Ischia, 1973

ON THE EVALUATION OF WORST CASE DESIGN WITH AN APPLICATION TO THE QUADRATIC SYNTHESIS TECHNIQUE

Pierre BERNHARD
Maître de Recherches
Centre d'Automatique
de l'Ecole Nationale Supérieure des Mines
FONTAINEBLEAU FRANCE

Gilles BELLEC
Ingénieur Elève
Ecole Nationale Supérieure des
Mines de Paris
PARIS FRANCE

ABSTRACT

A general philosophy for evaluating worst case design is proposed. It is applied to the linear quadratic problem, which requires some assumptions on the energy available to Nature. The sensivity of the results to that arbitrary parameter is obtained, thus providing a complete tool for the evaluation of the two design technique : stochastic optimal control, vs worst case control.

INTRODUCTION

Worst case design has early been identified as the potentially most interesting application of dynamical games theory. Early work in that direction may be found in [1] and [5], and in other papers. However, the developpement of these ideas has always been plagued by an apparent paradoxe. It is meaningful to dimension a system according to the results of a worst case design study only if the corresponding, game theoretic, control law is implemented, which is usually apparently meaningless mainly in view of the poor performance indicated by that theory.

We propose, here, what we believe to be the only consistent approach to an evaluation of worst-case control, with an application to the so called quadratic synthesis technique. This application is complicated by the appearance of a singular game . This difficulty is resolved through the use of an isoperimetric type constraint, on the available adverse energy. The sensitivity of the results to this arbitrary parameter is obtained in the course of the solution.

EVALUATION PHILOSOPHY

The control philosopies

Let the controlled system be described by the equation

$$\dot{x} = f(x, u, v, t), \qquad x(t_0) = x_0$$

$t \in T \subset \mathbb{R}$ is the time, t_0 is initial time,
$x \in \mathbb{R}^n$ is the state,
$u \in U \subset \mathbb{R}^m$, is the control,
$v \in V \subset \mathbb{R}^m$ is a perturbation.

A real functionnal of the trajectory, $J(x_0, t_0; u(\cdot), v(\cdot))$ is given, where $u(\cdot)$ and $v(\cdot)$ represent the control and perturbation histories, chosen among given families of admisible functions.

The "classical" approach to the minimization of J in the presence of the unknown perturbation v is to make some statistical assumptions on v, and then to minimize the expectation $E\{J\}$. It leads to a control law $u = u^\circ(x)$ and to a value $V^\circ = E\{J(u^\circ(\cdot), v(\cdot))\}$ The approach of "worst case design", on the contrary, seeks to make no assumption on v. The idea is to find a control law $u^*(x)$ ensuring

$$J(x_0, t_0; u^*(\cdot), v(\cdot)) \leq V^*(x_0, t_0) = J(x_0, t_0; u^*(\cdot), v^*(\cdot)),$$

with, of course, as low a V^* as possible. Therefore u^* should be chosen according to

$$\min_{u(\cdot)} \max_{v(\cdot)} J(x_0, t_0; u(\cdot), v(\cdot)) = V^*(x_0, t_0).$$

Assuming that Nature is blind, and must, therefore, chose its control law apriori allows to replace this by the more favourable value $V^{*\prime}$:

$$\max_{v(\cdot)} \min_{u(\cdot)} J(x_0, t_0; u(\cdot), v(\cdot)) = V^{*\prime}(x_0, t_0) \leq V(x_0, t_0)$$

To avoid going into that discussion, which is not our topic here, we shall assume that J has a saddle point in $u(\cdot), v(\cdot)$, so that $V^* = V^{*\prime}$.

The second control law has a very desirable feature of "security" since we know an absolute upper bound to the cost J incurred, with no risk of beeing fooled by a gross error in estimating the perturbation's statistics. Of course, one has $V^* \geq V^\circ$, and possibly much larger. However, discarding worst case control on these grounds makes no sense whatsoever, since completely different assumptions were made on v in computing V° and V^* making their comparison meaningless.

The evaluation

It is not the role of the control theoretist to decide how much security is worth, against expectation of returns. He should only provide the designer with a complete tool on which to base a decision. This tool should state what is the gain on one hand, and what is the "price" paid for it on the

"Superior numbers refer to similarly-numbered reference at the end of this paper."

other hand

The gain in security can be measured by comparing V^* against V^+ :

$$V^+(x_o,t_o) = \max_{v(\cdot)} J\,(x_o,t_o;u^o(\cdot),v(\cdot)) \geq V^*(x_o,t_o),$$

where the maximization is to be performed over the same set of admissible perturbations as in the computation of V^*, making the comparison meaningful. V^+ represents the "risk" taken in implementing u^o.

Similarily, the price paid for the security can be measured by comparing V^o against V^1 :

$$V^1(x_o,t_o) = E\{J(x_o,t_o;u^*(\cdot),v(\cdot))\} \geq V^o(x_o,t_o)$$

where the expectaction is to be taken with the same hypothesis on the a priori probabilities of v as in the computation of V , again making the comparison meaningful. In particular, it is V^1 which must be used in economic calculations about the implementation of u^*, and not the overly pessimistic value V^*.

One could suggest to make the evaluation in terms of a risk coefficient $\rho=(V^+-V^*)/V^*$ and a "loss" coefficient $\lambda=(V^1-V^o)/V^o$. While these are only one possibility, as regard the exact quantities to be taken into account (one could also consider the ratio of the larger value, etc...), the important claim is that only these comparisons, V^+ vs V^* and V^1 vs V^o are meaningful. In particular, although one will have, in general :

$$V^o < V^1 < V^* < V^+,$$

the comparison between the first two and the last two figures is most often of no signification.

QUADRATIC SYNTHESIS

The linear quadratic problem

Even the classical is theoretically solved only in the so called linear quadratic-gaussian case, described by the equations :

$$\dot{x}=F(t)x+G(t)u+D(t)v, \quad E\{v(t)v'(\tau)\}=C_v(t)\delta(t-\tau).$$

$$J= x'(T)Ax(T)+\int_{t_o}^{T}[x'(t)Q(t)x(t)+u'(t)R(t)u(t)]\,dt$$

Here, the accent means transposition. $A, Q(t), R(t)$ are given symmetric matrices, with $R(t) > 0 \quad \forall t$. We assume, here, that the state is available to the controller (Thus avoiding a major problem in worst case design)

We shall therefore limit our investigation to this family of problems. It is known to be of both practical and theoretical interest. The solution of the optimal stochastic control problem (the classical approach) is given by the following set of equations :

$$u^o(x)= -R^{-1}G'Px, \quad V^o(x_o,t_o)=x'_o\,P(t_o)x_o+p(t_o),$$

where $P(t)$ and $p(t)$ are given by (see P.FAURRE [2])

$$\dot{P}+PF+F'P-PGR^{-1}G'P+Q=0 \qquad P(T)=A,$$

$$\dot{p}+tr\,[PDC_v D']=0 \qquad p(T)=0.$$

However, one cannot solve the game problem

$$\min_{u(\cdot)} \max_{v(\cdot)} J\,(x_o,t_o,u(\cdot),v(\cdot))$$

because it is singular in v. By using arbitrarily large perturbations, Nature can make the cost arbitrarily large too. We shall therefore be obliged to make some further assumptions on v to rule out this possibility.

The energy constrained problem

Among the various assumptions that could be made to restrict the use of too large v's by Nature, the most natural and convenient, in the present framework, is to define the energy used as

$$\mathcal{E} = \int_{t_o}^{T} v'(t)S(t)v(t)\,dt, \qquad S(t)>0 \;\forall t$$

and to impose a limit on how much energy the perturbation can force into the system : $\mathcal{E} \leq \mathcal{E}_m$. Clearly, the "optimal" solution for v will yield $\mathcal{E}=\mathcal{E}_m$, since energy used is "free" although constrained.

By adding a new state variable x_{n+1} obeying the equation

$$\dot{x}_{n+1} = v'\,S\,v, \qquad x_{n+1}(t_o) = -\mathcal{E}_m, \quad x_{n+1}(T)=0\,,$$

one can easily check that the solution of the constrained game is the same as the solution of the unconstrained game with augmented cost :

$$\bar{J}= x'(T)\,A\,x(T) + \int_{t_o}^{T}(x'Qx+u'Ru-\lambda\, v'Sv)\,dt$$

where λ is a constant parameter to be adjusted so that $\mathcal{E}=\mathcal{E}_m$. This solution is given by the following set of equations (see [3])

$$u = -R^{-1}G'\Pi x, \qquad v = \frac{1}{\lambda}S^{-1}D'\Pi x,$$

$$V(x_o,t_o)= x'_o\,\Pi(t_o)\,x_o + \lambda\,\mathcal{E}_m$$

$$\dot{\Pi}+\Pi F+F'\Pi-\Pi GR^{-1}G'\Pi+\frac{1}{\lambda}\Pi DS^{-1}D'\Pi+Q=0, \quad \Pi(T)=A$$

Evaluation of $\mathcal{E}$, to adjust λ, may be performed directly, by use of the expression for v , but this obliges to compute the trajectory $x(t)$ at each iteration, or indirectly through the use of the formula.

$$\mathcal{E} = x'_o\,M(t_o)\,x_o$$

Where $M(t)$ can be computed together with $\Pi(t)$ through the equation

$$\dot{M}+M\left(F-GR^{-1}G'\Pi+\frac{1}{\lambda}DS^{-1}D'\Pi\right)+\left(F-GR^{-1}G'\Pi+\frac{1}{\lambda}DS^{-1}D'\Pi\right)'M+\frac{1}{\lambda^2}\Pi DS^{-1}D'\Pi=0 \qquad M(T)=0$$

We have, this way, obtained a sensible treatment of the problem at hand in a worst case sense. Further discussion of the restriction imposed on v, its meaning and the sensivity of the results to δ_m, will be given in another paragraph.

Evaluation of V^1 and V^+

The same techniques as used previously allow us to compute the expected cost V^1 under control u^* and the "risk" cost V^+ under control u°.

V^1 is directly obtained through the formula (see (2))

$$V^1 = x'_o L(t_o) x_o + l(t_o)$$

where $L(t)$ and $l(t)$ are given by

$$\dot{L} + L(F - GR^{-1}G'\Pi) + (F - GR^{-1}G'\Pi)'L + \Pi GR^{-1}G'\Pi + Q = 0 \qquad L(T) = A$$

$$\dot{l} + tr(\Pi D C_v D') = 0 \qquad l(T) = 0$$

V^+ is obtained as the solution of a classical linear quadractic maximization problem dealing with the isoperimetric constraint as in the game problem.

$$\dot{x} = (F - GR^{-1}G'P)\, x + Dv,$$

$$\max_v \; J = x'(T)Ax(T) + \int_{t_o}^{T} [x'(Q + PGR^{-1}G'P)x - \mu v'Sv]\, dt,$$

where μ is a scalar constant to be chosen so that $\delta = \delta_m$. This problem is solved by classical formulas

$$v^+ = \frac{1}{\mu} S^{-1}D'Kx, \qquad V^+ = x'_o K(t_o) x_o + \mu \delta_m$$

$$\dot{K} + (F - GR^{-1}G'P)'K + K(F - GR^{-1}G'P) + \frac{1}{\mu} KDS^{-1}D'K$$

$$+ Q + PGR^{-1}G'P = 0, \quad K(T) = A,$$

and again, δ may be computed either directly or through a formula

$$\delta = x'_o N(t_o)\, x_o$$

where $N(t)$ may be computed together with $K(t)$ through the equation

$$\dot{N} + (F - GR^{-1}G'\Pi + \frac{1}{\mu}DS^{-1}D'K)'N + N(F - GR^{-1}G'\Pi + \frac{1}{\mu}DS^{-1}D'K)$$

$$+ \frac{1}{\mu^2} KDS^{-1}D'K = 0$$

$$N(T) = 0$$

similar to that for M. The computation of K may be numerically difficult. A being positive while we seek to maximise J, for some values of μ, $N(t)$ will cease to be positive and the Riccati equation will diverge. However this would yield infinite energies hence a small enough value of μ exists that sends the conjugate point back in the past farther than t_o, and leads to the solution of our problem.

Discussion, sensitivity

We want to discuss further the restriction imposed on v. First, notice that we remain consistent in that the same assumption is used in the computation of V^* and V^+.

The choice of S may be rather arbitrary. However, in many cases, a quadratic form of v has the property of being the physical energy of the perturbation. Then, whe have a good basis on which to found our choice, and also a feeling of what energy is allowable. In that case, the choice of S is less arbitrary than that of C_v in many instances. In other cases it will often be found that a good choice for S is C_v^{-1}. This leads to a dimensionless energy , which is similar to a likelihood coefficient of the realization $v(t)$ with the previously assumed statistics. Although a likelihood coefficient is known to have no meaning for a continuous white noise, δ will be found to be exactly that in the discretized form of the problem, hence in any digital treatment of it. Our restriction amounts, then, to forbidding exceedingly unlikely realization of $v(t)$ (or rather of a sequence of finite increments of the generating brownian motion $b(t)$ such that $db(t) = v(t)dt$. This can compensate for the physical nonsense there often is in assuming gaussian distributions with arbitrary large potential realizations. In that case again, the choice of S is not more arbitrary than that of C_v.

We must also discuss the choice of the level δ_m chosen, which always appear as an arbitrary parameter. Of course, the problem can be treated with various values of δ_m which is always done in practice since we have to iterate until we find the right values of λ and μ. It is therefore advisable to retain all intermediary results in the iteration. But more interesting is the fact that λ and μ themselves are the very sensitivities of the results to the arbitrary parameter δ_m.

$$\lambda = \frac{\partial V^*}{\partial \delta_m} \qquad \mu = \frac{\partial V^+}{\partial \delta_m}$$

as is easily seen from the fact that λ and μ are the adjoint variables associated to the added state variable.

Unfortunately, $\partial V^1 / \partial \delta_m$ is not as easily obtained. More elaborate calculations can be made, leading to $\partial V^1/\partial\lambda$ and to $\partial\delta/\partial\lambda$ the last being helpful in the iteration. (Although numerical examples treated so far indicate no need for it).

CONCLUSION

We have seen how to evaluate two control philosophies in terms of expected cost and risk. This applies to the linear quadratic problem, and translates there into a set of differential equations readily integrable, although some iterations are needed. This application requires that an energy level be fixed rather arbitrarily. However, the same computational technique yields sensitivity of the results to the level chosen.

It seems that this evaluation of risk, and comparison with worst case design should be made in many applications of (stochastic) optimal control where large excursions away from the nominal can cause catastrophic failure.

A numerical example is being worked out. Results are not yet all available, but will be in the near future.

BIBLIOGRAPHY

(1) *DORATO and KESTENBAUM "Application of Game Theory to the Sensitivity Design of Optimal Systems" IEEE transactions AC 12, n° 1 pp 85-87 1967*

(2) *FAURRE "Commande Optimale Stochastique "Revue du CETHEDEC 1968*

(3) *HO, BRYSON and BARON "Differential Games and Optimal Pursuit Evasion Strategies" IEEE transactions AC 10 n° 4 pp 385-389 1965*

(4) *KALMAN "Contributions to the Theory of Optimal Control" Bol de la Soc. Mat. Mexicana second ser. vol 5 pp 102-119 1960*

(5) *SARMA and RAGADE "Some Condiderations in Formulating Optimal Control Problems as Differential Games" Intal- Jal of Control vol 4 pp 265-279 1966*