Differential Games : Lecture notes on the Isaacs-Breakwell Theory

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1 Introduction

1.1 An example : the obstacle tag chase game

Two children **P**eter and **E**lizabeth, play tag chase in a school courtyard. As every school courtyards, this one is flat and infinite. But in addition to the common one, it has a circular pool at its center, that the players are not able to cross. Otherwise, they run in the direction that they please, **P** at a speed a and **E** at a speed $b < a$, and they can turn instantly. The question is, how should they behave, **P** to capture **E** as quickly as possible, **E** to resist as long as she can. *Capture* is taken to mean that their positions coincide. To make the game non trivial, we assume that at the start of the game, the pool stands more or less between them.

Common sense tells us that this game has an obvious solution. Draw the shortest admissible path between **P** and **E**, made up of two pieces of straight line tangent to the pool and of an arc of circle joining them. **P** should run along that path, say until he sees **E** and can run directly at her. **E** should run in the direction of that path, but, of course, opposite to the direction of the pool. This is illustrated by fig. 1. The duration of this optimal chase is the initial distance between **P** and **E** as measured on the admissible path, divided through by $a - b$.

The shortest admissible path between the initial positions, which we have referred to, is unique unless they are exactly opposite with respect to the center of the pool, in which case **P** has the choice of which one to follow, choice to which **E** must adapt. It is a geodetic line of the playground, which is the not simply connected manifold made up of the plane with a circular hole in it. The optimal chase we have described happens entirely on this geodetic line, that both follow.

A second important remark about that geodetic solution is that it should be described carefully : **P** should run on the geodetic line *as long as he cannot run directly at* **E**. Which may happen at another time than planned if she has done the wrong thing. Put another way, the geodetic line between them may have changed because of nonoptimal play of **E**. And **P** should adapt in real time. This is symmetrically true for **E**, who should watch **P** and run away from him, whatever he does.

A third even more important remark is that the above strategies may be non optimal, as the counter example of figure 2 shows. On that figure, P_0 and E_0 represent the initial positions of both players, and P_1 and E_1 their positions after one unit of time, if they have played according to the geodetic strategy. It is apparent on the figure that the new geodetic line joining them is no longer the one they were following, and this has happened sometime before. This disqualifies the whole scheme.

This example was shown by Rufus Isaacs in the opening talk of the First International Conference on the Theory and Applications of Differential Games, held in Amherst, Massachussetts, in 1969, and later solved by John V. Breakwell. They are the two founders of the theory, and this course is dedicated to their memory.

1.2 Further examples

1.2.1 The Isotropic Rocket game

In the Isotropic Rocket game, again one proposed by R. Isaacs, the playground has no "hole", but the pursuer has control over its acceleration, say of fixed modulus Γ, the control being its angle u with the x axis, while the evader's control is the angle v of its velocity, of fixed modulus w, with the x axis. The pursuer being less agile than the evader, it can most probably not force exact coincidence of their positions, but *capture*, i.e. termination of the game, will be taken to mean that they are at a (euclidean) distance less than a given radius ℓ.

Taking a coordinate system with origin at **P**'s position, and inertial directions for the axes, this problem can be stated as follows.

The dynamical system is

$$\begin{aligned}\dot{x} &= w\cos v - \omega_x,\\ \dot{y} &= w\sin v - \omega_y,\\ \dot{\omega}_x &= \Gamma\cos u,\\ \dot{\omega}_y &= \Gamma\sin u.\end{aligned}$$

capture time is the first instant t_1 such that

$$x(t_1)^2 + y(t_1)^2 \leq \ell^2,$$

and the payoff, to be minimized by **P**, through the choice of u, and maximized by **E**, through the choice of v, is capture time, otherwise written as

$$\int_{t_0}^{t_1} dt.$$

Much more need to be said about the information available to both players to choose their controls. Clearly, **P**, at least, needs to "see" **E** to decide what to do (the blind cat does not catch mice), and the converse is also true if **E** is to be efficient. This reliance on *closed loop strategies*, or state feedback, will be discussed at greater length hereafter.

Also, what simultaneous minimization by one player and maximization by another one means requires further discussion. It will be assumed hereafter that the reader has some knowledge of classical minimax and game theory. And the solution concept looked for will be a saddle-point whenever possible. (In contrast to any course on classical game theory, the present lecture will avoid any use of mixed strategies).

1.2.2 The one-dimensional second-order servomechanism problem

A second order scalar plant is represented by the system

$$\ddot{y} = v, \qquad |v| \leq \beta.$$

One (say **E**) wishes to keep y close to a set point z, subject to drift. The drift of z is not known in advance (it is a disturbance of the system). The only information we have is its maximum velocity :

$$\dot{z} \leq \alpha.$$

The problem is to synthesize a control law that will keep, if possible, $|y - z|$ less or equal to a third given number γ for any possible disturbance. And if impossible, that will maximize the worst (earliest) possible escape time.

If one lets $\dot{z} = \alpha u$, $x_1 = y - z$, and $x_2 = \dot{y}$, the system may be written as

$$\begin{aligned} \dot{x}_1 &= x_2 - \alpha u, \qquad & |u| \leq 1 \\ \dot{x}_2 &= \beta v, & |v| \leq 1 \end{aligned}$$

game termination (or "capture") is defined by $|x_1| \geq \gamma$, and the payoff is as previously capture time.

We have stated explicitly our interest in determining whether termination can be avoided altogether. But of course, this question is also of interest in the previous game. We shall also have to deal carefully with the interplay between these questions.

This example is one of "worst case design". Let us just emphasize that solving it as a game where the disturbance minimizes escape time is not valid only under the pessimistic assumption that Nature (player **P** here) is antagonistic or "nasty". It is the necessary step to be able to ascertain that our control law will perform its task *under every circumstances.*

This concept has received much attention in the last decade with the advent of the so called H_∞ optimal control theory. Let us modify slightly this last example to make it an H_∞ optimal control problem.

1.2.3 The H_∞ one-dimensional second-order servomechanism problem

Let us assume now that final time is fixed. One is interested in keeping $y - z$ small, without spending too much control effort. (But we disregard the hard constraint on control magnitude). A measure of how badly we do in that respect may be taken as $\sqrt{J}$ with

$$J = \int_0^{t_1} (q x_1^2(t) + v^2(t))\, dt$$

where q is a design parameter allowing one to weigh the relative importance of the two objectives. Clearly we wish to keep J small.

Disregarding as well the hard constraint on the drift rate, we can safely say that the largest it is going to be, the largest J may be driven. A reasonable aim, therefore, is to control the *rate of growth* of J with respect to a measure of the intensity of the disturbance. We propose to take that measure as its L_2 norm $\|u\|$

$$\|u\|^2 = \int_0^{t_1} u^2(t)\,dt.$$

Therefore, given a desirable rate of growth γ, we would like to know whether there exists a control law $v = v(t,x)$ that guarantees that

$$\forall u(\cdot), \quad J < \gamma^2 \|u\|^2.$$

And this is true provided that

$$\inf_v \sup_u (J - \gamma^2 \|u\|^2) < 0.$$

where the minimization is to be performed over all admissible feedback laws.

This is again a differential game with the same dynamics as previously (less the hard constraints on u and v), and payoff

$$J_\gamma = \int_0^{t_1} (q x_1^2(t) + v^2(t) - \gamma^2 u^2(t))\,dt.$$

These examples were provided to start with so that the reader keeps in mind that these are the kind of problems we wish to formalize and solve from now on.

2 Two-person zero-sum perfect-information D.G.

2.1 The ingredients

We are given a two-person dynamical system of the form

$$\dot{x} = f(t,x,u,v) \tag{1}$$

where $x \in \mathcal{X} = \mathbb{R}^n$ is the state, $u \in \mathcal{U}$ and $v \in \mathcal{V}$ are the controls of the two players and $\mathcal{U} \subset \mathbb{R}^m$ and $\mathcal{V} \subset \mathbb{R}^p$ their control sets, and $t \in \mathbb{R}$ is the time. Admissible control functions $u(\cdot) : t \mapsto u(t)$ and $v(\cdot) : t \mapsto v(t)$ are measurable (or piecewise continuous, this will play no role) from $\mathbb{R}$, or a subinterval $[t_0, t_1]$, into $\mathcal{U}$ and $\mathcal{V}$ respectively. We shall call Ω_u and Ω_v these sets.

It is tacitly assumed that f satisfies regularity and growth hypotheses that guarantee existence of a solution of (1), over the time interval of interest, for every initial condition $(t_0, x_0) \in I\!\!R \times \mathcal{X}$ to be considered.

A *capture set* $\mathcal{C} \subset I\!\!R \times \mathcal{X}$ is defined, that specifies the *terminal time* of the game as the first instant t_1 such that

$$(t_1, x(t_1)) \in \mathcal{C}. \tag{2}$$

Interesting particular cases are

- $\mathcal{C} = \{T\} \times \mathcal{X}$, i.e., the game ends at time T whatever the controls are,
- $\mathcal{C} = I\!\!R \times C$, i.e., the game ends when x reaches the fixed subset C of $\mathcal{X}$.

Oftentimes, we shall assume that $\mathcal{C}$ is given by a set of inequalities

$$\gamma_i(t, x) \le 0, \quad i = 1, \dots r, \tag{3}$$

where γ is a given vector function from $I\!\!R \times \mathcal{X}$ into $I\!\!R^r$.

To each trajectory, generated from the initial condition (t_0, x_0) and the control functions $u(\cdot)$ and $v(\cdot)$, we associate a *payoff*

$$J(t_0, x_0; u(\cdot), v(\cdot)) = \begin{cases} K(t_1, x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), u(t), v(t))\, dt & \text{if } \exists t_1 \\ +\infty & \text{otherwise} \end{cases} \tag{4}$$

(We shall sometimes omit the first two arguments in J). This definition provides for the case where, $u(\cdot)$ and $v(\cdot)$ being defined over $[t_0, +\infty)$, capture never occurs, so that t_1 would not be defined. We shall come back to this topic later.

Here, K and L are real functions defined over the obvious sets, L assumed regular enough for the integral to exist.

The functions f, K, L as well as the boundary of the set $\mathcal{C}$ will be assumed to be of class C_1 almost everywhere, so that we can use their partial derivatives, even if we have to take special care about what happens at their points of non-differentiability. (This may require also that the sets of admissible strategies be restricted accordingly. As we shall see, this will be done implicitly). They are always assumed to be at least locally bounded.

It will always be understood that player **P**, the Pursuer, choosing the control u, wishes to minimize J, while player **E**, the Evader, choosing the control v, wishes to maximize it. As a result of our formulation (4), this means that, when possible, the Evader wishes above all to avoid capture, and if it cannot, maximize the payoff.

2.2 Strategies

An important feature is that we want to allow both players to have real time knowledge of the state, and use this information in chosing their controls. It must be emphasized that the situation here is very different from that in deterministic control because the state carries information on the past controls of the opponent. This is an important feature in stochastic control also, with the opponent replaced by chance. But in stochastic control, it makes sense to look for the optimal "open loop" control. Here the situation is worse: most often there is just no saddle-point in open loop controls.

We therefore wish to allow strategies of the form

$$\begin{aligned} u(t) &= \phi(t, x(t)) \\ v(t) &= \psi(t, x(t)) \end{aligned} \tag{5}$$

but this is the source of many questions and technical difficulties.

The first question is: why restrict oneself to (memoryless, or "markovian") *state feedback*, and not allow more general closed loop controls like $u(t) = \phi(t, x[t_0, t])$ and likewise for v, —where $x[t_0, t]$ stands for the whole restriction to $[t_0, t]$ of the time function $x(\cdot)$—, or at least initial state-dependant feedbacks? This question will be briefly taken up in the next subsection. Let us concentrate here on feedbacks of the form (5).

The natural thing to do would be to define sets Φ and Ψ of admissible feedbacks, and look for a saddle point (ϕ^*, ψ^*) over these sets, defined by the following inequalities (with a transparent abuse of notations : $J(t_0, x_0, \cdot, \cdot)$ was defined as a function over $\Omega_u \times \Omega_v$, and we use it over $\Phi \times \Psi$)

$$\forall \phi \in \Phi, \forall \psi \in \Psi, \quad J(t_0, x_0; \phi^*, \psi) \leq J(t_0, x_0; \phi^*, \psi^*) \leq J(t_0, x_0; \phi, \psi^*) \tag{6}$$

This poses the difficult problem of the right choice of admissible feedbacks Φ and Ψ. The main problem is with the existence of the solution to the dynamics with feedback control, i.e. to the differential equation

$$\dot{x} = f(t, x, \phi(t, x), \psi(t, x)), \qquad x(t_0) = x_0. \tag{7}$$

We know from simple examples that we cannot restrict admissible feedbacks to be continuous in x, let alone Lipshitz-continuous. As a result we have no convenient existence theorem. As a result, $J(\phi, \psi)$ might well not be defined for pairs (ϕ, ψ) that would not be compatible.

This difficulty has prompted much research work, some of the utmost relevance. Fleming, Friedman, Roxin, Varaiya, Krassovski and Soubbotin, Eliott and Kalton, and others, devised frameworks within which it was possible to give a precise mathematical meaning to such things, and prove existence of a Value, and most often that the value function in that particular framework was in some sense a solution of Isaac's equation (3) below. It took the recent work of Crandall and P.L. Lions on viscosity solutions of P.D.E's to establish, in a synthetic way, that all these values coincide.

We shall adopt a different approach. What makes it possible is that *we do not seek existence results* for the value or a saddle-point. We shall work in the framework of a sufficiency condition. We look for a pair of feedbacks ϕ^*, ψ^* which have the following properties:

- ϕ^* is *compatible with* Ω_v, i.e. the differential equation

  $$\dot{x} = f(t, x, \phi^*(t, x), v(t))$$

  has a solution for every $v(\cdot) \in \Omega_v$, generating an admissible open-loop control $u(t) = \phi^*(t, x(t))$, and this for every initial conditions.

- ψ^* is compatible with Ω_u.

- The following saddle-point inequalities hold:

$$\forall u(\cdot) \in \Omega_u, \forall v(\cdot) \in \Omega_v, \quad J(t_0, x_0; \phi^*, v(\cdot)) \leq V(t_0, x_0) \leq J(t_0, x_0; u(\cdot), \psi^*). \tag{8}$$

$V(t_0, x_0)$ is called the value of the game, and V the Value function. It is clear that if ϕ^* and ψ^* are compatible, then

$$J(t_0, x_0; \phi^*, \psi^*) = V(t_0, x_0).$$

The rationale for this definition is as follows. Notice first that any choice of admissible feedback sets Φ, Ψ, should satisfy the following properties.

Hypothesis H

H1 Φ and Ψ are compatible,

H2 $\Phi \supset \Omega_u$ and $\Psi \supset \Omega_v$.

The first hypothesis precisely means that, $\forall(\phi,\psi) \in \Phi \times \Psi$, the equation (7) has a unique solution $x(\cdot)$, and that the generated openloop controls $u(t) = \phi(t, x(t))$ and $v(t) = \psi(t, x(t))$ are in Ω_u and Ω_v respectively. The second one is, again, a slight abuse of notations to mean that, $\forall u(\cdot) \in \Omega_u$, the feedback defined by $\forall(t,x), \phi(t,x) = u(t)$ is admissible, and likewise for Ψ.

We have the simple but important following theorem.

Theorem 1 *If ϕ^* and ψ^* satisfy (8), they are a saddle-point in the sense of (6) over any pair of admissible sets $(\Phi \times \Psi)$ that satisfies the hypothesis* **H** *and contains (ϕ^*, ψ^*).*

Proof Because of H2, (6) clearly implies (8). Conversely, assume that (ϕ^*, ψ^*) satisfy (8). Take any $\bar{\phi} \in \Phi$. By H1 and the hypothesis that $\psi^* \in \Psi$, it follows that $(\bar{\phi}, \psi^*)$ together generate a well defined trajectory $\bar{x}(\cdot)$. Define $\bar{u}(\cdot) : t \mapsto \bar{u}(t) = \bar{\phi}(t, \bar{x}(t))$. Again, by H1 it is in Ω_u. The important point is that by its very definition, played against ψ^*, $\bar{u}(\cdot)$ generates the same trajectory as $\bar{\phi}$, ending in the same payoff :

$$J(\bar{u}(\cdot), \psi^*) = J(\bar{\phi}, \psi^*).$$

Then use (8) to get the rightmost inequality. And likewise for the left one.∎

As a result, it suffices to show (8), with no regard for the choice of admissible feedback sets. *Any* reasonable choice will do. It is worthwhile to notice that possible choices exist. One trivial one is as follows : take for Φ the closure under concatenation of $\{\phi^*\} \cup \Omega_u$, and likewise for Ψ. (We take the closure under finite concatenation because admissible strategy sets should always be closed under this operation. See next subsection). Less trivial choices can be proposed. For instance, take Φ as the largest set of state feedbacks compatible with ψ^* and Ω_v, and Ψ as the largest set compatible with Φ. This usually does not give the same sets as the symmetric construction.

A final remark is that in all the above, one of the feedbacks can be replaced by a *discriminating feedback*, i.e. a feedback of the form

$$u(t) = \phi(t, x(t), v(t)),$$

or symmetrically for v.

If the game has a value with discriminating feedback for **P** (but ordinary state feedback for **E**), it is called a *lower value*. Symmetrically, if the game has a value with a discriminating feedback for **E**, and ordinary state feedback for **P**, it is called an *upper value*. It is a simple matter to show that if a value exists in the sense of (8), it is also the lower value and the upper value.

It is also easy to see that a value defined as in (8) but with discriminating feedbacks for both players can be defined, but is of little interest. Take the following trivial example, where all variables are scalars.

Example

$$\dot{x} = 0, \qquad |u| \leq 1, \quad |v| \leq 1,$$

$$J = \int_0^1 |u + v| \, dt$$

$\phi^*(t, x, v) = -v$, $\psi^*(t, x, u) = \text{sign}(u)$ (with $\text{sign}(0) = 1$). One has, for any number $V \in [0, 1]$

$$J(\phi^*, v) = 0 \leq V \leq J(u, \psi^*) = 1 + \int_0^1 |u(t)| \, dt.$$

Exercise Find the upper saddle-point and the lower saddle-point of this game.

2.3 Memory and updating

This is only a brief summary of results that may be found in more detail in [2]. The right framework for this part is that of feedbacks *with perfect recall*, or any more general class of closed loop controls where each player's control may depend on past informations. Also, to avoid difficulties with the existence and unicity of a solution to the dynamical equation, it is possible to use a more abstract semi-group set up. We shall not be very precise here.

The main issue is the following one : is there a need for the players to remember past information, or does it *suffice*, in some sense, to use current value of the state? In classical control theory, Bellman's optimality principle says that current state is a sufficient information to play optimally, since the optimal control at time t and later is entirely specified as the solution of an optimal control problem depending on $t, x(t)$ (as its initial conditions) but not on past values of any of the variables. It is thanks to this remark that one may solve optimal control problems via dynamic programming, or synthesize an optimal state feedback from a field of optimal trajectories computed open loop.

To deal with that issue, we introduce *initial state dependant* feedbacks, without stating precisely what else they depend on, being understood that it is on values of the variables at times between initial time and current time. We write such strategies as $\phi(t_0, x_0)$ and $\psi(t_0, x_0)$.

We shall consider a family of optimal strategies $\{\phi^*(t, x), t \in I\!R, x \in \mathcal{X}\}$. That is, assume that the game considered has a value $V(t, x)$ for every initial conditions $(t, x) \in I\!R \times \mathcal{X}$, and

$$\forall v(\cdot) \in \Omega_v, \quad J(t, x; \phi^*, v) \leq V(t, x).$$

The interesting operation will be *updating* within a family. A strategy *updated at time* τ is the following one :

$$\phi_\tau(t_0, x_0)(t) = \begin{cases} \phi(t_0, x_0)(t) & \text{if } t < \tau, \\ \phi(\tau, x(\tau))(t) & \text{if } t \geq \tau. \end{cases}$$

Hence, updating the strategy at time τ within this family means that at time τ, the player forgets all previous information, looks at current time and state, and does as if these were the initial conditions of the game.

Dynamic programming is fundamentally based on the idea that the player may decide to update at some future time τ, and compute its current optimal strategy with integral cost only up to time τ, and final cost replaced by $V(\tau, x(\tau))$.

That this be allowable in differential games also is a consequence of the following *updating theorem.*

Theorem 2 *Let ϕ^* be a family of optimal feedbacks, then the feedbacks obtained by updating a finite number of times are also optimal.*

The new point is that this theorem is true only with the restriction to a finite number of updatings. This will be shown by the following example, where all variables are scalars [1]:

[1] This example is a joint work with J. Lewin, then at Raphael, now at the Technion Israel Institute of Technology

Example

$$\dot{x} = u + v, \qquad u \in [-1, 0], \quad v \in [-1, 0].$$

The capture set is $\{x \leq 0\}$, and the payoff as in (4) with $K = 0$ and $L(t, x, u, v) = 1 - x$.

Consider a game with $x_0 > 1$. In essence, since as long as $x > 1$ it "looses money", **E** should play $v = -1$, while to the contrary **P** should attempt to stay there as long as possible by playing $u = 0$. Conversely, as soon as $x \leq 1$, **P** should switch to $u = -1$ and **E** to $v = 0$. What we have described here is a pure state feedback. But let us pay attention to the open loop controls generated if both play that way. They are

$$\phi(t_0, x_0)(t) = \begin{cases} 0 & \text{if } t < t_0 + x_0 - 1 \\ -1 & \text{if } t \geq t_0 + x_0 - 1 \end{cases}$$

$$\psi(t_0, x_0)(t) = \begin{cases} -1 & \text{if } t < t_0 + x_0 - 1 \\ 0 & \text{if } t \geq t_0 + x_0 - 1 \end{cases}$$

It is possible to check that this pair of (open-loop) strategies is indeed a family of optimal strategies, with $V(t_0, x_0) = x_0(1 - x_0/2)$. However two families of strategies with potentially infinite updating can be described for **P** which are not optimal : either the strategy updated at constant time intervals, (for any such constant), or the "continuously updated" strategy, which is just the state feedback we started from. Both of these lead, for $x_0 > 1$, to $J(\phi_{\text{updated}}, 0) = +\infty$.

The above problem stems from the possibility that the game does not terminate. It is a fundamental aspect of the theory that we want to be able to deal with such games, typically pursuit-evasion games. It is true, however, that this problem does not show up for games that always terminate. We do not state it as a theorem, because it takes more care, and goes beyond the scope of the current lecture.

2.4 Playability

We have to say a word about a concept which we shall not use, but is often referred to in the literature. It is the concept of *playability*.

The objective is to "solve" simultaneously two problems that may interfere with the definition of the payoff in the presence of a pair of closed loop strategies : non-existence of a solution to the dynamics, and no termination —in the case where only the integral form of the payoff is given.

The concept is as follows. Pick first the sets Φ and Ψ of feedback strategies you wish to allow. Then, to any initial condition (t_0, x_0), associate the set of *playable pairs of strategies* $\mathcal{P}(t_0, x_0)$ as the set of all pairs $(\phi, \psi) \in \Phi \times \Psi$ such that with these strategies, the game has a unique trajectory, that terminates. A saddle point under this concept is a pair $(\phi^*, \psi^*) \in \mathcal{P}(t_0, x_0)$ such that

$$\forall \phi : (\phi, \psi^*) \in \mathcal{P}(t_0, x_0), \quad \forall \psi : (\phi^*, \psi) \in \mathcal{P}(t_0, x_0),$$

$$J(t_0, x_0; \phi^*, \psi) \leq J(t_0, x_0; \phi^*, \psi^*) = V(t_0, x_0) \leq J(t_0, x_0; \phi, \psi^*).$$

The nice point about this concept is that it lets one write and prove theorems. [4] is a good illustration of this point.

There are bad features, however. On the one hand, it is no longer a true saddle-point. In particular, two different "saddle-points" may exist, with *differing values* for the game. Let (ϕ_1^*, ψ_1^*) be one of them, and (ϕ_2^*, ψ_2^*) be another one, it is possible that

$$J(t_0, x_0; \phi_1^*, \psi_1^*) < J(t_0, x_0; \phi_2^*, \psi_2^*).$$

In that case, (ϕ_1^*, ψ_2^*) is not playable at (t_0, x_0).

On the other hand, and as a consequence, the saddle-point is not a normative concept anymore : it does not tell the players how they "should" play. In some sense they are supposed to "agree" on a playable pair of strategies, but in the case above, one does not see how they might reach such an agreement.

Our way around these problems is, on the one hand, to tell explicitly what is the payoff if the game does not terminate, on the other hand to insist that the strategy sets Φ and Ψ be compatible. We have seen that we may avoid to define them explicitly, but we do not avoid to assume that they are so chosen. In some sense, we insist that the game be played over a rectangle included into the set of playable strategies, at least if termination is not an issue.

3 Isaacs' Equation

3.1 Why not use Pontryagin's theorem?

Observe the right hand inequality in (8). It may be summarized in the following problem. Let

$$f^*(t,x,u) = f\left(t,x,u,\psi^*(t,x)\right)$$

and similarly for L^*. The problem is

$$\dot{x} = f^*(t,x,u), \qquad x(t_0) = x_0$$

$$J^*(u(\cdot)) = K(t_1, x(t_1)) + \int_{t_0}^{t_1} L^*(t,x,u)\, dt$$

Find ϕ^* such that

$$\forall u(\cdot) \in \Omega_u, \quad J^*(\phi^*) \le J^*(u(\cdot)).$$

(And if we assume (t_0, x_0) frozen, whether this optimal control problem is posed in open-loop or in closed-loop makes no difference.) Therefore, the control $u^*(\cdot)$ generated should satisfy Pontryagin's Maximum Principle. Symmetrically, the other inequality can be interpreted as a maximization problem. It is easy to see that the two problems lead to the same adjoint vector. This should be an efficient tool to synthesize the optimal strategies.

It is important to understand why this is not so. The point is that Pontryagin's theorem applies to a control problem with C_1 dynamics and integrand in the payoff. Here, we usually allow strategies less regular than C_1, because we need them to exhibit a saddle-point. As a consequence, f^* and L^* do not satisfy the requirements, and the necessary condition as stated in Pontryagin's theorem does not apply. Specifically, the gradient of the Value function, that will play a role similar to the adjoint vector, may happen to be *discontinuous along a trajectory*, a phenomenon forbidden in optimal control theory thanks to the Erdman-Weierstrass corner condition, embodied in Pontryagin's theorem.

Showing a true example of this problem requires to show such a discontinuity, an "equivocal line" or a "switch envelope" in the jargon of D.G's, a rather difficult task at this stage. Let us show first an example to the fact, at least, that integrating the "two-sided Pontryagin's equation" does not yield the value of the game, nor the open-loop min-max, nor the open-loop max-min, as sometimes stated in papers referring imprecisely to the "open-loop solution".

Example 1

All variables are scalar. The game dynamics are

$$\dot{x} = u + v, \qquad |u| \leq 2, \quad |v| \leq 1,$$

final time is $t_1 = 2$, and the payoff is purely final : $J = K(x(2))$ with

$$K(x) = |1 - |x||\,.$$

We shall look at the initial condition $t_0 = 0$, $x_0 = 3/2$. It is immediate to see that **P** can impose a slope to the trajectory between -1 and +1, so that from the said initial condition, it can force the state to reach $x(2) = 1$, insuring a payoff $J = K(1) = 0$ which is the absolute minimum of K. Hence the closed-loop value of the game is zero, as well as the open loop max-min. The open-loop min-max can easily be seen to be 1, since knowing $u(\cdot)$, **E** has a maneuverability of ± 2 over $x(2)$. So that **P** should choose $u(\cdot)$ so as to "center" that domain at zero, yielding the payoff said.

Let us try to apply a "two-sided Maximum Principle". The hamiltonian is

$$H(t, x, \lambda, u, v) = \lambda(u + v).$$

Thus we conclude that the optimal controls are $u = -\text{sign}(\lambda)$ and $v = \text{sign}(\lambda)$. The adjoint variable λ obeys the adjoint equation

$$\dot{\lambda} = 0,$$

hence the optimal controls are constant, and the transversality condition at $x(2) = -1/2$ is $\lambda(2) = -1$ yielding the trajectory $x(t) = 3/2 - t$, which goes through our initial condition. This trajectory provides a payoff $J = K(-1/2) = 1/2$, which is neither the closed-loop value of the game, neither any of the open-loop max-min or min-max.

Actually, the problem with that trajectory is that it crosses the equivalent of a conjugate point (a dispersal line) at $t = 3/2$, $x = 0$, and this contradicts its optimality. This would also happen in a control problem. It is why we give now a more elaborate example, of a purely game theoretic phenomenon.

But we shall be obliged to state the fact without proof, the complete theory going beyond the scope of this lecture. (This example is a modification of one of [7])

Example 2

All variables are scalar, the state has dimension 2, and is denoted (x, y). The dynamics are

$$\begin{aligned} \dot{x} &= y + v, \quad |v| \leq 1, \\ \dot{y} &= u, \qquad |u| \leq 1. \end{aligned}$$

The playing space is $y \geq 0$, and the capture set $\{x \geq 0,\ y = 0\}$. The payoff is

$$J = x(t_1) + \int_{t_0}^{t_1} dt.$$

Application of a "two-sided Maximum Principle" as above leads to

$$\begin{aligned} \lambda_x &= 1, \\ \lambda_y &= 2 + (t_1 - t) = 2 + y, \end{aligned}$$

$u = -1$, $v = 1$, trajectories of the form

$$\begin{aligned} x(t) &= x_1 - (t_1 - t) - \frac{1}{2}(t_1 - t)^2, \\ y(t) &= t_1 - t, \end{aligned}$$

yielding a Value function $V(x, y) = x + 2y + y^2/2$.

It turns out that these trajectories, and hence this expression of the value, are wrong to the left of a line defined by $x = y - y^2/2$ for $y \leq 1$ and $x = 2 + 2\ln y - y - y^2/2$ for $y \geq 1$. To the left of this line, the optimal controls are both opposite to those above. In terms of adjoints, on that line the adjoints jump from the value given above to

$$\begin{aligned} \lambda_x &= 1 - \frac{y}{y-1}, \\ \lambda_y &= 0. \end{aligned}$$

No local analysis can show that. It is why we shall have to turn to a global theorem, in the spirit of Dynamic Programming.

3.2 A simple form of Isaacs' equation

We now state a simple theorem, which in itself does not account for the more complex solutions showed above. But this is the starting point of all the analysis.

The notations are those of section (2.1) above. We define as in the examples above the *hamiltonian* of the game as the following function H from $I\!R \times I\!R^n \times I\!R^n \times \mathcal{U} \times \mathcal{V}$ into $I\!R$:

$$H(t, x, \lambda, u, v) = L(t, x, u, v) + (\lambda, f(t, x, u, v)) \tag{9}$$

We also use the notation ∇V for the gradient $\partial V / \partial x$ and V_t for $\partial V / \partial t$.

Theorem 3 (Isaacs) *If there exist a C_1 function V from $I\!R \times \mathcal{X}$ into $I\!R$ and two admissible, (in the sense defined just before (8)), feedbacks ϕ^*, ψ^* such that,*

$$\forall (t, x) \in I\!R \times \mathcal{X} \setminus \mathcal{C}, \quad \forall u \in \mathcal{U}, \quad \forall v \in \mathcal{V},$$

$$V_t + H(t, x, \nabla V, \phi^*(t, x), v) \leq 0 \leq V_t + H(t, x, \nabla V, u, \psi^*(t, x)), \tag{10}$$

$$\forall (t, x) \in \partial \mathcal{C}, \quad V(t, x) = K(t, x), \tag{11}$$

then (ϕ^*, ψ^*) constitutes a saddle point (in the sense of (8)) for all initial conditions (t_0, x_0) that ϕ^* drives to C against any control $v(\cdot)$ of **E**. *For these initial conditons, the Value of the game is $V(t_0, x_0)$.*

Proof Consider the time function $t \mapsto V(t, x(t))$ where $x(t)$ obeys (1), and let dV/dt denote its derivative, which depends on the trajectory followed. Notice that

$$V_t + H(t, x, \nabla V, u, v) = \left(\frac{dV}{dt} + L \right) (t, x, u, v).$$

Consider an initial condition (t_0, x_0) driven to C by ϕ^* against any control of **E**, and an arbitrary such control $v(\cdot)$. Consider the trajectory generated by this control and ϕ^*. According to the leftmost inequality of (10) and to the above remark, along that trajectory one has

$$\frac{dV}{dt} + L \leq 0.$$

By assumption that trajectory terminates at a (finite) time t_1. Integrate the above inequality from t_0 to t_1, to obtain

$$V(t_1, x(t_1)) - V(t_0, x_0) + \int_{t_0}^{t_1} L\, dt \le 0.$$

Now, $(t_1, x(t_1)) \in \partial\mathcal{C}$. Thus, using (11), we get

$$K(t_1, x(t_1)) + \int_{t_0}^{t_1} L\, dt \le V(t_0, x_0),$$

which shows the left hand inequality of the saddle-point.

Symmetrically, consider an arbitrary control $u(\cdot)$ of **P**, and the trajectory generated from (t_0, x_0) by this control together with ψ^*. If that trajectory does not terminate, by the definition (4), $J = +\infty \ge V(t_0, x_0)$. If the trajectory terminates, reproduce the above argument mutatis mutandis to obtain again the right hand side saddle-point inequality. ▮

The relation (10) above implies, among other things, that the hamiltonian has a saddle-point in (u, v) for fixed (t, x). Since the dependance of ∇V in (t, x) cannot be known in advance, this leads to the following hypothesis, that we shall make all along from now on

Isaacs' Condition For all (t, x, λ) in $I\!R \times \mathcal{X} \times I\!R^n$, the hamiltonian has a saddle-point in (u, v), i.e., there exists

$$\min_u \max_v H(t, x, \lambda, u, v) = \max_v \min_u H(t, x, \lambda, u, v) = \bar{H}(t, x, \lambda). \tag{12}$$

The above condition is satisfied by many an application. But the real problem is that in most games, (in almost all examples above), the Value function of the game is just not C_1, oftentimes not even C_0 (continuous). Hence the objective of the classical theory, as developed in [3] for instance, has been to extend this theorem to less regular Value functions, analyse the nature of the singularities, and give means to compute them. We shall only hint at that hereafter, concentrating on the discontinuities of that function. (See [1] for a short account.)

But let us first see what can be done where it is very regular.

3.3 Integrating Isaacs' equation

We can use, in an attempt to integrate Isaacs' equation, the technique of the characteristics. For that purpose, write

$$\lambda(t) = \nabla V(t, x(t)).$$

In any open set of $\mathbb{R} \times \mathcal{X}$ where V is C_2, it is a classical fact that along a trajectory generated by $\bar{u}(t) = \phi^*(t, x(t))$, $\bar{v}(t) = \psi^*(t, x(t))$, one has

$$\dot{\lambda} = -\frac{\partial H(t, x(t), \bar{u}, \bar{v})}{\partial x}. \tag{13}$$

The proof is easily made by differentiating with respect to x in

$$V_t + H(t, x, \nabla V, \phi^*(t, x), \psi^*(t, x)) = 0,$$

and recognizing that $V_{tx} + V_{xx} f = \dot{\lambda}$. Moreover, at least if the extremizing controls in (10) are unique, by Danskin's theorem, one may ignore the dependance of ϕ^* and ψ^* in x in taking the partial derivative. (If an extremizing control is not unique over an open time interval, one has a singular arc and it takes a closer look to check that the same applies.)

Clearly, (10) and (13) together *look like* a two-sided Maximum Principle. They are the same equations, and (11) allows one to recover the transversality conditions of Pontryagin's theorem. The difference of course is that we cannot assert that an optimal trajectory necessarily satisfies these equations with a continuous λ. Because we do not know before hand what the regularity of the Value function is going to be.

As a matter of fact, *if* the Value function is C_1, (10) can be shown to be necessary, and *if* it is C_2, (13) as well. But we don't know when and where this applies.

3.4 Example : the one-dimensional second-order servomechanism problem

Let us try to apply the above construction to our example of section (1.2.2). The boundary of the capture set is made up of the two "vertical" lines $x_1 = \gamma$ and $x_1 = -\gamma$. Let us look at the first one. The rest will be obtained by symmetry. Let us write λ_1 and λ_2 for $\partial V / \partial x_1$ and $\partial V / \partial x_2$ respectively. Notice also that since all the data of the game are stationary, the Value function does not depend on t. (The duration of an optimal game does not depend on its initial time.)

The hamiltonian of the game is

$$H = 1 + \lambda_1(x_2 - \alpha u) + \lambda_2 \beta v.$$

The extremizing controls are thus $\bar{u} = \text{sign}(\lambda_1)$, $\bar{v} = \text{sign}(\lambda_2)$.

The adjoint equations (13) become simply

$$\begin{aligned} \dot{\lambda}_1 &= 0, \\ \dot{\lambda}_2 &= -\lambda_1. \end{aligned}$$

The transversality condition (11) yields here that on the line $x_1 = \gamma$, $\lambda_2 = 0$. The value has to be positive in the playing region $x_1 < \gamma$, hence on the boundary, $\lambda_1 < 0$. (If they are not both zero). This suffices to see that, as long as this applies (as long as V is sufficiently regular), the optimal controls are both -1, yielding parabolic trajectories of the form

$$\begin{aligned} x_1(t) &= \gamma - (\alpha + x_2(t_1))(t_1 - t) - \frac{1}{2}\beta(t_1 - t)^2, \\ x_2(t) &= x_2(t_1) + \beta(t_1 - t). \end{aligned}$$

These and the symmetrical ones are shown on figure 3, for some typical values of the parameters, with $\alpha^2 < \beta\gamma$. The figure for other values of the parameters will appear later.

Something special happens at $x_2(t_1) = -\alpha$. At this point, the incoming trajectory is tangent to the capture set boundary. And at points "lower" than $-\alpha$ on that boundary, the trajectories we computed reach this boundary from the other side, i.e. from inside the capture set. Clearly these trajectories have no meaning, since they have been "captured" at an earlier time, while this calculation has been made with another assumption.

One may use the equation

$$\min_u \max_v H = 0$$

(since we have seen that $V_t = 0$) to recover λ_1 on $x_1 = \gamma$. This yields here

$$\lambda_1(t_1) = -\frac{1}{x_2(t_1) + \alpha}.$$

So we see that two problems occur : on the one hand, for $x_2(t_1) < -\alpha$, λ_1 has the "wrong" sign, since we had argued that it had to be negative. On the other hand, at $x_2(t_1) = -\alpha$, we cannot compute λ, we seem to have an "infinite" gradient.

It is this problem that we are going to investigate now.

4 Barriers

Notice that it is always possible to formulate a differential game of the form of section (2.1) as *stationary*, i.e. with all data : f, K, L, $\mathcal{C}$, independent from t. It suffices to introduce an extra state variable x_{n+1}, with the dynamics

$$\dot{x}_{n+1} = 1, \quad x_{n+1}(t_0) = t_0,$$

and to replace everywhere the dependance on t by one on x_{n+1}. This simply amounts to including the time as one of the state variables. For instance, the expression $V_t + H$ that appears in Isaacs' equation becomes under that transformation simply H, because what was V_t becomes ∇V_{n+1} and $f_{n+1} = 1$. Hence we may pretend that V only depends on x (which includes time) and set $V_t = 0$.

This simplifies the exposition. So in the sequel, we shall assume that, after doing that transformation if necessary, the game is stationary. And we shall keep the letter n for the dimension of the state space : $\mathcal{X} = \mathbb{R}^n$.

From now on, we shall also need a variant of Isaacs' condition (12), that we shall again refer to under the same name, because it causes no ambiguity:

Isaacs' condition for barriers For all (x, ν) in $\mathcal{X} \times \mathbb{R}^n$, there exists a saddle point

$$\min_u \max_v (\nu, f(x,u,v)) = \max_v \min_u (\nu, f(x,u,v)) = (\nu, f\,(x, \bar{u}(x,\nu), \bar{v}(x,\nu))) \tag{14}$$

(And remember that, if necessary, the time t has been included in the state x.)

4.1 Semipermeable surfaces

Theorem 4 (Isaacs) *Assume the Value function has a jump discontinuity on a hypersurface $\mathcal{B}$ in $\mathcal{X}$. Assume that this hypersurface is a C_1 manifold, and for all $x \in \mathcal{B}$, let $\nu(x)$ be a normal to $\mathcal{B}$, pointing towards the half space where V is largest. Assume that Isaacs' condition (14) holds. Then the following necessarily holds :*

$$\forall x \in \mathcal{B}, \quad \min_u \max_v (\nu, f(x,u,v)) = 0. \tag{15}$$

Proof Let $\bar{x}$ be a point of $\mathcal{B}$ where all the data are C_1. Assume that contrary to the theorem, in a neighborhood of x, there exists a control $\bar{u}$ such that, $\forall v \in \mathcal{V}$, the dot product in (15) is negative. Then, for x sufficiently close to $\bar{x}$, in the (local) half space of large V, **P** may force the state to cross $\mathcal{B}$ in an arbitrarily short amount of time, and since L is bounded in that neighborhood, at an arbitrarily small cost, in particular at a cost inferior to the jump in V across $\mathcal{B}$ at x. This is clearly a contradiction. And the same holds symmetrically for the other control. ∎

Definitions A hypersurface that satisfies (15) at each of its points is called *semipermeable*. A hypersurface that carries a jump discontinuity of the Value is called a *barrier*.

Therefore, the above theorem reads "barriers are semipermeable surfaces". It does *not* read "discontinuities of V happen on semipermeable (hyper)surfaces". Because discontinuities of V might be more complex than a simple jump, and, more often, happen on a set less regular than a C_1 manifold. We shall soon investigate a simple instance of that.

Let us immediately stress that the converse statement would be grossly wrong. One may easily find semipermeable surfaces which do not carry a discontinuity of V. As a matter of fact, we will show that generically, such a surface can be constructed passing through any $n-2$-dimensional differentiable manifold in the state space.

A further remark is that a particularly important discontinuity of V is that where it is infinite on one side of $\mathcal{B}$. Thus, hypersurfaces that separate a region where capture occurs against any strategy of **P** from regions where **P** can avoid capture forever, thus where $J=\infty$, are semipermeable where they are C_1.

Notice that equation (15) has a form similar to Isaacs' equation. Let the hypersurface be given by $W(x) =$ constant, then ∇W is a suitable normal $\nu(x)$, and (15) is exactly (10) with $L=0$, i.e. with a purely terminal payoff. As a matter of fact, this has been used by some authors (see [4]) to derive Isaacs' equation. One can always make the payoff purely terminal by adding yet another state variable (usually called x_0) governed by

$$\dot{x}_0 = L(x,u,v), \quad x_0(t_0) = 0,$$

$$J(u,v) = x_0(t_1) + K(x(t_1)) = \bar{K}(x(t_1)).$$

Then remark that the set of optimal trajectories that reach the set of points $K(x) = V$ for a constant V has to be semipermeable, otherwise the player who can do it would have the state cross that surface and reach a better point, providing it with a better payoff.

That approach seems very general, and seems to provide a necessary condition of optimality. For the same reason as previously, this is not so, because we have no way to insure that the "surface" is a differentiable manifold.

Yet, it is useful to remember that, for games with purely terminal payoff, these sets of trajectories (and each trajectory if the state space is of dimension 2), are semipermeable wherever they are C_1.

4.2 Construction of semipermeable surfaces

Notice that if $\mathcal{X} = I\!R^n$, $\nu(x)$ in (15) is defined by $n-1$ parameters, since it is defined up to a normalization. Now, (15) is one scalar condition on ν at each $x \in \mathcal{X}$. Therefore, generically, there exists a cone of solutions at each x. We shall call these solutions *semipermeable normals*, and the velocity $f(x, \bar{u}, \bar{v})$ that corresponds to it in (15) a *semipermeable velocity*.

In many games, the dynamics are *separated*, i.e. of the form

$$f(x, u, v) = h(x, v) - g(x, u). \tag{16}$$

(This happens each time we have a pursuit-evasion game, where the state is the relative positions, and possibly velocities, of two players acting otherwise independently of each other). In that case, there is an interesting geometric characterization of semipermeable normals and velocities.

Let $P(x) = g(x,\mathcal{U})$ and $Q(x) = h(x, \mathcal{V})$ be the holograph domains (or *vectograms* in Isaacs' wording) of **P** and **E** respectively. Consider *bitangent* hyperplanes, that are hyperplanes that intersect both P and Q but leave them both in the same closed half space. It is straightforward to see that any normal to such a hyperplane is a semipermeable normal, and a vector joining the intersection points of a bitangent hyperplane with P and Q (in that direction) is a corresponding semipermeable velocity.

Let us turn now to the construction of semipermeable surfaces. We shall denote by $\bar{u}(x, \nu)$ and $\bar{v}(x, \nu)$ the (a) pair of extremalizing controls in (15). Consider a $n-2$-dimensional differentiable manifold $\mathcal{S}$. We shall assume that it is parametrized by a map $x = \xi(s)$ where s ranges over an open set in $I\!R^{n-2}$. At each point of that manifold, look for a semipermeable normal which is furthermore normal to $\mathcal{S}$. According to our previous remark, and noticing that being normal to $\mathcal{S}$ amounts to the $n-2$ scalar constraints $(\nu, \partial\xi/\partial s_i) = 0$, there is generically one (or a finite number of) solution(s) to that condition. Pick a solution varying continuously with s, which is feasible at least locally. Let $\nu = n(s)$ be that solution. Then, consider the differential equations

$$\dot{x} = f(x, \bar{u}(x,\nu), \bar{v}(x,\nu)), \quad x(0) = \xi(s), \tag{17}$$

$$\dot{\nu}' = \nu' \frac{\partial f}{\partial x}(x, \bar{u}(x,\nu), \bar{v}(x,\nu)), \quad \nu(0) = n(s). \tag{18}$$

Let $x(t) = \mathcal{X}(s,t)$ and $\nu(t) = \mathcal{N}(s,t)$ be the solution of the above equations whenever it exists.

Theorem 5 *Assume that the extremalizing controls $\bar{u}, \bar{v}$ are unique along the solutions of (17). Then, the map $(s,t) \mapsto \mathcal{X}(s,t)$ defines a semipermeable hypersurface, with semipermeable normal $\mathcal{N}(s,t)$ for all (s,t) for which it is defined and such that the jacobian matrix of $\mathcal{X}(\cdot,\cdot)$ is of rank $n-1$.*

Sketch of proof The proof is not very difficult, but a bit cumbersome. We shall leave the details to the reader. One must check that ν remains normal to the set of x's, i.e. that the dot products $(\nu, f(x,\bar{u},\bar{v}))$ and $(\nu, \partial\mathcal{X}/\partial s_i)$, $i = 1, \ldots, n-2$ remain null. This is done by checking that, by construction, they are such at $t = 0$ (i.e. on $\mathcal{S}$), and that their time derivatives are zero all along the integration. The unicity of the extremalizing controls is needed to identify the r.h.s. of $\dot{\nu}$ above with $d/dx[\nu' f(x, \bar{u}(x,\nu), \bar{v}(x,\nu))]$ via Danskin's theorem. ∎

Several remarks are in order. The first one is that the proof sketched above only requires that $s \mapsto \xi(s)$ be differentiable, and that, for all s, $(n(s), \partial\xi/\partial s) = 0$. A particular case of interest where this is satisfied is one where ξ is a constant, and $n(s)$ is a parametrization of the local cone of semipermeable directions.

Two more remarks have to do with the requirement that the set $\mathcal{X}(s,t)$ be (locally) a differentiable manifold, i.e. that its jacobian be of rank $n-1$. On the one hand, this is necessary if $\mathcal{X}$ is to be an $(n-1)$-dimensional manifold. The locus of points where it fails is usually a cusp of the hypersurface. Beyond the cusp (in retrogressive integration) the hypersurface may again be semipermeable, but with its oriented normal ν pointing towards the region which was to be that of low V's before the cusp. Therefore, one must ignore this hypersurface beyond a cusp.

In the case where $n = 3$, the only way this can happen is that $\mathcal{X}_t = f$ be parallel to $\mathcal{X}_s$. The cusp appears as an envelope of the trajectories, which are smooth across it. It therefore takes some care to discover this cusp.

4.3 The B.U.P. and the Natural Barrier

Assume the capture set $\mathcal{C}$ is an open set in $I\!R^n$, with C_1 boundary $\partial\mathcal{C}$. Let x be a point of that boundary. Let $\nu(x)$ be an outward normal to $\mathcal{C}$ at x. Assuming **E** wants to avoid termination of the game, it can always do so unless $\min_u \max_v (\nu, f(t,x,u,v)) < 0$. The region of the capture set boundary where this condition is satisfied is called the *Usable Part* of the capture set. The boundary of this Usable Part will therefore be made up of points of $\partial\mathcal{C}$ where $\min_u \max_v (\nu, f(t,x,u,v)) = 0$. Generically this is a $n-2$- dimensional submanifold of $\partial\mathcal{C}$, called the *Boundary of the Usable Part*, or B.U.P.

On the B.U.P., the natural scheme to recover ∇V from the condition (11) fails. As a matter of facts, it consists in writing that $\nabla V = K_x + \alpha\nu$, computing α from $\min_u \max_v H = 0$, i.e.,

$$\alpha = \frac{L(t,x,\bar{u},\bar{v})}{(\nu, f(t,x,\bar{u},\bar{v}))}$$

By continuity, we see that $\|\nabla V\|$ goes to infinity as x approaches the B.U.P. in the Usable Part.

Along the B.U.P., according to its characterization, the outward normal ν to the capture set is semipermeable. We may construct a semipermeable surface joining on this B.U.P. according to the scheme of the previous subsection. This (hyper)surface will usually exist locally in a neighborhood of the B.U.P. Since on the B.U.P. it shares its normal with $\partial\mathcal{C}$, it is tangent to $\mathcal{C}$. It is a natural candidate to be a barrier, called the *Natural Barrier.* In particular it will usually be part of the complete barrier when the capture zone is bounded, and delimited by a set of semipermeable surfaces.

One must be careful however, it is possible to find examples where part or all of the (candidate) natural barrier must be dismissed in the complete solution of a game.

Example continued We take up the example of the second order one dimensional servomechanism problem. The boundary of the capture set is made up of the two lines $x_1 = -\gamma$ and $x_1 = \gamma$. The Usable Part on the first one is given by $\min_u \max_v [\nu_1(x_2 - \alpha u)] < 0$ with $\nu_1 > 0$, and both inequalities reversed on the second one. Thus the B.U.P. is made up of the two points $x_1 = -\gamma$, $x_2 = \alpha$, and $x_1 = \gamma$, $x_2 = -\alpha$.

The parabola through $x_1 = -\gamma$, $x_2 = \alpha$ is part of the natural barrier of this game. Another part is its symmetrical with respect to the origin, joining on the other part of the B.U.P.

Example 2 We consider here example 2 of section (3.1). Somewhat artificially, we have decided that the capture set ends at $x = 0$. Therefore, attempting to stay to the "left" of the origin makes sense for **E**. We try to construct a barrier through this point, and succeed! As a matter of fact, at the origin, $(\nu_x, \nu_y) = (-1, 1)$ is a semipermeable normal. Equations (17) read

$$\begin{aligned} \dot{\nu}_x &= 0, \qquad \nu_x(t_1) = -1, \\ \dot{\nu}_y &= -\nu_x, \quad \nu_y(t_1) = 1, \end{aligned}$$

$u = -\text{sign}(\nu_y)$, $v = \text{sign}(\nu_x)$. This yields the parabola $x = y - y^2/2$ up to $y = 1$, $x = 1/2$. Beyond that point, integration of the same equations yields a cusp, and we discard this further part.

This explains why, as we claimed in section (3.1), to the left of this line, the trajectories proposed in that section are no longer optimal. However, it does not tell us what *is* the optimal solution. As we said, this is beyond the scope of the present lecture.

4.4 Composite barriers

Let us come back to the example of the servomechanism. If $\alpha^2 < \beta\gamma$, we have a region delimited by two C_1 manifolds, which is a candidate to be a region of no termination: $V(x) = \infty$. These two manifolds are symmetric with respect to the origin. Each of them is made up of a barrier parabola and a line segment joining in a C_1 fashion, e.g.

$$x_1 = \begin{cases} \gamma - \frac{1}{2\beta}(x_2 + \alpha)^2 & \text{if } x_2 \geq -\alpha \\ \gamma & \text{if } x_2 \leq -\alpha \end{cases}$$

and the symmetrical one.

On these manifolds, one has $\max_v(\nu, f(t,x,u,v)) \geq 0)$, (the inequality being strict on the vertical part of them), and therefore by playing properly, **E** can indeed make sure that the state never crosses them. But the combined boundary is not a C_1 manifold, having "corners" at the points where each parabola cuts the opposite vertical line, i.e. at $(x_1, x_2) = (-\gamma, 2\sqrt{\beta\gamma} - \alpha)$ and the symmetrical point. We must investigate in more detail what happens at these points. As a matter of fact, the situation is simple, since at these points, $\dot{x}_2$ is positive at $-\gamma$ and negative at γ for any controls of **E**. (As well as **P** for that matter). Hence, **E** may choose her control in such a way as to make sure that the state does not cross the parabola, and she wins.

Therefore, for $\alpha^2 < \beta\gamma$, we have indeed shown a region of no capture, and the strategy that let the controller keep the state in the desired region, $|y| \leq \gamma$.

Let us examine now the case where $\alpha^2/2 < \beta\gamma < \alpha^2$. The upper barrier parabola cuts the x_2 axis at $x_2 = \sqrt{2\beta\gamma} - \alpha$. The two parabolas still delimit a region which we consider as a candidate for being a "no escape" region. Again, the only problem lies with the corners, at $x_2 = \pm\sqrt{2\beta\gamma - \alpha^2}$. Assume the state follows the upper parabola, and reaches the "south-east" corner. There, assume that **P** sticks with his control $u = -1$ which he uses on that parabola. Either **E** continues playing $v = -1$, but then the state continues on that same parabola, and thus leaves the region of interest, or **E** does anything else, but then the state crosses this same parabola upwards, again leaving the desired region. In J.V. Breakwell's terms, the corners "leak".

All we can say at this stage is that these two parabolas alone do not define a "no escape zone". As a matter of facts, it can be shown that all initial states may be captured (i.e. the perturbation can always drive the state outside of the desired region). But an optimal chase may be extremely complicated. J.V. Breakwell has shown that for any integer N, there is a set of parameters and an initial state such that an optimal chase has more than N velocity reversals for the perturbation.

For $\beta\gamma \leq \alpha^2/2$, the parabolas do not cross, and there is no possibility of a no escape zone. [2]

The above analysis can easily be generalized. We have the following

[2] The two symmetrical fields of parabolas fill the whole game space, but part of it is covered twice. These two fields must be separated by an "evader's dispersal line", computed as the line of equal optimal escape time on both fields. We leave it to the reader to complete the analysis. (The dispersal line is a straight line.)

result:

Theorem 6 *If a barrier is composed of two semipermeable C_1 manifolds that join in a non C_1 fashion, and if on trajectories of one of these manifolds incoming to the junction the semipermeable strategy of the* inferior player *(see below) is unique at the junction, then these trajectories do not cross the junction. (They may either be tangent to it or have a corner)*

Proof If the composite hypersurface is to be a barrier, one of the players, hereafter called the inferior player, has to be able to keep the state on the right side (for him) of both semipermeable manifolds, (and therefore be able to prevent crossing of either), while its opponent, hereafter called the superior player, has to be able to keep the state on *its* side of at least one, either one, of the two. Which depends on the geometry of the junction.

Assume that trajectories incoming to the junction cross it, i.e. cross the other semipermeable manifold. (Otherwise, the junction would be globally C_1, and there would be no problem). On a trajectory incoming to the junction, assume the superior player keeps playing its semipermeable strategy on this incoming trajectory. Then if the inferior player keeps its incoming strategy, the state will follow the incoming trajectory, and thus by hypothesis cross the junction. And since the two manifolds are assumed to be transverse to each other (non tangential), crossing the junction on one of them is crossing the other one. If the inferior player does anything else, the state will cross the incoming semipermeable surface, since the control that avoids that crossing is by assumption unique. In both cases, the inferior player has failed. Therefore that composite surface cannot be a barrier.∎

Notice that as a result of the above theorem, the incoming semipermeable manifold has a singularity on the junction: either an extremizing control in (15) is non unique, which usually means that it will switch between two values if one attempts to integrate further, or the trajectories are all tangent to the junction, thus have an envelope. In both cases the hypersurface made of these trajectories is not a C_1 manifold at the junction.

This condition sometimes allows one to actually compute a candidate junction on a semipermeable manifold previously computed, from which to integrate retrogressively the equations of the incoming semipermeable trajectories. But we shall not attempt to develop this point any further.

5 Conclusion

There exists a rather complete theory of the possible singularities of Isaacs' equation, somewhat along the lines of the simple case of the junction of barriers shown above. It should be emphasized that, while such an analysis is *local*, it serves a *global* theorem. Only when we have a complete set of optimal trajectories filling a *capture region* (possibly the whole space $\mathcal{X}\backslash\mathcal{C}$), separated from the *escape region* by a barrier do we have solved the game. This can be a formidable task as soon as the dimension of the state space gets large. We know many 2-D games completely solved that way, few 3-D ones, and essentially none in higher dimension. Moreover, applying that theory is more of an art than a technique. There is no hope to have something like a computer package to do it on a wide class of games.

It is why a new approach is currently being pursued, which attempts to more systematically tackle Isaacs' equation through discretization schemes, adapted from numerical analysis. At this time, that theory is not able to deal with barriers, and it is not clear how well it will do in terms of higher dimension. But at least in small dimension, it has the prospect of replacing art with technique. Yet, it remains an interesting aim to better understand the nature of the singularities, which that new approach does not do.

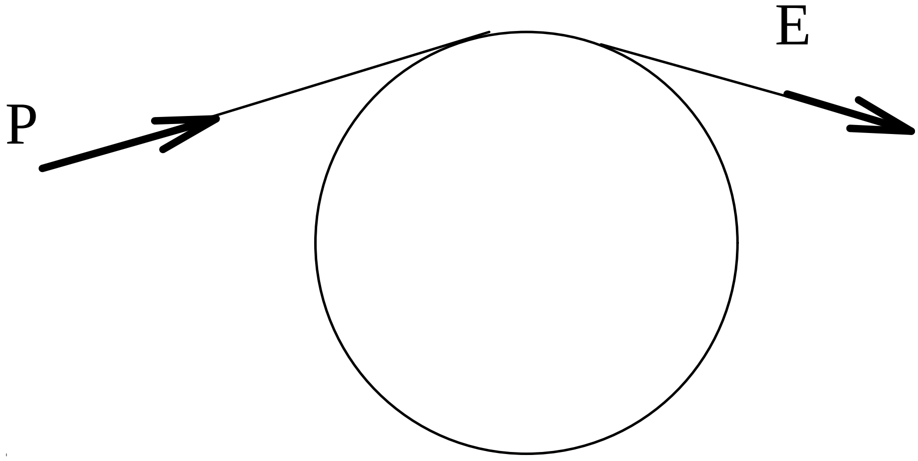

Figure 1:

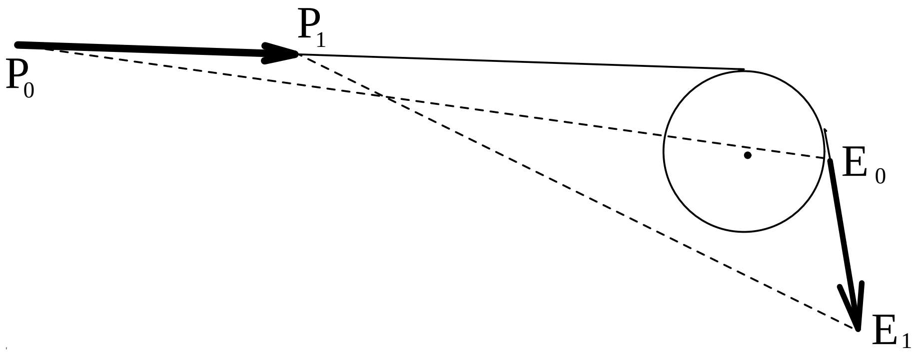

Figure 2:

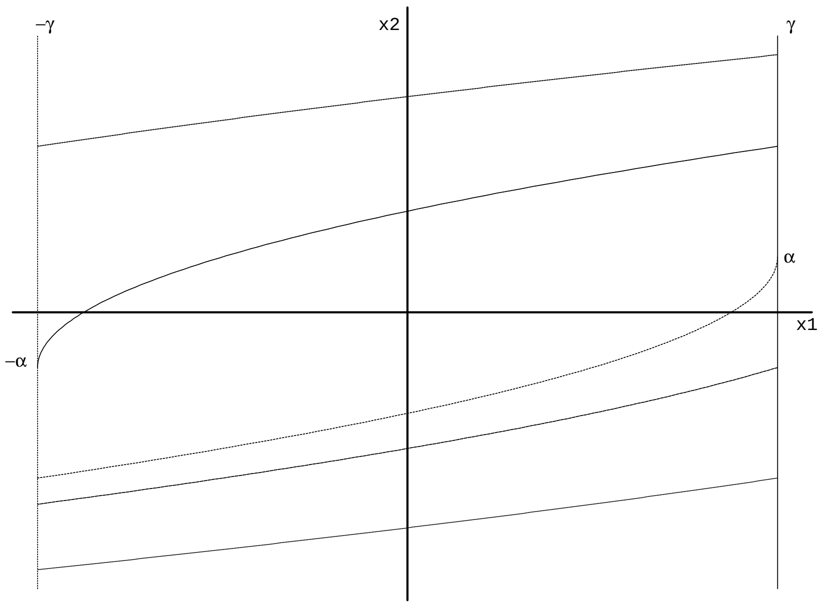

Figure 3:

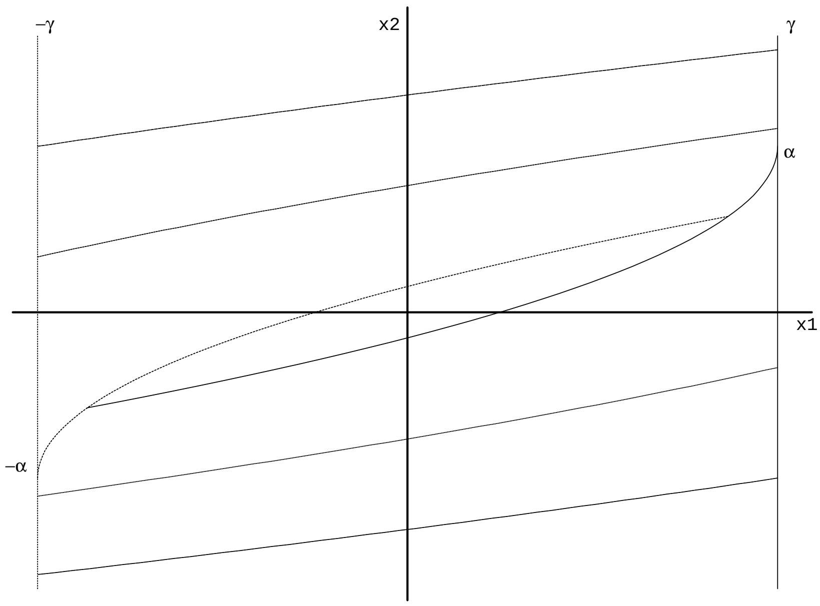

Figure 4:

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