# Minimax versus stochastic partial information control

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## Abstract

We review the current state of partial information finite time minimax control, stressing the parallel with stochastic control.

## 1 Introduction

Minimax control, or worst case design, as a means of dealing with uncertainty has gained a new popularity with the recognition, in 1988, of the fact that $H_\infty$-optimal control could be cast into that concept. Although some work in that direction existed long before (see [8]), this viewpoint has vastly renewed the topic. See [3] and related work.

Many have tried to extend this work to a nonlinear setup. Most prominent among them perhaps is the work of Isidori and coworkers, [11] [12], but also many others, such as [4] [5]. We shall mainly refer to [15] [16] [14] [13]. We also refer the reader to [18] and references therein. This has contributed to a renewed interest in nonlinear minimax control.

We revisit here recent work on finite time minimax control, (hence avoiding stability issues) mainly by ourselves (in particular [7]), James and coauthors, in the light of the parallel developed by Quadrat and coworkers (see [1] [2] [17]), between probabilities and maximization, yielding a parallel between minimax and stochastic control.

This parallel involves "Quadrat's morphism" induced by the (opposite) Cramer transform between the classical $(+,\times)$ algebra and the $(\max,+)$ algebra. It turns the convolution product into a supconvolution. In the use we make of it, it involves replacing products by sums, and sums, or integrals like expectations, by max. Thus, the counterpart of an expression such as $\int V(x)W(x)\,dx$ will be $\max_x[V(x)+W(x)]$.

We do not have space here to review their theory of *decision processes*. Let us only introduce the vocabulary of *cost measures* : let $\Omega$ be a topological space, $\mathcal{A}$ a $\sigma$-field of subsets, $K:\mathcal{A}\to\mathbb{R}\cup\{-\infty\}$ is called a cost measure if it satisfies the following axioms : $K(\Omega)=0$, $K(\emptyset)=-\infty$, and for any family of (disjoint) elements $A_n$ of $\mathcal{A}$, $K(\cup A_n)=\sup_n K(A_n)$. The function $c:\Omega\to\mathbb{R}\cup\{-\infty\}$ is called a *cost density* of $K$ if we have

$$\forall A\in\mathcal{A},\quad K(A)=\sup_{\omega\in\Omega}c(\omega)\,.$$

## 2 Discrete time control

### 2.1 The problem

We consider a partially observed two input control system

$$x_{t+1} = f_t(x_t,u_t,w_t)\,, \qquad (1)$$

$$y_t = h_t(x_t,w_t)\,, \qquad (2)$$

where $x_t\in\mathbb{R}^n$ is the state at time $t$, $u_t\in\mathsf{U}$ the (minimizer's) control, $w_t\in\mathsf{W}$ the disturbance input, and $y_t\in\mathsf{Y}$ the measured output. We shall call $\mathbf{U}$ the set of input sequences over the time horizon $[0,T]$: $\{u_t\}_{t\in[0,T]}$ usually written as $u_{[0,T]}\in\mathbf{U}$, and likewise for $w_{[0,T]}\in\mathbf{W}$. The initial state $x_0\in\mathsf{X}_0$ is also considered part of the disturbance. We introduce the combined disturbance, $\omega=(x_0,w_{[0,T]})$, and the set $\Omega=\mathsf{X}_0\times\mathbf{W}$ of such disturbances.

The solution of (1) (2) above shall be written as $x_t=\phi_t(u_{[0,T]},\omega)$, $y_t=\eta_t(u_{[0,T]},\omega)$.

Finally, we shall call $u^t$ a partial sequence $(u_0,u_1,\dots,u_t)$ and $\mathsf{U}^t$ the set of such sequences, likewise for $w^t\in\mathsf{W}^t$ and $y^t\in\mathsf{Y}^t$. Also, we write $\omega^t=(x_0,w^t)\in\Omega^t$.

The solution of (1) and (2) may alternatively be written as $x_t=\phi_t(u^{t-1},\omega^{t-1})$, $y_t=\eta_t(u^{t-1},\omega^t)$. Let also $x^t=\phi^t(u^{t-1},\omega^{t-1})$, $y^t=\eta^t(u^{t-1},\omega^t)$, refer to the partial sequences solution of (1) and (2)

Admissible controllers will be output feedbacks of the form $u_t=\mu_t(u^{t-1},y^{t-1})$, i.e. strictly causal. We denote by $\mathcal{M}$ the class of such controllers.

A performance index is given. One may absorb an integral cost into a purely terminal payoff of the form

$$J(x_0,u_{[0,T]},w_{[0,T]})=M(x_T)\,. \qquad (3)$$

We shall also, with no ambiguity, use the abusive notation $J(\mu,\omega)$. The aim of the control is to minimize

$J$, in some sense, "in spite of the unpredictable disturbance".

We want to compare here two ways of turning this unprecise statement into a meaningful mathematical problem. In the first one, *stochastic control*, we modelize the unknown disturbance as a random variable, more specifically here a random variable $x_0$ with a probability density $N(x)$ and an independant white stochastic process $w_{[0,T]}$ of known instantaneous probability distribution.

Let then

$$\bar{J}(u_{[0,T]}, w_{[0,T]}) = \int J(\xi, u_{[0,T]}, w_{[0,T]}) N(\xi)\, d\xi\,, \quad (4)$$

the criterion to be minimized over $\mathcal{M}$ is then

$$H(\mu) = \mathbb{E}_w \bar{J}(\mu, w_{[0,T]})\,. \quad (5)$$

In the second case, we do not modelize the perturbation otherwise than through the data of the set $\Omega$, and we wish to choose $\mu$ in such a way as to get the best possible guaranteed payoff. Let then

$$\bar{J}(u_{[0,T]}, w_{[0,T]}) = \max_{\xi} [J(\xi, u_{[0,T]}, w_{[0,T]}) + N(\xi)]$$

be the counterpart of (4), and the criterion to be minimzed over $\mathcal{M}$ be the counterpart of (5):

$$G(\mu) = \sup_{w_{[0,T]} \in \mathbf{W}} \bar{J}(\mu, w_{[0,T]})\,. \quad (6)$$

## 2.2 Dynamic programming

### 2.2.1 Stochastic dynamic programming

We quickly recall here for reference purpouses the classical solution of the stochastic problem via dynamic programming. One has to introduce the *conditional state probability measure*, and, assuming it is absolutely continuous with respect to the Lebesgue measure, its density $W$. Let, thus, $W_t(x)\, dx$ be the conditional probability measure of $x_t$ given $y^{t-1}$, and $W_t^{\eta}(x)\, dx$ be the conditional state distribution given $y^{t-1}$ and given that $y_t = \eta$.

Under suitable assumptions on the data (noticeably that $\partial h_t(x,w)/\partial w$ be invertible) one can compute $W_t$ from *past* data through a recursion starting with $W_0(x) = N(x)$, and that we summurize as $W_{t+1} = F_t(W_t, u_t, y_t)$.

The only properties one need to carry out the caculations are that, for any function $\psi(x)$,

$$\int \psi(x) W_{t+1}(x)\, dx = \int \mathbb{E}_w \psi(f_t(x, u_t, w)) W_t^{y_t}(x)\, dx, \quad (7)$$

(a dual propagation operator) where the conditional (or a posteriori) state probability density knowing $y_t = \eta$ : $W_t^{\eta}$, satsifies

$$\mathbb{E}_y \int \psi(x) W_t^y(x)\, dx = \int \psi(x) W_t(x)\, dx\,. \quad (8)$$

Let $\mathcal{W}$ be the set of all possible such functions $W_t$.

Via a standard dynamic programming argument, we can check that the Bellman return function $U$ is obtained by the recurrence relation: $\forall W \in \mathcal{W}$,

$$U_T(W) = \int M(x) W(x)\, dx\,, \quad (9)$$

$$U_t(W) = \inf_u \mathbb{E}_y U_{t+1}\left(F_t(W, u, y)\right)\,. \quad (10)$$

Moreover, assume that the minimum in $u$ is attained in (10) above at $u = \hat{\mu}_t(W)$. Then

$$u_t = \hat{\mu}_t(W_t) \quad (11)$$

defines an optimal feedback for the stochastic control problem. The optimal cost is $U_0(N)$

### 2.2.2 Minimax dynamic programming

Let us consider now the problem of minimizing $G(\mu)$. We have to introduce the *conditional state cost measure* and its *cost density* $W$. It is defined as the maximum possible past cost knowing the past information, as a function of current state. To be more precise, let us introduce the following subsets of $\Omega$. Given a pair $(u^t, y^t) \in \mathsf{U}^t \times \mathsf{Y}^t$, and a subset $A$ of $\mathbb{R}^n$, let

$$\Omega_t(A \mid u^t, y^t) = \{\omega \in \Omega \mid y^t = \eta^t(u^{t-1}, \omega^t), \text{ and } \phi_{t+1}(u^t, \omega^t) \in A\}$$

For any $x \in \mathbb{R}^n$, we shall write $\Omega_t(x \mid u^t, y^t)$, or simply $\Omega_t(x)$ when no ambiguity results, instead of $\Omega_t(\{x\} \mid u^t, y^t)$. And likewise for $\Omega_{t-1}(x)$.

The conditional cost measure of $A$ is defined as $\sup_{\omega \in \Omega_{t-1}(A)} N(x_0)$, and hence the conditional cost density function is

$$W_t(x) = \sup_{\omega \in \Omega_{t-1}(x)} N(x_0)\,.$$

This function again satisfies a recursion relation of the form $W_0(x) = N(x)$, $W_{t+1} = F_t(W_t, u_t, y_t)$. It suffices to know that for any function $\psi(x)$,

$$\max_x [W_{t+1}(x) + \psi(x)] = \max_{(x,w) \mid h_t(x,w) = y} [W_t(x) + \psi(f_t(x, u_t, w)]$$

and that hence

$$\max_y \max_x [W_{t+1}(x) + \psi(x)] = \max_{x,w} [W_t(x) + \psi(f_t(x, u_t, w)]\,,$$

the counterparts of (7) and (8) above.

As was probably first shown in [16], one can do simple dynamic programming in terms of this function $W$. The value function $U$ will now be obtained through the following relation : $\forall W \in \mathcal{W}$,

$$U_T(W) = \sup_x (M(x) + W(x))\,, \quad (12)$$

$$U_t(W) = \inf_u \sup_y U_{t+1}(F_t(W, u, y))\,. \quad (13)$$

Moreover, assume that the minimum in $u$ is attained in (13) above at $u = \hat{\mu}(W)$. Then it defines an optimal feedback (11) for the minimax control problem. The optimal cost is $U_0(N)$.

Of course, all our set up has been arranged so as to stress the parallel between (9),(10) on the one hand, and (12),(13) on the other hand.

## 2.3 Separation theorem

### 2.3.1 Stochastic separation theorem

We are here in the stochastic setup. The performance criterion is $H$ and $W$ stands for the conditional state probability density.

We introduce the *full information* Bellman return function $V_t$ defined by the classical dynamic programming recursion :

$$\forall x \in \mathbb{R}^n, \quad V_T(x) = M(x)\,,$$

$$\forall x \in \mathbb{R}^n, \quad V_t(x) = \inf_u \mathbb{E}_w V_{t+1}(f_t(x,u,w))\,.$$

Then we can state the following result.

**Proposition 1** *Let*

$$S_t(x,u) := \mathbb{E}_w V_{t+1}\left(f_t(x,u,w)\right) W_t(x)\,.$$

*If there exists a (decreasing) sequence of (positive) numbers $R_t$ with $R_T = 0$ such that,*

$$\forall t \in [0, T-1], \forall u_{[0,T]} \in \mathbf{U}, \forall \omega \in \Omega,$$
$$\int \min_u S_t(x,u)\,dx + R_t = \min_u \int S_t(x,u)\,dx + R_{t+1}\,,$$

*then the optimal control is obtained by minimizing the conditional expectation of the full information Bellman return function, i.e. choosing a minimizing $u$ in the right hand side above.*

**Proof** The proof relies on the following fact :

**Lemma 1** *Under the hypothesis of the proposition, the function*

$$U_t(W) = \int V_t(x) W(x)\,dx + R_t\,, \qquad (14)$$

*satisfies the dynamic programming equations (9)(10).*

The hypothesis of the theorem sounds in a large extent like wishfull thinking. It holds, as easily checked, in the linear quadratic case. (In that case, symmetry properties result in the certainty equivalence theorem.) There is little hope to find other instances. We state it here to stress the parallel with the minimax case.

### 2.3.2 Minimax separation theorem

This section is based upon [6] [7]. The same result is to appear independantly in [13].

We are now in the minimax setup. The performance criterion is $G$, and $W$ stands for the conditional state cost density.

We introduce the *full information* Isaacs Value function $V_t(x)$ (or here, *upper value*) which satisfies the classical Isaacs equation:

$$\forall x \in \mathbb{R}^n, \quad V_T(x) = M(x)\,,$$

$$\forall x \in \mathbb{R}^n, \quad V_t(x) = \inf_u \max_w V_{t+1}(f_t(x,u,w))\,.$$

**Proposition 2** *Let*

$$S_t(x,u) = \max_w [V_{t+1}\left(f_t(x,u,w)\right) + W_t(x)]\,.$$

*If there exists a (decreasing) sequence of numbers $R_t$, such that,*

$$\forall t \in [0, T-1], \forall u_{[0,T]} \in \mathbf{U}, \forall \omega \in \Omega,$$
$$\max_x \min_u S_t(x,u) + R_t = \min_u \max_x S_t(x,u) + R_{t+1}\,,$$

*then the optimal control is obtained by minimizing the conditional worst cost, future cost being measured according to the full information Isaacs Value function, i.e. taking a minimizing $u$ in the right hand side above.*

**Proof** The proof relies on the following fact :

**Lemma 2** *Under the hypothesis of the proposition, the function*

$$U_t(W) = \max_x [V_t(x) + W(x)] + R_t$$

*satisfies the dynamic programming equations (12)(13).*

The hypothesis of the proposition is not as unrealistic as in the stochastic case. It is satisfied in the linear quadratic case, but more generally, it can be satisfied if $S$ is convex-concave, for instance, with $R_t = 0$. Moreover, in that case, the same $u$ provides the minimum in both sides, yielding a certainty equivalence theorem.

# 3 Continuous time control

## 3.1 The problem

We now have a continuous time system, of the form

$$\dot{x} = f_t(x,u,w)\,, \qquad (15)$$
$$y = h_t(x,w)\,. \qquad (16)$$

The notations will be the counterpart of the discrete ones. In particular, $u^t$, will stand for the restriction to $[0,t]$ of the continuous time function $u_{[0,T]} : t \mapsto u_t$. We shall again let $\mathsf{U}^t$ designate the set of such segments of function. Likewise for $w^t \in \mathsf{W}^t$, $\omega^t \in \Omega^t$, and $y^t \in Y^t$. The state and output maps now read $x_t = \phi_t(u^t, \omega^t)$, $y_t = \eta_t(u^t, \omega^t)$, and similarly for the complete or truncated trajectories.

Admissible controllers will be of the form $u_t = \mu_t(u^t, y^t)$. This seems to be an implicit definition, since $u_t$ is contained in $u^t$. In fact, it is hardly more so than any feedback control. In any extent, we let $\mathcal{M}$ be the class of controllers of that form, such that they generate a unique trajectory for any $\omega \in \Omega$.

As in the discrete time case, we can always bring a classical integral plus terminal cost to the form (3). The two problems we want to investigate are again the minimization of $H(\mu)$ with a stochastic model for $\omega$ and that of $G(\mu)$ with the sole set description of $\omega$.

## 3.2 Hamilton Jacobi theory

### 3.2.1 Stochastic Hamilton Jacobi theory

In the continuous time case, the technicalities of diffusion processes and Ito calculus make the stochastic problem much more complex than its discrete counterpart, or, for that matter, than its continuous minimax counterpart. As far as we know, the classical litterature concentrates on simpler particular cases of the system (15),(16). Typically, classical nonlinear stochastic control deals with the system

$$\begin{aligned} dx_t &= b_t(x,u)\,dt + \sigma_t(x,u)\,dw_t\,, \\ dy_t &= c_t(x)\,dt + dv_t\,. \end{aligned}$$

where $v_t$ and $w_t$ are standard independent vector brownian motions, and the above equations are to be taken in the sense of stochastic integrals.

Under suitable regularity and growth assumptions, one can compute a conditional state probability distribution $W_t$ through a stochastic PDE (which can be derived, for instance, from Zakai's equation, see [10]) yielding $W_0 = N$ and, for any function $\psi(x)$,

$$d\int \psi(\xi) W_t(\xi)\,d\xi = \int (L_t(u)\psi)(\xi) W_t(\xi)\,d\xi\,dt + \int \psi(\xi) W_t(\xi)[c_t'(\xi) - \bar{c}_t']\,d\xi[dy_t - \bar{c}_t\,dt]\,.$$

where

$$(L_t(u)\psi)(\xi) = \frac{\partial \psi}{\partial x}(\xi) b_t(\xi,u) + \frac{1}{2} tr\left(\sigma_t' \frac{\partial^2 \psi}{\partial x^2}(\xi)\sigma_t\right),$$

and $c_t = \int c_t(z) W_t(z)\,dz$ stands for the conditional expectation of $c_t(x_t)$.

A full information control problem may be written in terms of that probability density as a state. We refer to [10] for a complete treatment.

### 3.2.2 Minimax Hamilton Jacobi theory

The minimax problem is not as complex as the stochastic one, at least to state formally, and as long as one only seeks sufficient conditions. It was independently developed in [6], and in [15] in a slightly less general context, but with a much more complete development in that it includes a first mathematical analysis of the resultant Isaacs equation.

For a given pair $(u^t, y^t) \in \mathsf{U}^t \times \mathsf{Y}^t$ and a subset $A$ of $\mathbb{R}^n$, let again

$$\Omega_t(A \mid u^t, y^t) = \{\omega \in \Omega \mid y^t = \eta^t(u^t, \omega^t), \text{ and } \phi_t(u^t, \omega^t) \in A\}$$

be the conditional disturbance subset of $A$, and again write $\Omega_t(\xi)$ instead of $\Omega_t(\{\xi\} \mid u^t, y^t)$. The conditional cost density function is now

$$W_t(x) = \sup_{\omega \in \Omega_t(\xi)} N(x_0)\,.$$

If it is $C^1$, $W_t$ satisfies a forward hamilton Jacobi equation. Let

$$\mathsf{W}_t(x \mid y) = \{w \in \mathsf{W} \mid h_t(x,w) = y\}\,,$$

then this forward equation is, for $u^t$ and $y^t$ fixed:

$$\frac{\partial W_t(x)}{\partial t} = \sup_{w \in \mathsf{W}_t(x|y_t)} \left[-\frac{\partial W(x)}{\partial x} f_t(x, u_t, w)\right] \tag{17}$$

which we write as

$$\frac{\partial}{\partial t} W_t = F_t(W_t, u_t, y_t)$$

and, together with the initial condition $W_0 = N$, it may define $W_t$ along any trajectory.

Then, the value function $U_t(W)$ is obtained through the following Isaacs equation. $U_T$ is again given by (12), and $\forall W \in \mathcal{W}$,

$$\frac{\partial U_t(W)}{\partial t} + \inf_{u \in \mathsf{U}} \sup_{y \in \mathsf{Y}} D_W U_t(W) F_t(W,u,y) = 0\,. \tag{18}$$

Moreover, assume that the minimum in $u$ is attained in (18) above at $u = \hat{\mu}_t(W)$, then (11) defines an optimal feedback for the minimax control problem. The optimal cost is $U_0(N)$.

Notice again that the easy task is to show a *sufficient* condition. If there exist $C^1$ functions $W$ and $U$ satisfying these equations, and if the feedback (11) is admissible, then we have a solution of the problem. It is worth noticing that the only existence result we are aware of is in [14], and is in a particular case somewhat similar to the set up we have outlined for the stochastic case.

## 3.3 Separation theorem

### 3.3.1 Stochastic separation theorem

We may take advantage of the linear character of the equation (14) to write, at least formally, the continuous time counterpart to the stochastic separation principle of section 2.3.1. We need first introduce the *full information* (state feedback) Bellman function $V_t(x)$ which satisfies the *stochastic Bellman equation* $V_T = M$, and $\forall (t,x) \in [0,T] \times \mathbb{R}^n$,

$$\frac{\partial V_t}{\partial t}(x) + \inf_u (L_t(u)V_t)(x) = 0\,.$$

We can then state the following result.

**Proposition 3** *Let*

$$S_t(x,u) = (L_t(u)V_t)(x)W_t(x)\,.$$

*If there exists a real (positive) $L^1([0,T])$ function $r_t$ such that,*

$\forall t \in [0,T], \forall u_{[0,T]} \in \mathbf{U}$, *almost surely*

$$\int \min_u S_t(x,u)\,dx + r_t = \min_u \int S_t(x,u)\,dx\,,$$

*then an optimal control is obtained by choosing the minimizing $u$, that we shall call $\hat\mu_t(W_t)$, in the right hand side above.*

**Proof**. The proof relies on the following fact

**Lemma 3** *Under the hypothesis of the proposition, the stochastic process $\alpha_t = U_t(W_t)$ is a submartingale for any admissible control, and a martingale if $u_t = \hat\mu_t(W_t)$, where the function $U_t$ is defined over the set $L^1(\mathbb{R}^n)$ by*

$$U_t(W) = \int_{\mathbb{R}^n} V_t(\xi)W(\xi)\,d\xi + R_t\,,$$

*and*

$$R_t = \int_t^T r_s\,ds\,. \tag{19}$$

Thus under the feedback control $\hat\mu_t(W_t)$, $\mathbb{E}U_T(W_T) = U_0(W_0)$, hence, recalling that $V_T = M$ and $W_0 = N$, $\mathbb{E}M(x_T) = \mathbb{E}V_0(x_0) + R_0$. And for any other admissible control, $\mathbb{E}U_T(W_T) \geq U_0(W_0)$, hence $\mathbb{E}M(x_T) \geq \mathbb{E}V_0(x_0) + R_0$.

The above result is formal in that we have not detailed the regularity and growth hypotheses under which these calculations are valid. But it can be made rigorous, and provide a proof of the separation theorem for the linear quadratic case for instance.

### 3.3.2 Minimax separation theorem

Introduce as in the discrete time case the *full information* Isaacs Value function $V$. It satisfies the Isaacs equation $V_T = M$ and, $\forall t, \forall x \in \mathbb{R}^n$,

$$\frac{\partial V_t(x)}{\partial t} = \min_{u\in\mathsf{U}} \max_{w\in\mathsf{W}} \left(\frac{\partial V_t(x)}{\partial x} f_t(x,u,w)\right)\,.$$

The use of weak solutions, the viscosity solution, is now well understood. However, for our purpouse here, which is to stress the formal duality, we shall assume that $V$ and $W$ are $C^1$. Assume also that the full information game admits a unique state feedback solution $u_t = \phi^*_t(x_t)$, argument of the min above.

As in [3], introduce also the *auxiliary problem*

$$\max_{x\in\mathbb{R}^n} [V_t(x) + W_t(x)]\,.$$

and assuming it has a (nonunique) solution, let $\hat X_t$ be the set of maximizing $x$'s, or *conditional worst states.*

We have the following fact:

**Proposition 4** *Let*

$$S_t(x,u) = \max_w \left[\frac{\partial V_t}{\partial x}(x) f_t(x,u,w)\right]\,.$$

*If there exists a real (positive) $L^1([0,T])$ function $r_t$ such that*

$\forall t \in [0,T], \forall u_{[0,T]} \in \mathbf{U}, \forall \omega \in \Omega,$

$$\min_{x\in\hat X_t} \min_u S_t(x,u) + r_t = \min_u \max_{x\in\hat X_t} S_t(x,u)\,,$$

*then an optimal control is obtained by minimizing the conditional worst rate of increase of the full information Value function among the conditional worst states, i.e. taking the minimizing $u$ in the right hand side above.*

**Proof** The proof relies on the following fact :

**Lemma 4** *Under the hypothesis of the proposition, the function*

$$U_t(W) = \max_x [V_t(x) + W_t(x)] + R_t$$

*with $R_t$ defined as in (19) satisfies the dynamic programming equations (12), and (18) replacing derivatives with right derivatives in time.*

The lemma hinges on Danskin's theorem to get for the right time derivative

$$\left(\frac{\partial U_t}{\partial t}\right)^+ (W) = \max_{x\in\hat X_t} \frac{\partial V_t}{\partial t}(x) - r_t$$

and for the directional derivative in a direction $dW \in \mathcal{W}$:

$$D_W U_t(W)\cdot dW = \max_{x\in\hat X_t} dW(x)\,.$$

Then it is a simple matter to place this in (18), notice that because all $x \in \hat{X}_t$ maximize the auxiliary problem, then at these points $-\partial W_t/\partial x = \partial V_t/\partial x$, and that, as in the case of mathematical expectations, the cascade $\max_y \max_{w \in W(x|y)}$ collapses in $\max_w$ to get the result.

**Remarks** The condition of the proposition depends on $W_t$, hence on $\omega_t$, through $X_t$.

We have quoted the proposition that way to stress a parallel with the other cases. (The parallel would have been better if we had not converted $-\max(-\cdot)$ in $\min(\cdot)$.) But its only reasonable use seems to be the following corollary, the now well known minimax certainty equivalence principle of [3],[9] :

**Corollary 1** *If* $\forall t \in [0,T]$, $\forall u_{[0,T]} \in \mathbf{U}$, $\forall \omega \in \Omega$, $\hat{X}_t$ *is a singleton* $\{\hat{x}_t\}$, *then an optimal control is obtained by replacing* $x_t$ *by* $\hat{x}_t$ *in the optimal state feedback of the full information problem, i.e. taking* $u_t = \phi_t^*(\hat{x}_t)$.

# 4 Conclusion

The parallel between stochastic and minimax control appears thus as striking, even if some technicalities make it less clear in the continuous time case than in the discrete time case. Some more work probably remains to be done to fully explain and exploit it. But it is clear that "Quadrat's morphism" is at the root of the problem.

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