# Comparison of two numerical schemes for barrier and value of a simple pursuit-evasion game

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**Abstract :** We investigate the barrier of a simple pursuit-evasion game for which we are able to compare two theoretical and numerical approaches. One is directly based on the capture time, and relies on the theory of viscosity envelope solutions of Isaacs equation. The second one, introduced by one of the authors, transforms the game in one of approach (or $L_\infty$ criterion). This second approach gives both a new characterization of barriers and a new, potentially more robust, numerical scheme for the determination of barriers. We provide a detailed analytical solution of the various problems thus raised, and use it as a benchmark for the numerical schemes.

## 1 Introduction

We revisit a well known one-dimensional second-order servomechanism problem, proposed by Bernhard in [5], with a new approach that transforms the game in one of approach (or $L_\infty$ criterion). This simple pursuit-evasion game allows us to compare the traditional approach with this new one, both on theoretical and numerical points of view.

We present numerical schemes for the computation of the value functions of the two versions of the game (the *game in time* and the *game in distance*), with a particular emphasis on the determination of the *barrier* of the pursuit-evasion game. Our methods use the theory of *viscosity solutions* for the Isaacs equation (see Barles [2] or Crandall, Ishii, Lions [10] for the state of the art), which is an alternative to the *viability* approach proposed by Cardaliaguet, Quincampoix, Saint-Pierre [7, 8] or the *minimax* solutions of Subbotin [18].

The first scheme is based on a finite difference approximation of the discounted capture time function, involving viscosity lower-envelope solutions of the Isaacs equation (cf the work of Bardi, Bottacin, Falcone [1]). The associated numerical scheme computes an approximation by discrete stochastic games, introduced by Pourtallier, Tidball [16] following the work of Kushner [14].

Nevertheless, when a barrier occurs in the capture-evasion game splitting the state space into capture and evasion areas, a detection of infinite value of the capture time function is required in order to characterize this manifold. (See Bernhard [6] for a state of the art of barriers of differential games). ¿From a numerical point of view, this previous scheme does not seem to be well suited for an accurate detection, since the barrier sought appears as the boundary of the set where the discounted value function is strictly less than one, a level it reaches with zero slope.

The second approach considers an approximation of the minimum oriented distance from the target, involving viscosity upper-envelope solutions of a variational inequality (see Rapaport [17]). The numerical scheme computes a monotone sequence of continuous solutions for a sequence of perturbated Hamiltonians, using again approximation by discrete stochastic games (see Crepey [11]). The barrier for the game in time is then determined by the zero level set of the value function for the game in distance, an intrinsically robust determination, as the gradient is not zero there. Moreover, this gradient also measures sensitivity of the barrier location with respect to the target.

Finally, we illustrate these methods with numerical experimentations, using the analytical solutions to benchmark the numerical results we obtain.

## 2 Presentation of the game

Consider a one-dimensional second-order plant :

$$\ddot{y} = \beta\, v, \qquad |v| \leq 1\,,$$

where the objective is to keep $y$ as close as possible to a set point $z$ subject to an unknown drift :

$$\dot{z} = \alpha\, u, \qquad |u| \leq 1\,.$$

More precisely, for a given positive number $\gamma$, we are looking for a (state feedback) control law $v^*()$ that guarantees $|y(t) - z(t)| \leq \gamma$ for all $t \geq 0$ whatever is the disturbance $u()$.

Considering the state vector :

$$x := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y - z \\ \dot{y} \end{pmatrix},$$

this problem can be formulated as a pursuit-evasion game, whose dynamics are :

$$\begin{cases} x(0) &= x_0\,, \\ \dot{x}(t) &= f(x(t), u(t), v(t)) := \begin{pmatrix} x_2(t) - \alpha u(t) \\ \beta v(t) \end{pmatrix}, \qquad |u(t)| \leq 1, \quad |v(t)| \leq 1\,, \end{cases}$$

with the target set :

$$\mathcal{T} := \{x \in \mathrm{I\!R}^2 \mid |x_1| \geq \gamma\}\,.$$

The usual way to study the existence of such a control law $v^*$ is to study the *game in time* (cf Isaacs [13]) :

$$V(x_0) = \sup_{\psi[]} \inf_{u(\cdot)} t^c(x_0, u, v)\,,$$

where $t^c(x_0, u, v) = \inf\{t \geq 0 \,|\, t \in \mathcal{T}\}$ is the capture time, $(v(\cdot) = \psi[u(\cdot)], u(\cdot))$ are admissible controls, and $\psi[]$ belongs to a set of strategies defined below. This game has been investigated in detail by Masle [15] and Bernhard in [5]. A particular emphasis is made on the existence and the characterization of the barrier, that splits the state space between initial positions for which there exists a strategy for the player $v$ guaranteeing a termination in finite time from its complementary.

Alternatively, we study another criterion related to the *game in distance* :

$$W(x_0) = \sup_{\psi[]} \inf_{u(\cdot)} \left[\inf_t d^o(x(t), \mathcal{T})\right],$$

where $(u, v)$ belong to the same sets of strategies as for the previous game and $d^o$ is the oriented distance function :

$$d^o(x, \mathcal{T}) = \begin{cases} d(x, \mathcal{T}) & \text{if } x \notin \mathcal{T}\,, \\ -d(x, \partial\mathcal{T}) & \text{otherwise}\,. \end{cases}$$

Here, $d^o(x, \mathcal{T}) = \gamma - |x_1|$, and we shall propose a new analytical resolution of this game. The barrier of the game in time is then determined by the set of points $\mathcal{B} = \{x \,|\, W(x) = 0\}$. Altough this criterion does not provide any information on the capture time, it characterizes the sensitivity with respect to the target, which is of complementary interest compared with the traditional approach.

# 3 Analytical solutions

## 3.1 Preliminaries

We shall define more precisely for which class of strategies the value functions $V$ and $W$ defined above should be considered :

**Definition 1** (VREK STRATEGIES) *Let $\mathcal{U}$, $\mathcal{V}$ be the sets of measurable functions from* $\mathbb{R}^+$ *to* $[-1,1]$ *or* open-loop *controls. $u(\cdot)$ is sought among $\mathcal{U}$ and $\psi[\,]$ among the* non-anticipative VREK strategies *:*

$$\{\ \psi[\,]\ : u \in \mathcal{U} \longmapsto \psi[u] \in \mathcal{V}\ \}$$

*such that :* $\forall u \in \mathcal{U}$, $[\forall t \leq t',\ u(t) = u'(t)] \Longrightarrow [\forall t < t',\ \psi[u](t) = \psi[u'](t)]$.

*Similarly, we can consider strategies for a reversed order of the players : $v(\cdot)$ is then sought among $\mathcal{V}$ and $\phi[\,]$ among the* non-anticipative VREK strategies *:*

$$\{\ \phi[\,]\ : v \in \mathcal{V} \longmapsto \phi[v] \in \mathcal{U}\ \}$$

*such that :* $\forall v \in \mathcal{V}$, $[\forall t \leq t',\ v(t) = v'(t)] \Longrightarrow [\forall t < t',\ \phi[v](t) = \phi[v'](t)]$.

These classes of strategies are well suited to characterize the value functions in terms of *viscosity solutions* (see Crandall, Ishii, Lions [10] and Barles [2]), for which we recall the definition :

**Definition 2** (VISCOSITY SOLUTIONS) *Consider a first order partial differential equation on a domain $\Omega$ :*

$$\begin{array}{c} H(x, V(x), \nabla V(x)) = 0, \quad x \in \Omega \\ (\textit{possibly with a boundary condition } V(x) = K,\ x \in \partial\Omega) \end{array} \tag{1}$$

*Let $D^+V(x)$ (*resp. *$D^-V(x)$) denote the Fréchet super-(*resp. *sub-)differential of the locally bounded function $V$ at $x$,* i.e. *the set of formal gradients $p$, such that*

$$V(y) \leq V(x) + < p, y - x > -\theta(||y - x||)$$
$$(\textit{resp. } V(y) \geq V(x) + < p, y - x > -\theta(||y - x||))$$

*for arbitrary small $\theta$ and $y$ close to $x$.*

*i) A* subsolution *(resp.* supersolution*) of $H$ on $\Omega$ is a* u.s.c. *(resp.* l.s.c.*) locally bounded function $V$* s.t. *$H(x, V(x), D^+V(x)) \geq 0$ (resp. $H(x, V(x), D^-V(x)) \leq 0$) on $\Omega$.*

*ii) A* Dirichlet subsolution *(resp.* Dirichlet supersolution*) of (1) on $\overline{\Omega}$ must satisfy also $V \leq K$ (*resp. *$V \geq K$) on $\partial\Omega$.*

*iii) If a* subsolution *(resp.* supersolution*) of $H$ on $\Omega$ satisfies at least $H(x, V(x), D^+V(x)) \geq 0$ (resp. $H(x, V(x), D^-V(x)) \leq 0$), wherever it fails to satisfy $V \leq K$ (resp. $V \geq K$) on $\partial\Omega$, we shall call it a* subsolution *(resp.* supersolution*) of (1) on $\overline{\Omega}$.*

*iv) A (resp. Dirichlet)* viscosity solution *means a function that is both a (resp. Dirichlet) sub- and a super-solution.*

*v) The viscosity upper (resp. lower) envelope solution on $\Omega$ (resp. on $\overline{\Omega}$) means the largest viscosity sub-solution on $\Omega$ (resp. the smallest Dirichlet viscosity super-solution on $\overline{\Omega}$).*

Alternatively, we shall also consider classes of *feedback strategies* :

**Definition 3** (FEEDBACK STRATEGIES) $\Phi \subset \{\phi : (t, x) \mapsto \phi(t, x) \in [-1, 1]\}$ *and* $\Psi \subset \{\psi : (t, x) \mapsto \psi(t, x) \in [-1, 1]\}$ *are admissible classes of* feedback strategies *if :*

*i) Open-loops are admissible :* $\mathcal{U} \subset \Phi$ *and* $\mathcal{V} \subset \Psi$.

*ii)* $\Phi$ *and* $\Psi$ *are closed by concatenation (* i.e. *switching from one strategy in the set to another one, at an intermediate instant of time, is allowed).*

*iii)* $\forall(\phi,\psi) \in \Phi \times \Psi$, $\forall x_0$, *there exists an unique solution of* $\dot{x} = f(x, \phi(., x), \psi(., x))$ *over* $\mathbb{R}^+$, *leading to measurable controls :* $u(.) = \phi(., x(.)) \in \mathcal{U}$ *and* $v(.) = \psi(., x(.)) \in \mathcal{V}$.

**Remarks**

1. These properties do not uniquely define the pair $(\Phi, \Psi)$.
2. It is clear that such classes exist and are sub-classes of VREK non-anticipative strategies.

### 3.2 Game in time

We sketch here the analysis of [15] and [5], according to the classical Isaacs-Breakwell theory. From dimensional analysis, it is easy to see that the only meaningfull parameter in that game is the ratio

$$p = \frac{\beta\gamma}{\alpha^2}\,.$$

First find the usable part of the capture set, here made up of two symmetric pieces: $\{x_1 = \gamma, x_2 > -\alpha\}$ and $\{x_1 = -\gamma, x_2 < \alpha\}$. The boundary of the usable part (BUP) is thus made up of the two points $(x_1 = \varepsilon\gamma, x_2 = -\varepsilon\alpha)$ for $\varepsilon = \pm 1$. From the BUP, attempt to construct a natural barrier. The semipermeable normal is $(\nu_1 = -\varepsilon, \nu_2 = 0)$. Given the hamiltonian of the game of kind,

$$H = \nu_1(x_2 - \alpha u) + \nu_2 \beta v\,,$$

we see on the one hand that the semipermeable controls are $u = \operatorname{sign}\nu_1$ and $v = \operatorname{sign}\nu_2$, and on the other hand that the adjoint equations give

$$\begin{aligned} \dot{\nu}_1 &= 0\,, \\ \dot{\nu}_2 &= -\nu_1\,. \end{aligned}$$

Initialized with the proposed semipermeable $\nu$'s on the BUP, this yields two parabolas with the controls $u = v = -\varepsilon$: (we call $t_1$ the final time)

$$\begin{aligned} x_1(t) &= \varepsilon[\gamma - \frac{\beta}{2}(t_1 - t)^2]\,, \\ x_2(t) &= \varepsilon[-\alpha + \beta(t_1 - t)]\,. \end{aligned}$$

These intersect the "other edge" of the game space, *i.e.* the straight line $x_1 = -\varepsilon\gamma$, at $x_2 = \varepsilon(-\alpha + 2\sqrt{\beta\gamma})$. We must now distinguish two cases depending on whether these points are in the usable part or the non usable part.

The simple case is when this intersection happens in the non usable part, which is the case if $p > 1$. In that case the two parabolas together with the pieces of (non usable) capture set boundary that join them (the thick lines in figure 2) indeed form a barrier, separating an escape zone "inside" from the capture zone outside.

Indeed that composite curve is a barrier. At all the points where it is smooth, the semipermeability condition holds (or, on the capture set boundary, a stronger inequality for the evader). At its points of non differentiability, the two intersections of the parabolas with the opposite capture sets, the evader may play according to the parabola's dictum, *i.e.* $v = \varepsilon$. This insures that the state remains inside the escape zone, since $\dot{x}_1$ has the desired sign whatever the controls.

Outside that region, we can construct a complete field of trajectories, that happen to be parabolas translated from the previous ones parallel to the $x_1$ axis. It is a simple matter to check that they define a value function

$$V(x) = \frac{1}{\beta}\left[\alpha + \varepsilon x_2 - \sqrt{(\alpha + \varepsilon x_2)^2 - 2\beta(\gamma - \varepsilon x_1)}\right] ,$$

with $\varepsilon = 1$ in the upper region and $\varepsilon = -1$ in the lower region. Inside the escape zone, of course $V = +\infty$. (We should emphasize that the value function computed here is Isaacs', not the function $V$ of the next paragraphs which is its Kruskov transform.)

In the case $p < 1$, the two parabolas intersect each other inside the game space, delineating what we shall call the *lens*. This lens is *not* an escape zone however : the corners "leak". Following the classical analysis of intersection of barriers, we have an intersection with incoming trajectories that cross it. Therefore the composite surface is *not* a barrier.

As a matter of fact, the lens is the *intersection* of the proposed safety zones defined by each parabola. Therefore, to stay in it, the state should cross none of the parabolas, a feat the pursuer cannot enforce since the required controls are +1 for one of the parabolas, -1 for the other one. Upon reaching such a corner, the pursuer can keep its optimal control according to the incoming parabola, and the state necessarily leaves the "lens".

In that case there is no escape zone. But the complete solution in terms of singularities of Isaacs'equation is extremely involved. John Breakwell, in a private communication, has even shown that the number of commutations of the optimal controls from +1 to -1 and conversely can be arbitrarily large, depending on the initial state and the value of $p$.

### 3.3 Game in distance

Following Rapaport [17], under technical assumptions, the value function $W$ for the game in distance is the viscosity upper-envelope solution (*i.e.* the largest u.s.c. sub-solution) of the following variational inequality :

$$H(x, W(x), \nabla W(x)) = \min\left[d^o(x, \mathcal{T}) - W(x)\, ,\, \min_u \max_v \nabla W(x).f(x, u, v)\right] = 0\, . \qquad (2)$$

When $W(x) < d^o(x, \mathcal{T})$, the characteritic fields of the considered game are obtained for $u^*(x) = \mathrm{sign}\, \partial_1 W(x)$ and $v^*(x) = \mathrm{sign}\, \partial_2 W(x)$ :

$$\left\{\begin{array}{lcl} x_1(t) & = & \pm\beta t^2/2 + (x_1(0) \mp \alpha)t + x_1(0) \\ x_2(t) & = & \pm\beta t + x_2(0) \end{array}\right. \qquad (3)$$

A necessary condition for $t^*$ to minimize $t \mapsto \gamma - |x_1(t)|$ is to have $\dot{x}_1(t^*) = 0$, which gives :

$$\gamma - x_1(t^*) = \gamma \pm x_1(0) - \frac{(x_2(0) \mp \alpha)^2}{2\beta}.$$

This leads us to consider the following candidate $Z$ solution of the variational inequality :

**Definition 4**

$$Z(x) = \left\{\begin{array}{ll} \min(\,\gamma + x_1\,,\, P^+(x)\,) & \textit{when } x_2 \geq \alpha\,, \\ \min(\,\gamma - x_1\,,\, P^-(x)\,) & \textit{when } x_2 \leq -\alpha\,, \\ \min(\,\gamma - \alpha^2/\beta\,,\, P^+(x)\,,\, P^-(x)\,) & \textit{when } |x_2| \leq \alpha\,, \end{array}\right.$$

*with*

$$\left\{\begin{array}{lcl} P^+(x) & = & \gamma - x_1 - \dfrac{(x_2 + \alpha)^2}{2\beta}\,, \\ P^-(x) & = & \gamma + x_1 - \dfrac{(x_2 - \alpha)^2}{2\beta}\,. \end{array}\right.$$

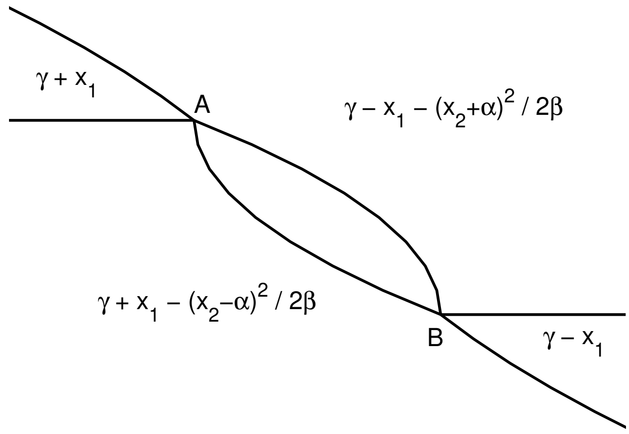


Figure 1: *Different areas defining the function $Z$.*

**Remarks**

1. $Z$ is maximum and constant equal to $\gamma - \alpha^2/\beta$ inside the "lens" delimited by two arcs of parabola :

$$\mathcal{L} := \{x \,|\, P^-(x),\, P^+(x) \geq \gamma - \alpha^2/\beta\} \,\cap\, \{|x_2| \leq \alpha\}.$$

2. The set of points where the function $Z$ is null is :

   i) void if $\gamma - \alpha^2/\beta < 0$,

   ii) otherwise equal to

$$\{P^+(x) = 0, x_1 \geq -\gamma, x_2 \geq -\alpha\} \,\cup\, \{P^-(x) = 0, x_1 \leq \gamma, x_2 \leq \alpha\}$$
$$\cup\, \{-\gamma\} \times [\alpha, 2\sqrt{\beta\gamma} - \alpha] \,\cup\, \{\gamma\} \times [-\alpha, \alpha - 2\sqrt{\beta\gamma}]$$

   (see figure 2).

We recognize in this last expression exactly the barrier found by Bernhard [5] for the game in time.

**Proposition 1** *$Z$ is a continuous viscosity solution of (2).*

**Proof** $Z$ is clearly continuous, below $d^o(., \mathcal{T})$. The constant value inside the lens $\mathcal{L}$ is equal to the common value kept by the three functions $\gamma - |x_1|$, $P^-(x)$ and $P^+(x)$ at points $x$ such that $|x_2| \leq \alpha$ and where they are equal, which are exactly the two points $A = (-\alpha^2/\beta, \alpha)$ and $B = (\alpha^2/\beta, -\alpha)$ (see figure 1).
Notice that requiring $Z$ to be a continuous viscosity solution of (2) is equivalent to :

$$\begin{cases} \min\limits_u \max\limits_v D^+Z(x).f(x,u,v) \geq 0\,, \\ Z(x) = d^o(x, \mathcal{T}) \text{ or } \min\limits_u \max\limits_v D^-Z(x).f(x,u,v) \leq 0\,. \end{cases}$$

Direct computation shows :

$$\begin{array}{lcl} x_2 \leq \alpha & \Rightarrow & \min\limits_u \max\limits_v \nabla P^-(x).f(x,u,v) = x_2 - \alpha - (x_2 - \alpha) = 0\,, \\ x_2 \geq -\alpha & \Rightarrow & \min\limits_u \max\limits_v \nabla P^+(x).f(x,u,v) = -x_2 - \alpha + (x_2 + \alpha) = 0\,, \\ x_2 \geq \alpha & \Rightarrow & \min\limits_u \max\limits_v \nabla d^o(x, \mathcal{T}).f(x,u,v) = x_2 - \alpha \geq 0\,, \\ x_2 \leq -\alpha & \Rightarrow & \min\limits_u \max\limits_v \nabla d^o(x, \mathcal{T}).f(x,u,v) = -x_2 - \alpha \geq 0\,, \end{array}$$

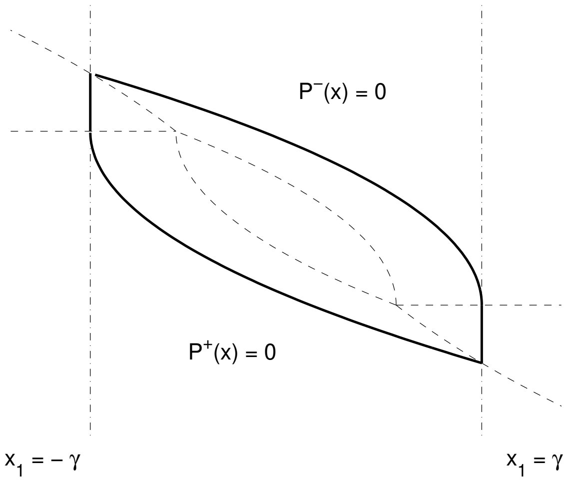


Figure 2: *The set of points $x$ where $Z(x) = 0$ (when $\gamma - \alpha^2/\beta > 0$).*

so $Z$ satisfies the variational inequality at its differentiable points.

Remind (see for instance Clarke [9]) that $\phi_1$, $\phi_2$ being two differentiable functions, at any point $x$ where $\phi_1(x) = \phi_2(x)$ and $\nabla\phi_1(x) \neq \nabla\phi_2(x)$, we have :

$$\begin{cases} D^- \min(\phi_1, \phi_2)(x) &= \emptyset\,, \\ D^+ \min(\phi_1, \phi_2)(x) &= \overline{\text{co}}\{\nabla\phi_1(x), \nabla\phi_2(x)\}\,. \end{cases}$$

We deduce that $D^+Z(x) = \emptyset$ at non-differentiable points $x$ and so that only the viscosity supersolution condition has to be checked at the boundary of the lens. But at such points $x$, $D^-(\gamma - \alpha^2/\beta)(x) = \{0\}$ then $D^-Z(x) = [0,1]\nabla P^{\pm}(x)$ and so $\min\limits_u \max\limits_v(x) D^-Z(x).f(x,u,v) = 0$. ∎

**Proposition 2** *$Z$ is the value function with feedback strategies (for the game in distance).*

**Proof** Consider the state space divided into the three domains :

$$\begin{array}{l} \mathcal{S} := \{\, x \mid Z(x) = P^+(x) \text{ or } Z(x) = \gamma - x_1 \,\} \\ \mathcal{I} := \{\, x \mid Z(x) = P^-(x) \text{ or } Z(x) = \gamma + x_1 \,\} \\ \mathcal{L} := \mathbb{R}^2 \setminus (\mathcal{S} \bigcup \mathcal{I}) \qquad (\text{ the “lens” as already defined above}) \end{array}$$

and the following strategies :

$$\tilde{u}(x) = \left| \begin{array}{ll} 1 & \text{if } x \in \mathcal{I} \\ -1 & \text{otherwise} \end{array} \right. \quad \text{and} \quad \tilde{v}(x) = \tilde{u}(x).$$

We first show that $Z$ is the minimal oriented distance function for the pair of strategies $(\tilde{u}, \tilde{v})$. Take an initial condition $x_0 = (x_{10}, x_{20}) \in \mathcal{S}$, the corresponding trajectory in $\mathcal{S}$ is supported by a parabola $\mathcal{P}$, parallel to the upper border of the lens $\mathcal{L}$, in the direction of decreasing $x_2$ and so touches at a finite time $t_{\mathcal{I}}$ the border of the domain $\mathcal{I}$. As the apogee of such parabolas is reached for $x_2 = -\alpha$, we have :

$$\min_{t \in [0, t_{\mathcal{I}}]} \{\gamma - x_1(t)\} = \left| \begin{array}{ll} P^+(x_0) & \text{if } x_{20} \geq -\alpha \\ \gamma - x_{10} & \text{otherwise} \end{array} \right.$$

On this part, we then have :

$$\begin{aligned}\min_{t\in[0,t_{\mathcal{I}}]}\{\gamma-|x_1(t)|\} &= \min\left[\min_{t\in[0,t_{\mathcal{I}}]}\{\gamma+x_1(t)\}\,,\ \min_{t\in[0,t_{\mathcal{I}}]}\{\gamma-x_1(t)\}\right]\\ &= \min\left[\gamma+x_{10}\,,\ \min_{t\in[0,t_{\mathcal{I}}]}\{\gamma-x_1(t)\}\right]\\ &= Z(x_0)\end{aligned}$$

(as $\dot{x}_1(t)\geq 0$ when $x_2(t)\geq -\alpha$ and $x_1(t)\geq 0$ when $x_2(t)\leq -\alpha$). On the domain $\mathcal{I}$, the trajectory is then supported by another parabola closer to the lens $\mathcal{L}$ than $\mathcal{P}$ and so cannot improve the cost $\min_{t\in[0,t_{\mathcal{I}}]}\{\gamma-|x_1(t)|\}$. By symmetry, we have the same result when $x_0\in\mathcal{I}$.

Consider finally $x_0\in\mathcal{L}$. The trajectory follows first a parabola parallel to the upper border of the lens, in the direction of decreasing $x_2$, and so touches in finite time the border of the domain $\mathcal{I}$. Then, it stays on it, moving towards the point $A$, then down again towards the point $B$ and so on. So, $\min_{t\geq 0}\{\gamma-|x_1(t)|\}$ is reached equivalently at points $A$ or $B$ for a value of the criterion equal to $\gamma-\alpha^2/\beta=Z(x_0)$.

Consider now the pair $(\tilde{u},v)$ for an arbitrary open-loop control $v\in\mathcal{V}$.
Take $x_0\in\mathcal{S}$. Necessarily $\dot{x}_1=x_2+\alpha$ and $\dot{x}_2\geq-\beta$ as long as the trajectory remains outside $\mathcal{I}$. This first part of the trajectory is so above the parabola $\mathcal{P}$ generated by the constant controls $(-1,-1)$, up to a certain time $t_{\mathcal{I}}$ (possibly infinite) when it enters $\mathcal{I}$ (necessarily with $x_2\leq-\alpha$) :

$$x_1(t)\geq x_1^{\mathcal{P}}(t),\ \forall t\leq t_{\mathcal{I}}\quad\text{and}\quad x_1^{\mathcal{P}}(t_{\mathcal{I}})\in\mathcal{I}.$$

We deduce that :

$$\inf_{t\geq 0}\{\gamma-|x_1(t)|\}\leq\min_{t\in[0,t_{\mathcal{I}}]}\{\gamma-x_1(t)\}\leq Z(x_0).$$

If $x_0\in\mathcal{I}$, we obtain by symmetry the same conclusion.
Consider now $x_0\in\mathcal{L}$. Remember that inside the lens $x_2\geq-\alpha$, so let $l=\inf_{t\geq 0}\{x_2(t)+\alpha\}$. Inside the lens, the dynamics in $x_1$ is : $\dot{x}_1=x_2+\alpha>l$. If $l>0$, the trajectory leaves $\mathcal{L}$ in finite time. So, we conclude that either the trajectory reaches $\partial\mathcal{L}$ in finite time, either it converges asymptotically towards the corner point $B$. In any case, the trajectory reaches the boundary of $\mathcal{L}$ (possibly in infinite time), which belongs also to the boundary of $\mathcal{S}$ or $\mathcal{I}$. Then, we have :

$$\inf_{t\geq 0}\{\gamma-|x_1(t)|\}\leq\max_{\xi\in\partial\mathcal{L}}Z(\xi)=Z(x_0),\quad\forall v\in\mathcal{V}$$

(as $Z$ is continuous on $\partial\mathcal{L}$). So, we conclude that :

$$Z(x_0)\geq\sup_{v\in\mathcal{V}}\left[\inf_{t\geq 0}d^o(x(t),\mathcal{T})\right]\geq\inf_{\phi\in\Phi}\sup_{\psi\in\Psi}\left[\inf_{t\geq 0}d^o(x(t),\mathcal{T})\right],\quad\forall x_0.\qquad(4)$$

Consider now the pair $(u,\tilde{v})$ for an arbitrary open-loop control $u\in\mathcal{U}$.
Take $x_0$ outside $\mathcal{L}$ and consider the $Z$-level set containing $x_0$ : $\mathcal{R}(x_0):=\{\,x\in\mathbb{R}^2\,|\,Z(x)=Z(x_0)\,\}$. According to the previous remark, $\mathcal{R}(x_0)$ is exactly the barrier of the game in time but for the dilated target :

$$\mathcal{T}'=\{\,x\in\mathbb{R}^2\,|\,|x_1|\geq\gamma-Z(x_0)\,\}.$$

So, with the strategy $\tilde{v}$, the compact set $\mathcal{C}(x_0):=\{\,x\in\mathbb{R}^2\,|\,Z(x)\geq Z(x_0)\,\}$ is $u$-invariant and we deduce that :

$$\inf_{t\geq 0}d^o(x(t),\mathcal{T})\geq\inf_{x\in\mathcal{C}(x_0)}d^o(x,\mathcal{T})=Z(x_0),\quad\forall u\in\mathcal{U}.$$

This last result is obtained for any $x_0$ such that $\mathcal{C}(x_0)$ contains the lens $\mathcal{L}$ and so is also true for $x_0 \in \partial\mathcal{L}$. Then $\mathcal{L}$ is also $u$-invariant (for the strategy $\tilde{v}$). For $x_0$ belonging to $\mathcal{L}$, we have :

$$\inf_{t\geq 0} d^o(x(t),\mathcal{T}) \geq \inf_{x\in\mathcal{L}} d^o(x,\mathcal{T}) = \gamma - \frac{\alpha^2}{\beta} = Z(x_0), \quad \forall u \in \mathcal{U}.$$

So, we conclude that :

$$Z(x_0) \leq \inf_{u\in\mathcal{U}} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right] \leq \sup_{\psi\in\Psi} \inf_{\phi\in\Phi} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right], \quad \forall x_0. \tag{5}$$

¿From (4) and (5), we finally conclude that :

$$Z(x_0) \; = \; \sup_{\psi\in\Psi} \inf_{\phi\in\Phi} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right] \; = \; \inf_{\phi\in\Phi} \sup_{\psi\in\Psi} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right].$$

∎

**Corollary 1** *If there exists a saddle point in VREK strategies :*

$$W(x_0) = \sup_{\psi[\,]} \inf_{u\in\mathcal{U}} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right] = \inf_{\phi[\,]} \sup_{v\in\mathcal{V}} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right], \quad \forall x_0$$

*then* $W = Z$.

**Proof** We have, clearly :

$$W(x_0) = \sup_{\psi[\,]} \inf_{u\in\mathcal{U}} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right] \geq \sup_{\psi\in\Psi} \inf_{\phi\in\Phi} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right] = Z(x_0),$$

$$Z(x_0) = \inf_{\phi\in\Phi} \sup_{\psi\in\Psi} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right] \leq \inf_{\phi[\,]} \sup_{v\in\mathcal{V}} \left[\inf_{t\geq 0} d^o(x(t),\mathcal{T})\right] = W(x_0),$$

and we conclude that $W(x_0) = Z(x_0)$, $\forall x_0$. ∎

**Remark** By definition, the barrier of capture-evasion game with feedback strategies is nothing else than the zero level set of the function $Z$. So, existence condition and determination of the barrier are both derived explicitly from $Z$.

# 4 Numerical schemes

We shall study numerical approximations of the value functions $V$ and $W$ on a given subset of the state space $\mathcal{E} = \mathbb{R}^2$. For the game in time, the domain of definition of the value function $V$ is then $\Omega = \mathcal{E} \setminus \mathcal{T}$ (for the game in distance, we shall simply say that $\Omega = \mathcal{E}$).

## 4.1 Preliminaries

**Definition 5** (Kruzkov transformation) *Let $U$ denote the Kruzkov transform of the value function of the game, where by definition Kruzkov transform is : $\phi(\xi) = 1 - \exp(-\xi)$.*

We recall how $U$ is related to the discounted version of the game :

**Proposition 3** *$U$ is the value function (in the same meaning) of the discounted differential game, where by discounted game we mean the game with the same dynamics as before and the criterion $t^c$ or $\inf_t d^o$ replaced by $\phi(t^c)$ or $\inf_t \phi(d^o)$, by monotonicity of $\phi$.*

¿From now on, $V$ and $W$ will refer to the value functions of the discounted games, which have the numerical advantage to be bounded from above by 1.

**Definition 6** (DISCOUNTED HAMILTONIANS) *To the discounted games, we associate the following Hamiltonians :*

*i) for the game in time :*

$$H(x,s,p) = \min_u \max_v < p, f(x,u,v) > +1 - s\,, \tag{6}$$

*ii) for the game in ditance :*

$$H(x,s,p) = \min\left[\phi(d^o(x,\mathcal{T})) - s, \min_u \max_v < p, f(x,u,v) >\right]\,. \tag{7}$$

We introduce a finite difference scheme to approximate values of both differential games (game in time or game in distance). This scheme is nothing else than a classical *upwind* finite difference scheme for first-order PDEs, adapted by Kushner to optimal control problems [14] and later to differential games by Pourtallier, Tidball [16]. This scheme can also be interpreted as approximation by discrete stochastic games.

**Definition 7** (STARRED MESH) *By* starred mesh *of step $h$ on $\mathcal{E}$, we mean:*

1. *A discrete set of nodes $\mathcal{E}^h \subseteq \mathcal{E}$.*
2. *A local triangulation of the space around each node.*

The last point means that about each node $x \in \mathcal{E}^h$ we choose a finite set of $r$-simplices, or *cells*, with edges linking the nodes of $\mathcal{E}^h$ (a typical instance is the square mesh we effectively use in the algorithms). These simplices must meet at $x$, and fit together to cover the space about $x$ just once. Roughly said about each node $x$ a finite sequence of boxes is constructed on the nodes of $\mathcal{E}^h$, so that these boxes meet at $x$ and partition the space around $x$. The set of vertices of cells at $x$ (included) will be noted $\mathcal{V}^h(x)$.

More precisely, we shall consider families $(\mathcal{E}^h)_{h>0}$ of starred meshes that fill $\mathcal{E}$, in the sense that the union of the nodes of all $\mathcal{E}^h$ $(h > 0)$ is dense in $\mathcal{E}$. Moreover we shall assume the non degeneracy of these families :

**Assumption 1** (NON DEGENERACY) *There exist $\underline{\pi} \in (\pi/2, \pi)$, positive functions $\underline{\delta}(h)$ and $\overline{\delta}(h)$ going to 0 with $h$, such that for every $h > 0, x \in \mathcal{E}^h$ and $y \in \mathcal{V}^h(x)$ :*

*i) $\underline{\delta}(h) \leq ||y - x|| \leq \overline{\delta}(h)$.*

*ii) The angle between an edge $\overrightarrow{xy}$ of a cell at $x$ and the opposite face is less or equal than $\underline{\pi}$.*

**Remark** The classical square mesh satisfies all these requirements.

**Proposition 4** *For every $(x,u,v) \in (\mathcal{E}^h \times \mathcal{U} \times \mathcal{V})$, there is a unique family of nonnegative $f^y(x,u,v)$ $(y \in \mathcal{V}^h\{x\})$* s.t.*:*

$$f(x,u,v) \;=\; \sum_{y \in \mathcal{V}^h(x)} f^y(x,u,v)(y-x),$$

*and $f^y(x,u,v) = 0$ if $y \in \mathcal{V}^h(x)$ does not belong to the intersection of the cells at $x$ which meet $x + \mathbb{R}^+_\star f(x,u,v)$.*

**Proof** It is a decomposition of a vector on a base. ▮

Let then $\Delta t(x,u,v)$ be a notation for $\left(\sum_{z\in\mathcal{V}^h(x)} f^z(x,u,v)\right)^{-1}$ when $f(x,u,v)\neq 0$, $+\infty$ otherwise.

We shall need the concept of *weak limit* introduced by Barles, Perthame [3] :

**Definition 8** (Weak limits) *For any family of functions $V_h$ on $\mathcal{E}^h$ $(h>0)$, we introduce lower and upper weak limits $\underline{V},\overline{V}:\mathcal{E}\to\overline{\mathrm{IR}}$ when $h$ tends to 0 :*

$$\underline{V}(x)=\liminf_{\substack{x^h\in\mathcal{E}^h,x^h\to x\\ h\to 0}} V_h(x^h)\leq \limsup_{\substack{x^h\in\mathcal{E}^h,x^h\to x\\ h\to 0}} V_h(x^h)=\overline{V}(x). \tag{8}$$

¿From Bardi, Bottacin, Falcone, definition 2.2 [1] :

**Definition 9** (Double convergence) *We say that there is* double convergence *of the family $(V_{\epsilon,h})_{\epsilon,h>0}$ towards $V$ at $x\in\mathcal{E}$, where $V_{\epsilon,h}$, $V$ are real functions on $\mathcal{E}^h$, and write*

$$V(x)=\lim_{\substack{x^h\in\mathcal{E}^h,x^h\to x\\ h(\epsilon)\to 0}} V_{\epsilon,h}(x^h),$$

*if for any $\gamma>0$ there exists a function $\overline{h}:(0,+\infty)\to(0,+\infty)$, and $\overline{\epsilon}>0$, such that*

$$|V_{\epsilon,h}(x^h)-V(x)|\leq\gamma,$$

*for all $\epsilon\leq\overline{\epsilon}$, $h\leq\overline{h}(\epsilon)$, and $x^h\in\mathcal{E}^h$ s.t. $||x-x^h||\leq\overline{h}(\epsilon)$.*

### 4.2 Game in time

**Definition 10** (Discrete stochastic game) *On the discrete space $\mathcal{E}^h$ we define a stochastic game (cf Filar, Raghavan [12]), composed of the following elements.*

*i) A discrete target : $\mathcal{T}^h=\mathcal{E}^h\cap\mathcal{T}$ and domain : $\Omega^h=\mathcal{E}^h\setminus\mathcal{T}^h$.*

*ii) Transition probabilities :*

$$p(x,y\mid u,v)=\begin{cases} f^y(x,u,v)\Delta t(x,u,v) \text{ if } x\in\Omega^h, y\in\mathcal{V}^h(x),\\ 1 \text{ if } y=x\in\mathcal{T}^h,\\ 0 \text{ otherwise.}\end{cases}$$

*For the particular case when $f(x,u,v)=0$, we take :*

$$p(x,y\mid u,v)=\begin{cases} 1 \text{ if } y=x,\\ 0 \text{ otherwise.}\end{cases}$$

*iii) Instantaneous reward and discount factor :*

$$k(x,u,v)=\begin{cases}\phi[\Delta t(x,u,v)] & \text{when } x\in\Omega^h,\\ 0 & \text{when } x\in\mathcal{T}^h,\end{cases}$$

$$\beta(x,u,v)=\begin{cases}\exp[-\Delta t(x,u,v)] & \text{when } x\in\Omega^h,\\ 0 & \text{when } x\in\mathcal{T}^h.\end{cases}$$

Classically, the value $V_h$ of the discrete stochastic game so defined satisfies the following discrete averaged dynamic programming equation, known as Shapley equation:

$$V_h = T_h V_h, \tag{9}$$

where by definition $T_h$ is the following non-linear operator from the metric complete space of all bounded real sequences $\mathbb{R}_b^{\mathcal{E}^h}$ into itself:

$$[T_h V_h](x) = \min_u \max_v \{k(x,u,v) + \beta(x,u,v) E_x^{u,v} V_h\}. \tag{10}$$

Here $E_x^{u,v} V_h$ means the expected value of $V_h$ viewed as a functional on the Markov random field (9). $E_x^{u,v} V^h(x) = \sum_{y \in \mathcal{V}^h(x)} p(x, y \mid u, v) V_h(y)$. In particular for $x \in \mathcal{T}^h$ (9) gives:

$$V_h(x) = 0$$

The following proposition is drawn from Prop. 3.1 by Pourtallier, Tidball [16].

**Proposition 5** *$T_h$ is contractive from $\mathbb{R}_b^{\mathcal{E}^h}$ to itself, so that Shapley equation (9) admits a unique solution $V_h$.*

Now, we relate this Shapley solution with the viscosity Dirichlet lower envelope solution of the Isaacs equation on $\overline{\Omega}$ :

$$\begin{cases} H(x, V(x), \nabla V(x)) = 0, & x \in \Omega \\ V(x) = 0, & x \in \partial\Omega \end{cases} \tag{11}$$

Denoting $\underline{V}$ and $\overline{V}$ the weak limits when $h \to 0$ of $V_h$, solutions of Shapley fixed point equations (9), have the fundamental result :

**Proposition 6** *$\overline{V}$ (resp. $\underline{V}$ ) is a viscosity subsolution (*resp. *supersolution) of Isaacs equation (11) on $\overline{\Omega}$.*

**Proof** See Pourtallier, Tidball [16] or Crepey [11] for application to differential games, following ideas of Barles, Souganidis [4]. ∎

When the discounted VREK value function $V$ is continuous, it can be infered that this scheme converges towards $V$ *i.e.* $\lim_{h \to 0, x^h \to x} V_h(x^h) = V(x)$. But when the value function turns out to be discontinuous, which is the case when a barrier occurs, we have to consider a double approximating scheme, adding a dilatation of the target according to the ideas introduced in Bardi, Bottacin, Falcone [1] :

**Definition 11** (Double approximating scheme) *For $\epsilon > 0$, define $\mathcal{T}_\epsilon = \{x \in \mathcal{E} \mid d(x, \mathcal{T}) \le \epsilon\}$, while $\Omega_\epsilon$ is $\mathcal{E} \setminus \mathcal{T}_\epsilon$.*
*Let $V_{\epsilon,h}$ be the value function of the stochastic game with $\Omega := \Omega_\epsilon$, and $\mathcal{T} := \mathcal{T}_\epsilon$.*

Following Th. 2.5 in Bardi, Bottacin, Falcone [1], we have :

**Proposition 7** *$V_{\epsilon,h}$ doubly converges towards the viscosity lower Dirichlet envelope solution of (11) on $\overline{\Omega}$.*

**Proof** See the work of Bardi, Bottacin, Falcone [1] or Crepey [11]. ∎

**Remarks**
1. Bardi, Bottacin, Falcone have shown that the viscosity lower Dirichlet envelope solution of the Isaacs equation on $\overline{\Omega}$ is the value function for Friedman-like strategies [1]. For capture-time problems, proving that it is also the VREK value function is still an open problem, excepted in the case where the VREK value function is continuous.
2. The dependancy $h(\epsilon)$ between the sequences $h \to 0$ and $\epsilon \to 0$ required to guarantee the practical convergence of the scheme is also an open problem.

### 4.3 Game in distance

Following Rapaport [17], for a given positive number $\epsilon$, we consider the $\epsilon$-game :

$$W^\epsilon(x_0) = \sup_{\psi} \inf_{u(\cdot),t} \left\{ d^o(x(t), \mathcal{T}) + \int_0^t \epsilon \, d\tau \right\} .$$

**Proposition 8** *$W^\epsilon$ is a non-increasing sequence of bounded continuous functions, unique viscosity solutions of the variational inequalities :*

$$\min \left[ d^o(x, \mathcal{T}) - W^\epsilon(x) \, , \, \min_u \max_v \nabla W^\epsilon(x).f(x,u,v) + \epsilon \right] = 0, \quad \forall x \in \mathcal{E}.$$

**Proof** See Rapaport [17]. ∎

The Hamiltonian associated to the discounted version of this $\epsilon$-game is then :

$$H^\epsilon(x,s,p) = \min \left[ \phi(d^o(x,\mathcal{T})) - s \, , \, \min_u \max_v < p, f(x,u,v) > +\epsilon(1-s) \right] . \quad (12)$$

The scheme described in previous section to compute numerically a continuous value can be adapted here to approximate $W^\epsilon$. More precisely, the dynamic programming for an appropriate approximation $W^\epsilon_h$ of $W^\epsilon$ on a grid of $\mathcal{E}_h$ yields the Shapley-like equation :

$$W^\epsilon_h = T^\epsilon_h W^\epsilon_h \, , \quad (13)$$

with

$$\begin{aligned} [T^\epsilon_h W^\epsilon_h](x) &= \min \left[ \phi[d^o(x,\mathcal{T})] - W^\epsilon_h(x) \, , \, \min_u \max_v k(x,u,v) + \beta(x,u,v) E^{u,v}_x W^\epsilon_h \right] , \\ k(x,u,v) &= \phi(\epsilon \Delta t(x,u,v)) \quad \text{and} \quad \beta(x,u,v) = exp(-\epsilon \Delta t(x,u,v)) \end{aligned} \quad (14)$$

(remind that there is no boundary condition for this game).

$T^\epsilon_h$ is a contractive operator on $\mathbb{R}^{\mathcal{E}^h}_b$, as $T_h$ defined through Shapley equation used to be for the game in time (noticing that whatever are three real numbers $a, b, c$, we have $|\min(a,b) - \min(a,c)| \leq |b-c|$). So the fixed point equation (13) defines a unique $W^\epsilon_h$.

**Theorem 1** *$W^\epsilon$ converges doubly to the viscosity upper-envelope solution of (2) on $\Omega$.*

**Proof** See Crepey [11]. ∎

**Remark** It is still an open problem to know when the viscosity upper-envelope solution coincides with the VREK value function of the game.

# 5 Algorithms

To approximate the capture time or the minimum oriented distance, we are led to solve the Shapley equation (10) with a target dilated by $\epsilon$ or (13). But these equations are infinite algebraic systems, since an infinite number of edges are needed to cover the whole state space $\mathcal{E}$. So, their numerical resolutions require to localise a bounded *window* of interest. All transitions from $x$ in the window to $y$ outside the window are replaced by transitions from $x$ to $x$. Moreover, we need also to discretize the control sets into $U_f, V_f$. We use a rough discretization as usual for such problems, without prejudice on the quality of the results. Indeed most of the optimal controls are *bang bang* or median.

In the following experiments, we have used the set of parameters $(\alpha, \beta, \gamma) = (3, 2, 5)$, a truncation of $20 \times 20$ centered, a domain $\mathcal{E}^h_b$ of about $10^5$ nodes and sets of discretized controls of 5 values (experiments with more values have been made without any significant improvement on the precision of the results).

### 5.1 Game in time

A first possible algorithm to solve the fixed point equation (9) with $\Omega$ replaced by $\Omega_\epsilon$ is the Shapley one *i.e.* iterations on the values :

$$V_h^{n+1}(x) = \min_{u\in U_f}\max_{v\in V_f}\{k(x,u,v)+\beta(x,u,v)E_x^{u,v}V_h^n\},\ x\in\mathcal{E}_b^h\,.$$

This is a gradient method, as remarked by Filar, Raghavan [12]. Therefore its convergence is quite slow, consequently it is not the algorithm that we shall use in practice.

Another possible algorithm is the Hoffman-Karp one, making iterations on the policies, which consists in solving iteratively the linear systems ($n\in\mathbb{N}$) :

$$V_h^n(x) = k(x,u^{n-1}(x),v^{n-1}(x)) + \beta(x,u^{n-1}(x),v^{n-1}(x))E_x^{u^{n-1}(x),v^{n-1}(x)}V_h^n,\ x\in\mathcal{E}_b^h\,,$$

where $(u^{n-1}(x),v^{n-1}(x))\in U_f\times V_f$ minimaximizes $\{k(x,u,v)+\beta(x,u,v)E_x^{u,v}V_h^{n-1}\}$, and $V_h^0$ is arbitrary in $\mathbb{R}_b^{\mathcal{E}^h}$.

It is of Newton-Raphson type (see Filar, Raghavan [12]), converging much faster than the Shapley one, altough its convergence is not proved in general. It is the one we shall use in practice.

Figure 3 shows the value $\widehat{V}$ obtained for small values of $\epsilon$ and $h$, after an hundred of Newton-Raphson iterations, which was the required amount of iterations to obtain the stabilization of the algorithm. On figure 4, the results are presented in terms of level curves. Curves of level less than 0.9 are represented in light color, while those of greater level are darker. The lens that can be seen on this figure is the area $\{x\,|\,\widehat{V}(x)\geq 0.9\}$, therefore it approximates the evasion zone. The existence and the general shape of this evasion zone are consistent with the analytical results obtained for this game by Bernhard ([5] and section 3.2).

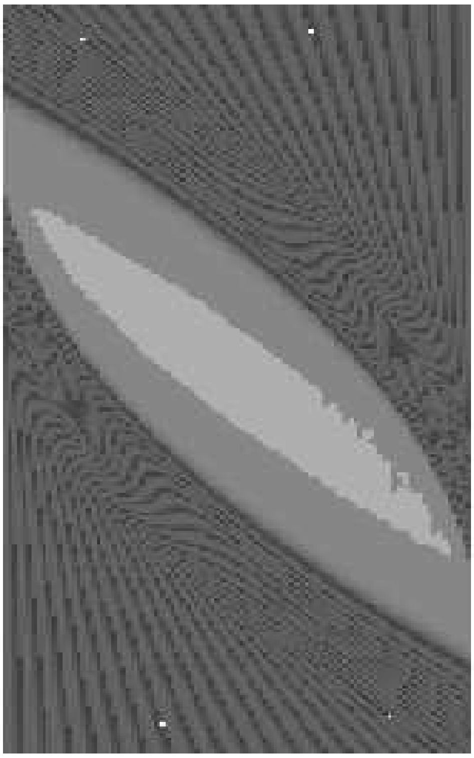

Figure 3: *Iso-values for the game in time.*

### 5.2 Game in distance

Let $(W_h^n)_{n\in\mathbb{N}}$ be the sequence $W_h^n = T_h^\epsilon W^{n-1}$ $(n \in \mathbb{N}^*)$, where $W_h^0$ is arbitrary in $\mathbb{R}_b^{\mathcal{E}^h}$. By Picard fixed point theorem, $W_h^\epsilon$ is the uniform limit of $W_h^n$ when $n \to \infty$. But as before we prefer to use a Newton-Raphson algorithm on the policies adapted from Hoffman- Karp, *i.e.* we solve iteratively the linear systems :

$$\begin{aligned} W_h^n(x) = \min[ \quad & \phi[d(x,\mathcal{T})] - W_h^n(x)\,, \\ & k(x,u^{n-1}(x),v^{n-1}(x)) + \beta(x,u^{n-1}(x),v^{n-1}(x)) E_x^{u^{n-1}(x),v^{n-1}(x)} W_h^n\ ], \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathcal{E}_b^h \end{aligned}$$

where $(u^{n-1}(x), v^{n-1}(x)) \in U_f \times V_f$ minimaximizes $k(x,u,v) + \beta(x,u,v)E_x^{u,v}W_h^{n-1}$ and $W_h^0 = \phi(d(\cdot,\mathcal{T}))$ on $\mathcal{E}^h$.

The figure 4 shows the obtained discounted value $\widehat{W}$ : The curves of negative level are represented in light color, while those of positive level are darker. These numerical results are consistent with the theoretical study of section 3.3, except in the anti-first diagonal corners of the window, where the edge effects are important. Looking at figure, we recognize the optimal fields (3), and their separation along the abstract target $\mathcal{T}^\star := \{x \,|\, W(x) = d^o(x,\mathcal{T})\}$. The *oppidum* that should split the optimal fields is clearly visible, even if it is not entirely formed at this stage of the computation. It corresponds to the inner lens whose upper border is already well drawn, while its lower border is still *in construction.* $\widehat{W}$ is roughly constant at its maximum value in this approximate *oppidum*, as expected.

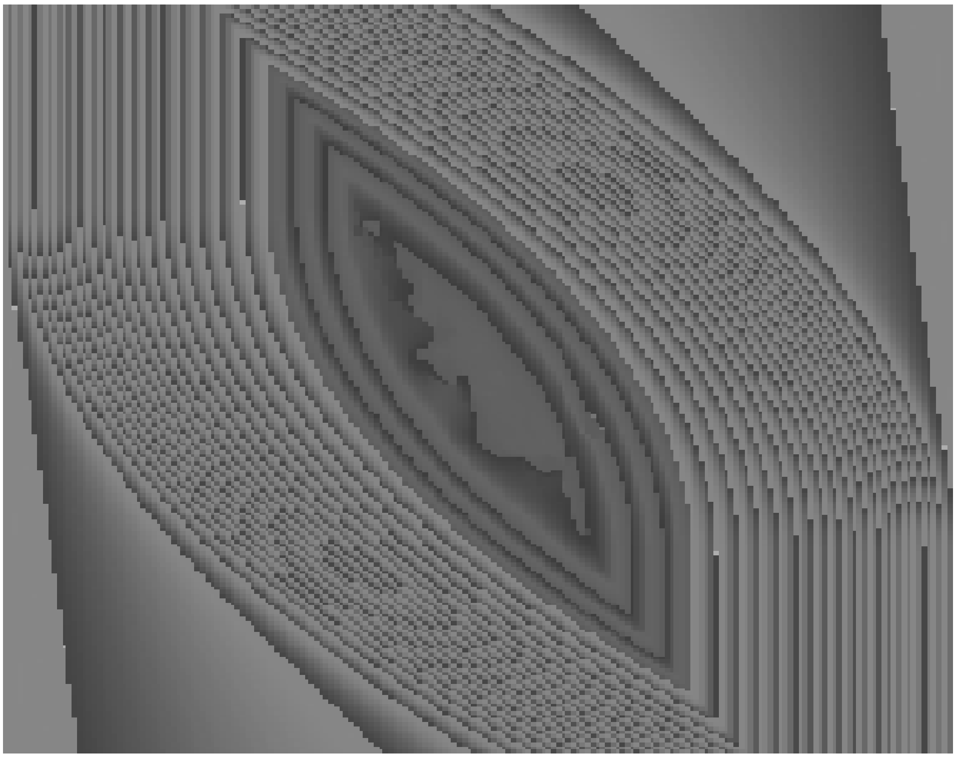

Figure 4: *Iso-values for the game in distance.*

### 5.3 Comparison

The figures 3 and 4 allow one to compare the time and distance approaches, as far as the determination of the barrier of the capture-evasion game is concerned. As already mentioned, the domain $\{x \mid \widehat{V}(x) < 1\}$ could be rather different from $\{x \mid \{V(x) < 1\}$. Indeed, if we consider level sets of $\widehat{V}$ less than $1 - \mu$ (for small $\mu$), these domains depend strongly on the arbitrary value of $\mu$, so they are numerically very sensitive. Put in another way, we can see on figure 3 that the level curves are very sparse inside the dark lens.

On the opposite, consider once again the figure 4 illustrating the approach in distance. This time, the level curves are very close to each other about the border of the lens $\{x \mid \widehat{W}(x) < 0\}$. Indeed, there is no reason why $W$ should be flat about the level curve 0. Therefore it is not a surprise that the lens $\{x \mid W(x) < 0\}$ be less numerically sensitive than $\{x \mid V(x) < 1\}$.

## 6 Conclusion

Viscosity solutions of Isaacs equation provide two complementary viewpoints on the solution of our capture-evasion game. The PDE equation allows one to investigate the capture time, while the variational inequality is an efficient way to investigate the barrier.

Both approaches allow us to construct candidate solutions, and lead to similar numerical approximation schemes. However, when the grid mesh goes towards 0, we do not know, at least theoretically, how to have the penalty decreasing, in the first case on the dilatation of the target, and in the second one on the time, so that the schemes do converge. In this respect, the viability framework is more satisfactory from a numerical point of view (see [7, 8]). Maybe a mixed approach could be fruitful : using construction techniques from Isaacs-Breakwell theory and completing the results obtained by numerical investigations thanks to viability theory.

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