)\|_\infty$$

is less than $2\sqrt{\beta\varepsilon}$, and that thus the equation

$$\mathrm{D}_4 H^m(t, W_x^{m|M}(t,x), u, u) = 0$$

has a solution u^M close to u^N, leading to the conclusion that the sequence $\{\varphi^M\}$ is Cauchy and thus convergent.