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Application of the min–max certainty equivalence principle to the sampled data output feedback H^∞ control problem

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Abstract We apply our certainty equivalence principle to the solution of the sampled data output feedback H^∞ control problem As expected, the solution bears close resemblance to a Kalman filter design

Keywords H^∞ control; sampled data; min–max; certainty equivalence, dynamic games.

Introduction

In [7], we derived a certainty equivalence principle for min–max control problems with incomplete information. It says that one should, at each instant of time, compute the worst perturbation compatible with the currently available information, and use the current state on the corresponding trajectory as the 'estimate' of the state, and place it in the optimal state feedback strategy, obtained as the saddle point of a full information two-person zero-sum game.

On the other hand, it has recently been discovered that the so called H^∞ control problem is fundamentally a min–max control problem. See [11] and [12] for a review of this aspect. This has allowed several authors to use a game theoretic approach to solve these problems. In the most classical cases, this gives back the state space solutions, such as derived in [8] for instance. As a matter of fact, this approach even predates H^∞ control theory (see [5]). It also allows one to solve new problems, such as the time varying finite time problem for instance, and also more significant extensions such as the delayed measurement or the sampled data problems for example. See [1,2,3] in particular.

However, up to recently, this powerful approach could not be applied to the output feedback problem, or 'four block problem', for the lack of a theory of min–max control with partial information. State space formulas were nevertheless obtained by other means, see [8,10]. In [13], a solution was found to the classical four-block problem by a min–max approach, completing the squares. This is not yet a general way of tackling such problems

In this paper, we show that application of our certainty equivalence principle very easily yields the solution of the sampled data output feedback problem. Since this problem seems to have eluded attempts to solve it via other means, this is an indication that the new theory is indeed powerful.

1. The certainty equivalence principle

1.1. The continuous measurement problem

We quickly recall here our main theorem of [7]. Let a two-player dynamical system be given by

$$\dot{x} = f(x, u, v, t), \quad x(t_0) = x_0. \tag{1}$$

with

$$t \in [t_0, T], \quad x(t) \in \mathbb{R}^n, \quad u(t) \in U, \quad v(t) \in V.$$

Let Ω_u and Ω_v be the set of open loop controls. Adequate regularity and growth conditions have been assumed on f to guarantee existence and unicity of the solution of (1) over $[t_0, T]$ for any x_0 and any $(u, v) \in U \times V$.

A criterion, to be minimized by the first player and maximized by the second, is given by

$$J_{x_0,t_0}(u, v) = M(x(T)) + \int_{t_0}^{T} L(x(t), u(t), v(t), t)\, \mathrm{d}t + N(x_0). \tag{2}$$

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We assume that the corresponding full information zero-sum two-person differential game, *without the $N(x_0)$ term* that we add for future use, has, in an adequate setting, a pure feedback strongly time consistent saddle point

$$u(t) = \phi^*(x(t), t), \qquad v(t) = \psi^*(x(t), t), \tag{3}$$

and a piecewise C^1 value function $V(x, t)$.

Introduce now a partial measurement or output

$$y(t) = h(x(t), v(t), t) \tag{4}$$

We also allow for an uncertain initial state $x_0 \in X_0$. The problem we tackle is to find a causal controller $u = K(y)$ that will solve the following problem

Problem $\mathscr{P}$.

$$\min_K \max_{\substack{v \in \Omega_v \\ x_0 \in X_0}} J_{x_0, t_0}(K(y), v)$$

The solution to this problem as given in [7] involves, for each $t_1 \in [t_0, t]$, a family of *auxiliary problems* $\mathscr{Q}^{t_1}(u, y)$ parametrized, beyond t_1, by the past control and observation histories $u[t_0, t_1]$ and $y[t_0, t_1]$, with payoff function

$$\begin{aligned} G^{t_1}(x_0, u[t_0, t_1], v[t_0, t_1]) &= V(x(t_1), t_1) \\ &\quad + \int_{t_0}^{t_1} L(x(t), u(t), v(t), t)\, \mathrm{d}t + N(x_0). \end{aligned} \tag{5}$$

The auxiliary problem is one of maximization under constraint, defined as

Problem $\mathscr{Q}^{t_1}(u[t_0, t_1], y[t_0, t_1])$

$$\max_{x_0 \in X_0} \max_{v \in \Omega_v^{t_1}} G^{t_1}(x_0, u[t_0, t_1], v[t_0, t_1])$$

In the above, $\Omega_v^{t_1}$ stands for the set $\Omega_v^{t_1}(x_0, u[t_0, t_1], y[t_0, t_1])$ of all perturbations $v[t_0, t_1]$ that, together with x_0 and the past controls $u[t_0, t_1]$, generate the past outputs $y[t_0, t_1]$. Let $\hat{v}^{t_1}(\ )$ be the solution of this problem, assumed to exist and be unique, and $\hat{x}^{t_1}(\cdot)$ be the corresponding trajectory The theorem says that, on the one hand, under these assumptions an optimal controller is obtained by using

$$u(t) = \phi^*(\hat{x}^t(t), t), \tag{6}$$

and that on the other hand, if for some t_1^* the auxiliary problems have an infinite supremum for all (u, y), then the original problem had an infinite supremum for all causal controllers K

The proof is indeed very simple, and is based upon showing that the return function

$$W^{t_1}(u, y) = \max_{x_0 \in X_0} \max_{v \in \Omega_v^{t_1}} G^{t_1}(x_0, u[t_0, t_1], v[t_0, t_1])$$

is decreasing with time t_1 as soon as u is chosen according to (6). This also proves that this controller is strongly optimal in that sense that it not only solves problem $\mathscr{P}$, but moreover, if one uses any control until some intermediary t_1, and then (6) from then on, it guarantees the best possible value to J given the information up to time t_1. (In [7] we called this property time consistency, but it seems to be a bad choice)

1.2 The sampled data problem

It is straightforward to see that the principle of [7] extends to the set up where the available measurements occur at a sequence of time instants $\{\tau_1, \tau_2, \ldots, \tau_N\}$, where $t_0 \le \tau_1 < \tau_2 < \cdots < \tau_N < T$. We must now split the perturbation v into (v, w) where v is a continuous part and w a discrete sequence (or impulsive part) $\{w_k\}$ Let (4) be replaced by

$$y_k = h_k(x(\tau_k), w_k). \tag{7}$$

We must also include a term in w in the cost, replacing (2) by

$$\begin{aligned} J_{x_0, t_0}(u, v, w) &= M(x(T)) + \int_{t_0}^{T} L(x(t), u(t), v(t), t)\, \mathrm{d}t \\ &\quad + \sum_{k=1}^{N} \tilde{L}_k(x(\tau_k), w_k) + N(x_0) \end{aligned} \tag{8}$$

To make things simple, we shall assume that, $\forall x$,

$$\tilde{L}^k(x, 0) = 0, \qquad \tilde{L}^k(x, w) < 0, \quad \forall w \neq 0. \tag{9}$$

Then, in the full information game, the obvious solution is $w = 0$, leaving the optimal strategies and the value unchanged

The same term in $\tilde{L}$ must also be added to G^{t_1} in the definition (5) of the auxiliary problem, limiting the summation to all indices k such that $\tau_k < t_1$. We shall often call ι the largest such index. $\tau_\iota < t_1 \le \tau_{\iota+1}$.

Between two measurement instants τ_k, the proof of [7] remains unchanged At an instant τ_ι, consider the auxiliary problem taking y_ι into account. Either $\hat{w}_\iota^{\tau_\iota} = 0$. Then on the trajectory $\hat{x}^{\tau_\iota}$, G^{t_1} is continuous, and the theory of [7] applies. Or $\hat{w}_\iota^{\tau_\iota} \neq 0$, and G^{t_1} has a jump decrease. In both cases, it is decreasing, and we may still conclude that W^{t_1} is decreasing. This is the crux of the theorem

2. The sampled data output feedback H^∞ control problem

2.1. The problem

We are given the following linear system in $\mathbb{R}^n$ over the time interval $[t_0, T]$, in which a sequence $\{\tau_k\}$ is given, $t_0 \le \tau_1 < \tau_2 < \cdots < \tau_N < T$.

$$\dot{x} = F(t)x + G(t)u + E(t)v, \quad x(t_0) = x_0,$$
$$y_k = H_k x(\tau_k) + J_k w_k,$$
$$z = C(t)x + D(t)u$$

We set the following notations:

$$\begin{pmatrix} C' \\ D' \end{pmatrix}(C \quad D) = \begin{pmatrix} Q & S \\ S' & R \end{pmatrix}, \qquad J_k J_k' = N_k \tag{10}$$

A causal controller

$$u(t) = K(y_1, y_2, \dots, y_\iota)(t), \quad \text{where } \tau_\iota < t \le \tau_{\iota+1},$$

is said to have an attenuation level γ if it guarantees

$$\forall (x_0, v, w) \in X_0 \times \Omega_u \times \Omega_w,$$
$$\|z\|^2 + \|x(T)\|_A^2$$
$$\le \gamma^2\left(\|v\|^2 + \|w\|^2 + \|x_0\|_{B^{-1}}^2\right).$$

The norms have to be understood with respect to the appropriate spaces: L^2 spaces for the continuous variables z and v, $\mathbb{R}^{nN}$ for w, and $\mathbb{R}^n$, with weighting positive definite matrices for x_0 and $x(T)$ The H^∞ control problem consists in characterizing the infimum γ^* of all possible attenuation levels, and for any $\gamma > \gamma^*$ in finding a controller that achieves that attenuation level.

2.2. Applying the general theory

It is now classical to associate to the above problem the criterion

$$J_\gamma(x_0, u, v, w)$$
$$= \|x(T)\|_A^2 + \int_{t_0}^T \|z\|^2 \,\mathrm{d}t$$
$$- \gamma^2\left(\int_{t_0}^T \|v\|^2 \,\mathrm{d}t + \sum_{k=\iota}^N \|w_k\|^2 + \|x_0\|_{B^{-1}}^2\right), \tag{11}$$

and to seek whether the problem

$$\min_K \max_{x_0, v, w} J_\gamma(x_0, K(y), v, w)$$

has a solution. (It will then be zero)

A saddle point of the complete information differential game in uniformly Lipschitz feedback strategies exists if and only if the following Riccati equation has a solution over $[t_0, T]$ (see [6]):

$$\dot{P} + P(F - GR^{-1}S') + (F' - SR^{-1}G')P$$
$$- P(GR^{-1}G' - \gamma^{-2}EE')P + Q - SR^{-1}S' = 0,$$
$$P(T) = A. \tag{12}$$

The saddle point is then given by

$$\begin{aligned} u(t) &= -R^{-1}(G'P + S')x(t), \\ v(t) &= \gamma^{-2}E'Px(t), \end{aligned} \tag{13}$$

and the value function is $V(x, t) = \|x\|_{P(t)}^2$.

To express the auxiliary problem, we turn to an operator form Let $t_1 \in (\tau_\iota, \tau_{\iota+1}]$. We write u, v, z for $u[t_0, t_1]$, $v[t_0, t_1]$, and $v[t_0, t_1]$, and $z[t_0, t_1]$, that belong to appropriate L^2 spaces, and y and w for $\{y_k\}_{k \le \iota}$ and $\{w_k\}_{k \le \iota}$ in appropriate Euclidean spaces The system restricted to $[t_0, t_1]$ may be written in the form

$$y = \mathscr{A}u + \mathscr{B}v + \mathscr{T}w + \eta x_0,$$
$$z = \mathscr{C}u + \mathscr{D}v + \zeta x_0, \tag{14}$$
$$x(t_1) = \lambda u + \mu v + \nu x_0$$

The lemma in [7] extends to the following one:

Lemma. *The system*

$$\alpha = \mathscr{A}^* p + \mathscr{C}^* q + \lambda^* \xi_1,$$
$$\beta = \mathscr{B}^* p + \mathscr{D}^* q + \mu^* \xi_1, \qquad \rho = \mathscr{T}^* p, \tag{15}$$
$$\xi(t_0) = \eta^* p + \zeta^* q + \nu^* \xi_1,$$

admits the following internal representation:

$$-\dot{\xi} = F'\xi + C'q, \quad \xi(t_1) = \xi_1,$$
$$\xi(\tau_k^-) = \xi(\tau_k^+) + H_k' p_k, \tag{16}$$
$$\alpha = G'\xi + D'q, \qquad \beta = E'\xi, \qquad \rho_k = J_k' p_k.$$

Proof. Compute

$$\xi'(t_1)x(t_1) - \xi'(t_0)x(t_0)$$
$$= \int_{t_0}^{t_1} (\dot{\lambda}'x + \lambda'\dot{x})\,\mathrm{d}t + \sum_{k=1}^{l} (\xi'(\tau_k^+) - \xi'(\tau_k^-))x(\tau_k)$$

substituting from the above equations, and identify to finally get

$$\xi_1' x(t_1) + \int q'z + \sum p_k' y_k = \xi'(t_0)x_0 + \sum \rho_k w_k + \int \beta v + \int \alpha u$$

This proves the lemma.

In these notations, the auxiliary problem reads

$$G^{t_1} = \|\mathscr{C}u + \mathscr{D}v + \zeta x_0\|^2 + \|\lambda u + \mu v + \nu x_0\|_{P_1}^2 - \gamma^2(\|v\|^2 + \|w\|^2 + \|x_0\|_{B^{-1}}^2),$$

with the constraint

$$\mathscr{A}u + \mathscr{B}v + \mathscr{T}w + \eta x_0 = y$$

Form the Langrangian with a multiplier $2p$, differentiate with respect to v, w, and x_0 to obtain.

$$\mathscr{B}^* p + \mathscr{D}^* \hat{z} + \mu^* P_1 \hat{x}(t_1) = \gamma^2 \hat{v}, \qquad \mathscr{T}^* p = \gamma^2 \hat{w},$$
$$\eta^* p + \zeta^* \hat{z} + \nu^* P_1 \hat{x}(t_1) = \gamma^2 B^{-1} \hat{x}_0,$$

where $\hat{z} = Cu + \mathscr{D}\hat{v} + \zeta\hat{x}_0$, and $\hat{x}(t_1) = \lambda u + \mu\hat{v} + \nu\hat{x}_0$.

As in [7], we use the representation (15), (16) to express these conditions more concretely. We easily get $\hat{v} = \gamma^{-2}E'\xi$, $\hat{w}_k = \gamma^{-2}J_k'p_k$, with $p_k = \gamma^2 N_k^{-1}(y_k - H_k\hat{x}(\tau_k))$. Overall, the necessary conditions read:

$$\dot{\hat{x}} = F\hat{x} + \gamma^{-2}EE'\xi + Gu, \quad \hat{x}_0 = \gamma^{-2}B\xi(t_0), \tag{17}$$
$$\dot{\xi} = -Q\hat{x} - F'\xi - C'Du, \quad \xi(t_1) = P_1\hat{x}(t_1), \tag{18}$$
$$\xi(\tau_k^-) = \xi(\tau_k^+) + \gamma^2 H_k' N_k^{-1}(y_k - H_k\hat{x}(\tau_k)). \tag{19}$$

If we can find a solution of this boundary value problem, and if the auxiliary problem is concave, then the optimal controller we propose is $u(t_1) = -R^{-1}(G'P + S')\hat{x}(t_1)$ for each t_1, or more explicitly, recalling that $\hat{x}$ in the above equations stands for $\hat{x}^{t_1}$,

$$u(t) = -R^{-1}(G'P + S')\hat{x}^t(t) \tag{20}$$

2.3 Recursive formulas

The apparent difficulty to obtain recursive formulas similar to those in [7] comes from the fact that the jumps in ξ are in terms of $\hat{x}^{t_1}(\tau_k)$, therefore depending on t_1. The way around this difficulty will be to introduce jumps in Σ so as to get jumps in terms of the recursive variable $\check{x}$. Let us therefore introduce the matrix $\Sigma(t)$ defined, when it exists, by

$$\dot{\Sigma} = F\Sigma + \Sigma F' + \gamma^{-2}\Sigma Q \Sigma + EE', \quad \Sigma(t_0) = B, \tag{21a}$$

$$\Sigma(\tau_k^+) = \Sigma(\tau_k^-)\left[I + H_k' N_k^{-1} H_k \Sigma(\tau_k^-)\right]^{-1} = \left[\Sigma^{-1}(\tau_k^-) + H_k' N_k^{-1} H_k\right]^{-1} \tag{21b}$$

Notice that according, for instance, to [13], if $\Sigma(\tau_{k-1}^+)$ is positive definite, it remains so over (τ_{k-1}, τ_k). Then our jump condition at τ_k amounts to a positive jump on the inverse. Therefore the inverse will stay positive definite after the jump, and thus Σ also. Hence, B being positive definite, Σ will be so as long as it exists.

Introduce the variable

$$\check{x} = \hat{x} - \gamma^{-2}\Sigma\xi.$$

Some staightforward calculations show that it satisfies the following equations

$$\dot{\check{x}} = (F + \gamma^{-2}\Sigma Q)\check{x} + (G + \gamma^{-2}\Sigma S)u, \quad \check{x}(t_0) = 0, \tag{22a}$$

$$\check{x}(\tau_k^+) = \check{x}(\tau_k^-) + \Sigma(\tau_k^+)H_k' N_k^{-1}(y_k - H_k\check{x}(\tau_k^-)) \tag{22b}$$

But now, these equations do not depend on t_1. Therefore they give a recursive formula for $\check{x}^{t_1}(t_1)$, from which we recover $\hat{x}^{t_1}(t_1)$ using the final condition of (18) (expressed in terms of t instead of t_1)

$$\hat{x}^t(t) = \left[I - \gamma^{-2}\Sigma(t)P(t)\right]^{-1}\check{x}(t). \tag{23}$$

We can now state the theorem.

Theorem. *Let γ^* be the optimal attenuation level for the sampled data output feedback H^∞ control problem. Then a necessary condition for $\gamma \geq \gamma^*$ is that the Riccati equations* (12) *and* (21) *have a solution over $[t_0, T]$, satisfying $\rho(\Sigma(t)P(t)) \leq \gamma^2$, $t_1 \in [t_0, T]$. If this last inequality is strengthened to a strict one, then $\gamma > \gamma^*$, and a strongly optimal controller is given by equations* (12), (21), (22), (23), *and* (20) *to be placed back into* (22).

Proof. The proof goes exactly as in [7], with some obvious modifications that we shall indicate here. We still look at the homogeneous problem, where $(u, y) = (0, 0)$ The generating matrices of the extremals are now defined by

$$\dot{\Phi} = F\Phi + EE'\Psi, \quad \Phi(t_0) = B,$$

$$\dot{\Psi} = -\gamma^{-2}Q\Phi - F'\Psi, \quad \Psi(t_0) = I$$

$$\Psi(\tau_k^+) = \Psi(\tau_k^-) + H_k'N_k^{-1}H_k\Phi(\tau_k).$$

As previously, we see that $\Sigma = \Phi\Psi^{-1}$ The fundamental identity of the extremals $x(t) = \Phi(t)\mu$, $\xi(t) = \gamma^2\Psi(t)\mu$, now reads

$$\xi'(t_1)x(t_1) - \xi'(t_0)x(t_0)$$
$$+ \int_{t_0}^{t_1}\left(\|x\|_Q^2 - \gamma^2\|\hat{v}\|^2\right) dt$$
$$- \gamma^2\sum_{k=1}^{i}\|\hat{w}_k\|^2 = 0.$$

so that on an extremal, we have

$$G^{t_1} = \|x(t_1)\|_{P_1}^2 - \xi'(t_1)x(t_1)$$
$$= \|x(t_1)\|_{P(t_1) - \gamma^2\Sigma^{-1}(t_1)}^2,$$

the second expression holding true when $\Sigma(t_1)$ exists. This lets us show trivially that invertibility of Ψ, hence existence of Σ, is indeed necessary, as well as $P(t_1) - \Sigma^{-1}(t_1) \leq 0$, for all $t_1 \in [t_0, T]$.

When this last inequality is strict, a Hamilton–Jacobi–Caratheodory theory under the constraint

$$H_k x(\tau_k) + J_k w_k = 0, \quad k = 1, \ldots, i,$$

can be developped. As it requires some more care, we give some details here (Exactly the same theory could be made on the particular problem at hand using completion of the squares, hiding the general nature of the approach.)

Let a system be given by

$$\dot{x} = f(x, v), \quad x(t_0) = x_0,$$

together with a criterion

$$G(x_0, v, w) = M(x(t_1)) + \int_{t_0}^{t_1} L(x, v)\, dt$$
$$+ \sum_{k=1}^{i} \tilde{L}(x(\tau_k), w_k) + N(x_0)$$

We are interested in the problem of maximizing G with respect to x_0, v, and w, subject to the constraint that for a sequence of time instants $\{\tau_k\}$,

$$h_k(x(\tau_k), w_k) = 0.$$

Assume there exists a function $W(x, t)$, and an admissible feedback $\hat{v} = \psi(x, t)$ satisfying the following equations.

$\forall x \in X_0$, $W(x, t_0) = -N(x_0)$,

$\forall t \neq \tau_k$, $\forall x \in \mathbb{R}^n$,

$$\frac{\partial W}{\partial t} + \frac{\partial W}{\partial x}f(x, v) + L(x, v) \leq 0,$$

with equality for $v = \hat{v}$,

for $k = 1, \ldots, i$, $\forall x \in \mathbb{R}^n$,

$$W(x, \tau_k^+) = W(x, \tau_k^-) - \hat{L}_k(x),$$

where $\hat{L}$, assumed to exist, is defined by

$$\hat{L}_k(x) = \max_w \tilde{L}_k(x, w)$$

under the constraint that $h_k(x, w) = 0$,

the max being reached at $w = \hat{w}_k(x)$.

Then, for all admissible x_0, v, w,

$$G(x_0, v, w) \leq M(x(t_1)) - W(x(t_1), t_1),$$

the equality being reached for $v = \hat{v}$ and $w = \hat{w}$.

It suffices to integrate by part along a trajectory the partial differential equation, noticing that the jump condition on W at τ_k can be written

$$W(x, \tau_k^+) - W(x, \tau_k^-) + \tilde{L}_k(x, w) \leq 0,$$

with an equality for $w = \hat{w}$

Applying this theory to the auxiliary problem exactly yields the last expression for $G^{\prime\gamma}$ above as an upper bound, which is reached on the extremals. This ends the proof.

As a final remark, notice that the equations for $\check{x}$ above look very much like a Kalman filter Its differential equation part is similar to that of the continuous measurment problem

$$\dot{\check{x}} = F\check{x} + \gamma^{-2}\Sigma C'\check{z} + Gu$$

the term in $y - H\check{x}$ entering of course in the jumps. It would be particularily interesting to know whether the same filter shows up in the LEQG problem, strengthening the similarity between these two fields (see [4,5])

Conclusion

This short developement shows the power of the min-max certainty equivalence principle in solving various forms of the four block H^∞ control problem. Clearly, many particular cases of this deserve closer examination. One is the case where the measurements are produced by integrating devices, that is, of the form $y_k = y(\tau_k) + w_k$, with

$$\dot{y} = Hx + Jv, \quad y(t_0) = y_0$$

This is actually a particular case of the above, taking $(x', y')'$ as the state. Either y_0 is assumed to be unknown; then one should add a cost on it in the payoff, presumably with a very large weight B_y^{-1}, hence a very small B_y, to mean that it has to be small, or it is assumed to be zero A slight extension of the theory shows that this is obtained by making the corresponding B_y zero.

Many other classical problems could be solved along the same lines, including delayed measurement, mixed sampled data and continuous measurements, etc.

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