partial observation, whose application to the linear quadratic control problem provides the solution to the H^∞ four block control problem. We first solve the finite time continuous problem with continuous measurement. The solution agrees with recently published versions of the "central controller", although not exactly with the classical one, whose equivalence with our formulas is less than obvious. The new formulation bears closer resemblence with the Kalman filter. Moreover this establishes the "central controller" of the literature as the one that best exploits the available information. We also use the same approach to solve the continuous dynamics, sampled data output feedback problem, and the discrete time one. Both are new results as far as we know. The former is completely proved, as the proofs are very similar to the continuous case. Complete proofs for the latter are not included, but will be given in a monography to appear soon. In every case, standard tools of control theory and stationary Riccati equations allow one to extend the results to the stationary infinite time problem. Again details are not given here but in the aforementioned monography.

Un principe d'équivalence à la certitude minimax et son application aux problèmes de commande H$^\infty$ optimale continu, à mesures échantillonnées et discret.

Résumé. On démontre un principe d'équivalence à la certitude pour des problèmes de commande minimax, dont l'application au problème linéaire quadratique donne la solution de problèmes de commande "H^∞ optimale" dits "à quatre blocs". (En feedback de sortie). On résoud d'abord le problème continu, retrouvant des résultats récents auxquels on apporte quelques améliorations. Puis on résoud le problème continu à mesures échantillonnées et le problème en temps discret, deux résultats nouveaux à notre connaissance.

Abstract. We derive a certainty equivalence principle for min-max control problems with partial observation, whose application to the linear quadratic control problem provides the solution to the H^∞ four block control problem. This approach yields the solution of the discrete or continuous time, finite horizon time varying or stationary problem. This establishes the "central controller" of the literature, in continuous time, as the one that best exploits the available information. We also find new formulas whose equivalence with the previous ones is by no means obvious. In addition, our results also apply to game problems other than the H^∞ control problem.

Introduction.

Worst case design as an approach to sensitivity reduction is an old idea. A whole session of an IFAC symposium held in 1973 was devoted to that topic. In ref [6], presented at that symposium, we derived the now called H^∞ controller for the full information linear quadratic problem, and began a discussion of its merits as compared to the classical H^2 controller.

In [6], the problem considered was explicitly to minimize the largest possible value of a quadratic performance index under an L^2 norm constraint on the perturbation, therefore explicitly minimizing the (square of) the operator norm of the resulting closed loop system. The fact that the so caled H^∞ control problem is equivalent to that, with an added stability constraint in infinite time, is now well understood. Moreover, the literature has introduced the so-called suboptimal H^∞ problem, which amounts to a min-max problem with a fixed criterion. See [2]–[4], [11], [13]–[15], and the bibliography of [11] and [14] in particular. This has given rise to solutions of that problem in the state space domain, in both finite and infinite time.

However, the partial information case, or the four block problem in the jargon of H^∞ control, is less easy to solve using the tools of min-max control, for the lack of a theory of partial information min-max control. In [13], such a solution is given, using ad hoc tools for the linear quadratic case, and *assuming apriori* that the admissible controllers are information case in [12].

In the continuous time case, we show that the controller obtained is in some sense strongly time consistent, (see [1]), implying that it optimally exploits the available information. The equivalent property for the discrete time controller turns out to require a somewhat more complex controller than the classical one. To keep the exposition simple, we shall omit it here, deferring this topic to a later paper.

In section 1, we derive the certainty equivalence principle in a nonlinear setup, as well as the converse theorem we need to apply it to H^∞ control, both in continuous and discrete time. In section 2, we apply this principle to the continuous time and the discrete time H^∞ control problems respectively.

1. The certainty equivalence principle.

1.1 The Problem.

Let a two player dynamical system in $\mathbb{R}^n$ be given by

$$\dot{x} = f(x, u, v, t), \qquad x(t_0) = x_0, \tag{1}$$

where

$$x(t) \in \mathbb{R}^n, \quad u(t) \in U, \quad v(t) \in V, \quad t \in [t_0, T].$$

The time instants t_0 and T are fixed, U and V may be taken for instance as closed subsets in euclidean spaces. We denote by Ω_u and Ω_v the sets of admissible control functions (or "open loop controls") from $[t_0, T]$ into U and V respectively. (Say, measurable functions.) The function f is assumed to satisfy regularity and growth conditions that insure existence and unicity of the solution of (1) over $[t_0, T]$ for any (x_0, u, v) in $\mathbb{R}^n \times \Omega_u \times \Omega_v$.

We shall often need restrictions $u[t_0, t]$ and $v[t_0, t]$ of u and v to $[t_0, t]$, and shall again write $u[t_0, t] \in \Omega_u$ and likewise for v, as this abuse does not result in an ambiguity. (It would be more consistent with the rest of our notations to write $u[t_0, t] \in \Omega_u^t$.)

We shall consider a differential game problem whose dynamics wil be (1), where feed-

Here, $y(t) \in Y$ where, for instance, $Y = I\!R^p$. We shall consider the problem of devising a controller of the form

$$u = K(y)$$

where K has to be *causal*. Id est

$$u[t_0, t] = K(y[t_0, t]).$$

The admissible controllers are all the causal controllers that are compatible with Ψ, i.e such that $\forall \psi \in \Psi$, the differential equation

$$\dot{x} = f(x, K(y), \psi(x, t), t)$$

$$y(t) = h(x(t), \psi(x(t), t), t)$$

has a unique solution, satisfying $K(y) \in \Omega_u$.

A criterion or payoff function is given, of the form

$$J(u, v) = M(x(T)) + \int_{t_0}^{T} L(x(t), u(t), v(t), t)\, dt, \tag{3}$$

where L and M are given real functions, L regular enough to insure existence of J. We shall also use the abusive but unambiguous notations $J(\phi, v)$, $J(u, \psi)$, and $J(K, v)$ or $J(K, \psi)$. Also, in a classical way, we shall imbed the game problem so defined in a family letting x_0 and t_0 vary, so that (1)(3) define $J_{x_0, t_0}(u, v)$.

The min-max control problem is usually defined as

$$\min_{K} \max_{v \in \Omega_v} J(K, v).$$

However, this logically yields a controller K that depends on x_0, and this may be unacceptable in a framework where the state is not available to form the control u, but only

1.2 The auxiliary problem.

The system (1)(3) may be viewed as defining a zero-sum two-person differential game, with u as the minimizer and v as the maximizer. Assume it admits a unique state feedback (i.e. full information) saddle point (ϕ^*, ψ^*), and a Value function V piecewise C^1. That is, (see [9])

$$\forall (u,v) \in \Omega_u \times \Omega_v, \quad \forall (\xi,\tau) \in I\!R^n \times [t_0, T],$$

$$J_{\xi,\tau}(\phi^*, v) \leq J_{\xi,\tau}(\phi^*, \psi^*) = V(\xi,\tau) \leq J_{\xi,\tau}(u, \phi^*).$$

The Value function V satisfies Isaacs'equation, stated in terms of

$$H(x,u,v,t) = \frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t) f(x,u,v,t) + L(x,u,v,t)$$

as : $\forall (u,v) \in U \times V$, $\forall (x,t) \in I\!R^n \times [t_0, T]$,

$$H(x, \phi^*(x,t), v, t) \leq H(x, \phi^*(x,t), \psi^*(x,t), t) = 0 \leq H(x, u, \psi^*(x,t), t).$$

Let now $t_1 \in [t_0, T]$, $x_0 \in X_0$, $u[t_0,t_1] \in \Omega_u$, $y[t_0,t_1]$ be fixed, and define a subset of Ω_v (restricted to $[t_0,t_1]$) as

$$\Omega_v^{t_1}(x_0, u[t_0,t_1], y[t_0,t_1]) = \left\{ v[t_0,t_1] \in \Omega_v \;\middle|\; h\big(x(t), v(t), t\big) = y(t), \quad \forall t \in [t_0,t_1] \right\} \tag{5}$$

where $x(t)$ is the solution of (1) generated by $v[t_0,t_1]$ with the given x_0 and $u[t_0,t_1]$, and $y(t)$ is also given. (These are all the perturbations compatible with x_0 and the available past data up to time t_1.) Introduce also the auxiliary cost function

$$G^{t_1}(x_0, u, v) = V(x(t_1), t_1) + \int_{t_0}^{t_1} L(x(t), u(t), v(t), t)\, dt + N(x_0). \tag{6}$$

The auxiliary problem associated to these data is

PROBLEM $Q^{t_1}(u[t_0,t_1], y[t_0,t_1])$

The optimal controller will be shown to be given by the fixed point $u^* = K^*(y)$ of the equation

$$u^* = \hat{K}(u^*, y). \tag{8}$$

This means that at each instant of time $t_1 \in [t_0, T]$, one solves the auxiliary problem Q^{t_1} with the control $u[t_0, t_1]$ that has actually be applied up to that time, and the observed output $y[t_0, t_1]$ as its parameters. Then, take the final (i.e. at time t_1) value of the state on the optimal trajectory of the auxiliary problem as the "estimate" of current state, and apply a certainty equivalence principle to this estimate.

1.3 The certainty equivalence principle.

We are now ready to state and prove the two main theorems.

Theorem 1. *If the auxiliary problems $Q^{t_1}(K^*(y), y)$ have a unique solution for every t_1 and every y, then K^* as defined by (7)(8) is a solution of problem $\mathcal{P}$, and the guaranteed payoff is $V(x_0^*, t_0) + N(x_0^*)$.*

Proof. Introduce the function

$$W^{t_1}(u[t_0, t_1], y[t_0, t_1]) = \max_{x_0 \in X_0} \max_{v \in \Omega_v^{t_1}(x_0, u, y)} G^{t_1}(x_0, u, v).$$

Notice that on a trajectory of the system, one has

$$\frac{d}{dt_1} G^{t_1}(x_0, u, v) = H(x(t_1), u(t_1), v(t_1), t_1).$$

Therefore, on the trajectory $\hat{x}^{t_1}$, on which $u(t_1) = \phi^*(x(t_1), t_1)$, we have

$$\frac{d}{dt_1} G^{t_1}(\hat{x}_0, \hat{K}, \hat{v}) \leq 0.$$

And by definition, $G^{t_1}(\hat{x}_0, \hat{K}, \hat{v}) = W^{t_1}(u, y)$. Using Danskin's theorem, from the assumed

$$\forall u \in \Omega_u, \qquad J(x_0^*, u, \psi^*) \geq V(x_0^*, t_0) + N(x_0^*)$$

it follows that no controller can insure a cost smaller than $V(x_0^*, t_0) + N(x_0^*)$, i.e. that

$$\inf_K \max_{x_0 \in X_0} \max_{v \in \Omega_v} \bar{J}(x_0, K, v) \geq V(x_0^*, t_0) + N(x_0^*).$$

Comparing this inequality to (9) proves the theorem.

As a matter of fact, if $x_0 = x_0^*$ and $v(t) = \psi^*(x(t), t)$, then we shall have $K^*(y)(t) = \phi^*(x(t), t)$ for all t. Therefore, K^* is a representation of x_0^*, ϕ^* (see [1]), in the game $\min_u \max_{x_0, v} \bar{J}$. And the strong time consistency of the feedback saddle point is crucial here.

The controller $\hat{K}$ is also strongly time consistent in the following sense.

Corollary 1. *For each $t_1 \in [t_0, T]$, for every $u[t_0, t_1]$ and $y[t_0, t_1]$, use of the control $u(t) = \hat{K}(u, y)(t)$ for all $t \geq t_1$ guarantees*

$$\forall v \in \Omega_v, \qquad \bar{J}(x_0, u, v) \leq W^{t_1}(u[t_0, t_1], y[t_0, t_1]),$$

and no controller can guarantee a better bound.

Proof. By definition, for every t we have $G^t \leq W^t(u[t_0, t], y[t_0, t])$. We also have that $G^T = \bar{J}$, and we have shown that under the control $u(t) = \hat{K}(u, y)(t)$, W^t decreases with t. Hence the first claim. Moreover, the perturbation generated by

$$x_0 = \hat{x}_0^{t_1}$$

$$v(t) = \begin{cases} \hat{v}^{t_1}(t), & \text{if } t \in [t_0, t_1] \\ \psi^*(x(t), t), & \text{if } t \in (t_1, T) \end{cases}$$

insures $\bar{J} \geq W^{t_1}$ for any $u(\cdot)$ over $[t_1, T]$. Hence the second claim.

This property insures that in some sense, the controler $\hat{K}$ optimally uses the available

$$v^*(t) = \begin{cases} \bar{v}(t) & \text{if } t \in [t_0, t_1^*], \\ \psi^*(x(t), t) & \text{if } t \in (t_1^*, T). \end{cases}$$

Because of the saddle point property of ψ^*, it results in

$$\bar{J}(\bar{x}_0, K, v^*) \geq G^{t_1^*}(\bar{x}_0, u_0, \bar{v}) \geq w.$$

As w was chosen arbitrarily, the theorem follows.

1.4 The discrete time case.

The previous theory extends to the discrete time case. We limit ourselves here to a "week" extension, where the technique used to prove the optimality of the controller does not give its strong time consistency. (In the sense of corollary 1 above). As we stated earlier, this topic is deferred to a later paper, together with that of causal but not strictly causal controllers. We shall also need here some more regularity on the data, due to the technique of proof used, which shall not be needed in the more elaborate treatment.

Let the system be

$$x(t+1) = f\big(x(t), u(t), v(t), t\big), \qquad x(t_0) = x_0. \tag{10}$$

The output is still given by (2), and the augmented payoff by

$$\bar{J}(x_0, u, v) = M\big(x(T)\big) + \sum_{t=t_0}^{T-1} L\big(x(t), u(t), v(t), t\big) + N(x_0). \tag{11}$$

The functions f, L, M, and N are all assumed to be C^1. The sets Ω_u and Ω_v are now those of all sequences of length $T - t_0$ in U and V respectively.

We shall consider the problem of finding a *strictly causal* controller K, i.e. such that

$$u[t_0, t] = K(y[t_0, t-1])$$

Let $\hat{x}_0^{t_1}$, $\hat{v}^{t_1}$ be a solution of Q^{t_1}, and $\hat{x}^{t_1}$ the associated state trajectory. The controller we propose here is given by

$$\hat{K}(u[t_0, t_1 - 1], y[t_0, t_1 - 1])(t_1) = \phi^*(\hat{x}^{t_1}(t_1), t_1),$$

and $u^* = K^*(y)$ as the fixed point of equation (8) as previously. Of course the interpretation of this formula is the same as in the continuous time case. We can state the main theorem.

Theorem 3. *If for all $t_1 \in [t_0, T]$ the auxiliary problem Q^{t_1} is strictly concave and has a solution, then K^* is an optimal strictly causal controller, and the guaranteed payoff is $V(x_0^*, t_0) + N(x_0^*)$. If for some t_1^*, $Q^{t_1^*}$ has an infinite supremum for all $(u[t_0, t_1 - 1], y[t_0, t_1 - 1])$, then the problem $\mathcal{P}$ has an infinite supremum for all strictly causal controllers.*

Proof. Notice first that the maximization in x_0 in $\mathcal{P}$ can be imbeded into a classical control problem by introducing a step -1 into the system, with

$$x(-1) = 0, \qquad f(x, u, v, -1) = v, \qquad L(x, u, v, -1) = N(v).$$

Now the choice of x_0 is equivalent to that of $v(-1)$. So, for this proof, let v include x_0. We use the classical argument of Başar [1], slightly detailed. Remark that K^* is a representation of ϕ^*. As a matter of fact, let $x^*(\cdot)$ be the trajectory generated by (ϕ^*, ψ^*), and u^*, v^*, and y^* be the controls and output associated to that trajectory. It is known that v^* maximizes G^{t_1} over all $v \in \Omega_v$, thus a fortiori over $\Omega_v^{t_1 - 1}(u^*, y^*)$ for each t_1. Therefore, $\hat{x}^t(t) = x^*(t)$ for all t, and thus $\hat{K}(u^*, y^*)(t) = u^*(t)$.

Let

$$\hat{J}(v) = \inf_{u \in \Omega_u} \bar{J}(u, v).$$

From Danskin's theorem, it follows that

We consider a two-input two-output linear system over $\mathbb{R}^n$ given by

$$\dot{x} = Fx + Gu + Ev, \qquad x(t_0) = x_0, \tag{13}$$

$$y = Hx \qquad + Jv, \tag{14}$$

$$z = Cx + Du. \tag{15}$$

The system is considered over a fixed time interval $[t_0, T]$. The matrices F, G, E, H, J, C, and D are of appropriate dimension, possibly time varying, say piecewise continuous, with $J(t)$ surjective and $D(t)$ injective for all t. We take Ω_u and Ω_v as the sets of square integrable functions from $[t_0, T]$ into $\mathbb{R}^m$ and $\mathbb{R}^{m'}$ respectively.

As in section 1, y is the measured output, while z is the output to be controlled. (As a matter of fact, we shall also control $x(T)$.) The input u is the control, v the perturbation. (Again, together with x_0).

We are looking for a controller $u = K(y)$ with the only restrictions that K be causal, and that it insure existence of the solution of the differential equation (13) for every measurable control v or linear feedback $v(t) = L(t)x(t)$ with a bounded piecewise continuous gain $L(t)$.

It is now well understood that the equivalent to the so called suboptimal four block H^∞ control problem is to find the admisible controller K that minimizes the supremum of $\|z\|^2 - \gamma^2\|v\|^2$. Since we are here in finite time, we do not have stability conditions, but we must include the initial state in the perturbation, and we add a term depending on the final state to the output norm by symetry. Moreover, we wish to characterize the infimum γ^* of all γ for which a solution exists. We call it the *optimum attenuation level* of the H^∞ control problem.

We shall endow the output and input spaces with the norms

$$\left(\|x(T)\|_A^2 + \int_{t_0}^{T} \|z(t)\|^2\, dt\right)^{1/2}, \qquad \text{and} \qquad \left(\|x_0\|_{B^{-1}}^2 + \int_{t_0}^{T} \|v(t)\|^2\, dt\right)^{1/2}.$$

respectively, where A and B are positive definite matrices. This naturally leads us to

Then $P(t)$ is positive definite for all t, the saddle point strategies are

$$\phi^*(x,t) = -R^{-1}(G'P + S')x,$$
$$\psi^*(x,t) = \gamma^{-2}E'Px.$$

and the value is

$$J_\gamma(\phi^*,\psi^*) = \|x_0\|^2_{P(t_0)} - \gamma^2\|x_0\|^2_{B^{-1}} = \|x_0\|^2_{P_0-\gamma^2 B^{-1}}. \tag{22}$$

If (20) has no solution, its maximum extension interval being $(t_1^*, T]$ with $t_1^* > t_1$, then the supremum in v of $J(\phi, v)$ is infinite for all admissible ϕ.

We shall apply the previous theory to the problem. To that end, it will be necessary to consider restrictions of (13)(14)(15) to a subinterval $[t_0, t_1]$. We shall omitt many superscripts t_1. It will be convenient to rewrite the system (over $[t_0, t_1]$) in an operator form as

$$\begin{aligned} y &= \mathcal{A}u + \mathcal{B}v + \eta x_0,\\ z &= \mathcal{C}u + \mathcal{D}v + \zeta x_0,\\ x(t_1) &= \lambda u + \mu v + \nu x_0. \end{aligned} \tag{23}$$

Here y, z, u, and v live in appropriate L^2 spaces, x_0 in $\mathbb{R}^n$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, η, ζ, λ, μ, and ν are linear operators, whose adjoints will be denoted with a *.

We recall the following classical fact

Lemma. *The system*

$$\begin{aligned} \alpha &= \mathcal{A}^*p + \mathcal{C}^*q + \lambda^*\xi_1,\\ \beta &= \mathcal{B}^*p + \mathcal{D}^*q + \mu^*\xi_1,\\ \xi(t_0) &= \eta^*p + \zeta^*q + \nu^*\xi_1 \end{aligned} \tag{24}$$

admits the following internal representation :

$$\begin{aligned} -\dot\xi &= F'\xi + H'p + C'q, \qquad \xi(t_1) = \xi_1,\\ \alpha &= G'\xi \qquad\qquad + D'q,\\ \beta &= E'\xi + J'p. \end{aligned} \tag{25}$$

Here, $P_1 = P(t_1)$, where $P(\cdot)$ is the solution of (20).

To solve this problem, dualize the constraint with a Lagrange multiplier p necessarily in $L^2([t_0, t_1] \to Y')$, differentiate successively with respect to v and x_0 to obtain

$$\mathcal{B}^* p + \mathcal{D}^*(\mathcal{C}u + \mathcal{D}v + \zeta x_0) + \mu^* P_1(\lambda u + \mu v + \nu x_0) = \gamma^2 v,$$

$$\eta^* p + \zeta^*(\mathcal{C}u + \mathcal{D}v + \zeta x_0) + \nu^* P_1(\lambda u + \mu v + \nu x_0) = \gamma^2 B^{-1} x_0.$$

According to (24)(25), the left hand sides of the above two equations can be expressed as the second output and the initial state respectively of a system of the form (25) excited by p and the second output of a system of the form (24) (that we shall write in $\hat{x}$), and whose final state is $P_1 \hat{x}(t_1)$. Let us do that. It comes

$$\dot{\hat{x}} = F\hat{x} + Gu + Ev, \tag{26}$$

$$\hat{z} = C\hat{x} + Du, \tag{27}$$

as our system in $\hat{x}$, whose output and final state are used in the system in ξ :

$$-\dot{\xi} = F'\xi + H'p + C'\hat{z}, \qquad \xi(t_1) = P_1 \hat{x}(t_1). \tag{28}$$

The necessary conditions of optimality now read

$$\gamma^2 v = E'\xi + J'p, \tag{29}$$

$$\gamma^2 B^{-1} x_0 = \xi(t_0). \tag{30}$$

The constraint is

$$y = Hx + Jv. \tag{31}$$

From (29) we get

$$\hat{v} = \gamma^{-2}(E'\xi + J'p). \tag{32}$$

where $\hat{x}$ is the solution of the two point boundary value problem (34)(35).

Of course, this formula is not recursive, and thus not very usefull as such, since the two point boundary value problem (34)(35) has to be solved for each time instant $t_1 \in [t_0, T]$ to apply (36). Finding a nicer form of this controller will be a matter of calculation, and is the topic of the next subsection.

2.2 The continuous time case: recursive formulas.

We shall use the classical device of the Riccati equation to transform the two point boundary value problem. Let, if it exists, $\Sigma(t)$ be the solution of the following Riccati equation.

$$\dot{\Sigma} = \bar{F}\Sigma + \Sigma\bar{F}' - \Sigma(H'N^{-1}H - \gamma^{-2}Q)\Sigma + \bar{M}, \quad \Sigma(t_0) = B. \tag{37}$$

It is a simple matter to check that the variable $\check{x} = \hat{x} - \gamma^{-2}\Sigma\xi$ satisfies the equation

$$\dot{\check{x}} = \left[\bar{F} + \Sigma(\gamma^{-2}Q - H'N^{-1}H)\right]\check{x} + (\Sigma H' + L)N^{-1}y + (G + \gamma^{-2}\Sigma C'D)u, \quad \check{x}(t_0) = 0. \tag{38}$$

And from the condition on $\xi(t_1)$ in (35) and the definition of $\check{x}$, it follows that, if the inverse exists,

$$\hat{x}(t_1) = \left(I - \gamma^{-2}\Sigma(t_1)P(t_1)\right)^{-1}\check{x}(t_1). \tag{39}$$

Equation (38) is the same for all auxiliary problems, (all t_1), therefore together with (39) it does give a recursive formula for $\hat{x}$, that can be used in the controller (36).

We now state the theorem that gives a first solution of the H^∞ control problem.

Theorem 4. *Let γ^* be the optimum attenuation level for the H^∞ control problem defined by (13) to (15). Then, a necessary condition for $\gamma \geq \gamma^*$ is that the Riccati equations (20) and (37) have a solution over $[t_0, T]$, and that $\rho(\Sigma(t)P(t)) \leq \gamma^2$ for all t. Furthermore, strengthening this last inequality to a strict one suffices to insure that $\gamma > \gamma^*$. In that case, an optimal controller is given by (20)(37)(38)(39), and (36) which defines u to be placed into (38). This controller is strongly time consistent in the sense of corollary 1 ab*

We must now prove that the auxiliary control problems are concave, for all t_1, only if the matrix Σ exists over $[t_0, T]$, and satisfies together with P the spectral radius condition. We are dealing with the minimization of a quadratic form under an affine constraint, i.e. over an affine subspace. It is easily seen, for instance by using the formulation (23), that concavity of the problem does not depend on the non homogeneous terms, here (u, y). And if the problem is not concave, it indeed has an infinite supremum whatever they are. (Notice also that if the problem is concave, but not strictly so, replacing γ by $\gamma - \epsilon$ for any positive ϵ adds a term $\epsilon\|v\|^2$ to the criterion, making it surely not concave, so that in that case, $\gamma \leq \gamma^*$).

Let us therefore investigate the homogeneous problem, i.e. with $(u, y) = (0, 0)$.

We shall first show the necessity of the existence of Σ. It is known that, since $\bar{M} = E'(I - J^{\dagger} J)E$ is positive semidefinite, and B positive definite, then Σ is positive definite when it exists. (See, e.g., [16] for a concise proof).

Let us rewrite the necessary conditions (34) (35) for the homogeneous problem, in terms of $\lambda = \gamma^{-2}\xi$:

$$\dot{x} = Fx + \bar{M}\lambda, \qquad x(t_0) = B\lambda(t_0),$$

$$\dot{\lambda} = (H'N^{-1}H - \gamma^{-2}Q)x - F'\lambda, \qquad \lambda(t_1) = \gamma^{-2}P_1 x(t_1),$$

with $\hat{v} = -J'N^{-1}Hx + (I - J'N^{-1}J)E'\lambda$. We call *extremals* solutions of the differential system, without the boundary conditions. They are trajectories of the system generated by $\hat{v}$. On such trajectories, we have the fundamental identity

$$\gamma^2(\lambda'x)^{\bullet} = -\|x\|_Q^2 + \gamma^2\|\hat{v}\|^2,$$

and thus integrating,

$$\gamma^2\lambda'(t_1)x(t_1) - \gamma^2\lambda'(t_0)x(t_0) + \int_{t_0}^{t_1} (\|x\|_Q^2 - \gamma^2\|\hat{v}\|^2)\, dt = 0.$$

large. Therefore, invertibility of Φ, hence existence of Σ, for all t is seen to be necessary. But then the same calculation yields, for any t_1

$$G^{t_1} = \|x(t_1)\|^2_{P_1 - \gamma^2\Sigma^{-1}(t_1)}.$$

Since Φ is now invertible, $x(t_1)$ can be chosen arbitrarily. Therefore, we see that a necessary condition for the problem to be concave is $P(t_1) - \gamma^2\Sigma^{-1}(t_1) \leq 0$, and this must hold for all t_1. This is equivalent to the condition $\rho(\Sigma(t)P(t)) \leq \gamma^2$. The necesary condition is proved.

To prove the sufficiency part, we just do a classical Hamilton Jacobi Caratheodory theory under the constraint $Hx + Jv = 0$, initializing the return function $W(x,t)$ at $W(x,t_0) = \gamma^2\|x\|^2_{B^{-1}}$. We find that the solution of the forward Hamilton Jacobi equation is $W(x,t) = \gamma^2\|x\|^2_{\Sigma^{-1}(t)}$. This establishes the above expression for G^{t_1} as an upper bound. The continuity of the solution of the Riccati equations with respect to γ imply that the three conditions are satisfied in an open set, thus in that case $\gamma > \gamma^*$. This ends the proof of the theorem.

There are two final remarks to be made. The first one is that solving the stationary version of the problem with an added constraint that the system should be internally stable is now straightforward. As a matter of fact, the stability constraint is exactly the one that allows one to state the equivalent results in infinite time. (see, e.g. [10]) Hence we recover the theorem that the standard H^∞ problem has a solution for a given γ if and only if the stationary versions of the Riccati equations have positive definite solutions P and Σ, and if the eigenvalues of $P\Sigma$ are smaller than γ^2.

The second remark is that the controller obtained seems different from the one proposed in the literature, as in [11] for instance. It is worhtwhile to rewrite it here, redevelopping in terms of the original matrices

$$\begin{aligned}\dot{\hat{x}} = \big[F + \Sigma(\gamma^{-2}C'C - H'(JJ')^{-1}H) - EJ'(JJ')^{-1}H \\ -(G + \gamma^{-2}\Sigma C'D)(D'D)^{-1}(G'P + D'C)(I - \gamma^{-2}\Sigma P)^{-1}\big]\hat{x} \\ +(\Sigma H' + EJ')(JJ')^{-1}y, \qquad (40)\end{aligned}$$

Nerode equivalence class of the controller —, on the past u,y. It is a striking fact that it satisfies *the same equations* (37) and (43) as the sufficient statistics in the LEQG, or "risk sensitive", control, as given in [5], equations (1.6) and (1.7). (At least in the case considered there, where EJ' and $C'D$ are both zero.)

On the other hand, it is possible to compute the time derivative of

$$\hat{x} = (I - \gamma^{-2}\Sigma P)^{-1}\check{x}$$

replacing the derivatives of Σ and P by their expressions in the Riccati equations and taking that of $\check{x}$ above. We should point out that this time derivative is not that given by equation (34). That one was $(d/dt)\hat{x}^{t_1}(t)$, while we are in effect computing here $(d/dt)\hat{x}^{t}(t)$, differentiating with respect to both occurences of the variable t. An elementary (but bulky) calculation yields the following equations, which exhibit a strange "duality" with (40)(42) :

$$\begin{aligned}\dot{\hat{x}} = \big[F + (\gamma^{-2}EE' - G(D'D)^{-1}G')P - G(D'D)^{-1}D'C \\ -(I - \gamma^{-2}\Sigma P)^{-1}(\Sigma H' + EJ')(JJ')^{-1}(H + \gamma^{-2}JE'P)\big]\hat{x} \\ +(I - \gamma^{-2}\Sigma P)^{-1}(\Sigma H' + EJ')(JJ')^{-1}y,\end{aligned} \tag{44}$$

$$\hat{x}(t_0) = 0, \tag{45}$$

$$u = -(DD')^{-1}(G'P + D'C)\hat{x}. \tag{36}$$

These formulas coincide with that of [11], at least for the case considered there where $C'D$ and EJ' are both zero.

2.3. The discrete time case.

We consider now the discrete time system

$$P = (F - GR^{-1}S')\Delta(F' - SR^{-1}G') + Q - SR^{-1}S', \qquad P(T) = A, \tag{50}$$

where Δ has many equivalent forms, two of them being :

$$\Delta = \left[P_+^{-1} + (\,G \quad E\,) \begin{pmatrix} R^{-1} & 0 \\ 0 & -\gamma^{-2}I \end{pmatrix} \begin{pmatrix} G' \\ E' \end{pmatrix} \right]^{-1} \tag{50a}$$

and

$$\Delta = P_+ - P_+(\,G \quad E\,) \begin{pmatrix} R + G'P_+G & G'P_+E \\ E'P_+G & E'P_+E - \gamma^2 I \end{pmatrix}^{-1} \begin{pmatrix} G' \\ E' \end{pmatrix} P_+. \tag{50b}$$

The saddle point feedback strategies are now given by the simple, but apparently non causal formulas

$$u = -R^{-1}(G'P_+x_+ + S'x),$$
$$v = \gamma^{-2}E'P_+x_+,$$

or equivalently in an explicitly causal form as

$$u = -R^{-1}(G'\Delta(F - GR^{-1}S') + S')x, \tag{51}$$
$$v = \gamma^{-2}E'\Delta(F - GR^{-1}S')x. \tag{52}$$

Call again u, v, y, and z the sequences $\{u(t), t \in [t_0, t_1 - 1]\}$ and likewise for the other three, and the system restricted to $[t_0, t_1]$ can be written with the same equations (23) as in the continuous time case.

The lemma on duality stands provided that (25) be replaced by its discrete counterpart

$$\xi = F'\xi_+ + H'p + C'q, \qquad \xi(t_1) = \xi_1,$$
$$\alpha = G'\xi_+ \qquad\quad + D'q,$$
$$\beta = E'\xi_+ + J'p.$$

The same calculation as in the continuous time leads to the two-point boundary value

Let again $\check{x} = \hat{x} - \gamma^{-2}\Sigma\xi$, carefully conducting the calculations leads to the following formulas :

$$\check{x}_+ = \bar{F}\check{x} + \bar{F}\Sigma(\, H' \quad C'\,)\begin{pmatrix} N + H\Sigma H' & H\Sigma C' \\ C\Sigma H' & C\Sigma C' - \gamma^2 I \end{pmatrix}^{-1}\begin{pmatrix} y - H\check{x} \\ -\check{z} \end{pmatrix} + LN^{-1}y + Gu$$

$$\check{x}(t_0) = 0. \tag{56}$$

where

$$\check{z} = C\check{x} + Du,$$

and as in the continuous case, using the final condition of (54), for the variable $\hat{x}^t(t)$,

$$\hat{x} = (I - \gamma^{-2}\Sigma P)^{-1}\check{x}. \tag{57}$$

We may therefore state the following theorem.

Theorem 5. *If the Riccati equations (50) and (55) have solutions P and Σ defined over $[t_0, T]$ that furthermore satisfy $\rho(\Sigma P) < \gamma^2$, formulas (50),(55),(56),(57), and (51) define an optimal min max controller for the system (46) to (49). (Hence an H^∞ controller for (46)–(48) with atenuation level γ).*

We did not state here that the conditions of the theorem are necessary. It will be shown elsewhere that $\Lambda > 0$ and the spectral radius condition are indeed necessary, as well of course as the existence of P, when the matrix [F E] is surjective and the matrix [F' C']' is injective, with a slight further hypothesis on the data.

We did not either try to find an alternate form of the controller where the dynamical equation would bear upon $\hat{x}$ and not $\check{x}$. Anyway, we feel that the form in $\check{x}$ is more interesting. Notice only that in (56) again, y enters the equation only through the combination $y - H\check{x}$.

Conclusion.

4th International Conference on Differential Games and Applications, Helsinky, Finland, 1990

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Abstract. We apply our certainty equivalence principle to the solution of the sampled data, output feedback H^∞ control problem. As expected, the solution bears close resemblance to a Kalman Filter design.

Keywords. H^∞ control, sampled data, min-max, certainty equivalence, dynamic games.

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Introduction.

In [7], we derived a certainty equivalence principle for min-max control problems with incomplete information. It says that one should, at each instant of time, compute the worst perturbation compatible with the currently available information, and use the current state on the corresponding trajectory as the "estimate" of the state, and place it in the optimal state feedback strategy, obtained as the saddle point of a full information two-person zero-sum game.

On the other hand, it has recently been discovered that the so called H^∞ control problem is fondamentally a min-max control problem. See [11] and [12] for a review of this aspect. This has allowed several authors to use a game theoretic approach to solve these problems. In the most classical cases, this gives back the state space solutions, such as derived in [8] for instance. As a matter of fact, this approach even predates H^∞ control theory. (See [5]). It also allows one to solve new problems, such as the time varying finite time problem for instance, and also more signifcant extensions such as the delayed measurement or the sampled data problems for example. See [1],[2],[3] in particular.

However, up to recently, this powerfull approach could not be applied to the output feedback problem, or "four block problem", for the lack of a theory of min-max control

$$t \in [t_0, T], \quad x(t) \in I\!R^n, \quad u(t) \in U, \quad v(t) \in V.$$

Let Ω_u and Ω_v be the set of open loop controls. Adequate regularity and growth conditions have been assumed on f to guarantee existence and unicity of the solution of (1) over $[t_0, T]$ for any x_0 and any $(u, v) \in U \times V$.

A criterion, to be minimized by the first player and maximized by the second, is given by

$$J_{x_0,t_0}(u, v) = M(x(T)) + \int_{t_0}^{T} L(x(t), u(t), v(t), t)\, dt + N(x_0).$$

We assume that the corresponding full information zero-sum two-person differential game, *without the* $N(x_0)$ *term* that we add for future use, has, in an adequate setting, a pure feedback strongly time consistent saddle point

$$u(t) = \phi^*(x(t), t), \tag{3a}$$

$$v(t) = \psi^*(x(t), t), \tag{3b}$$

and a piecewise C^1 value function $V(x, t)$.

Introduce now a partial measurement or output

$$y(t) = h(x(t), v(t), t). \tag{4}$$

We also allow for an uncertain initial state $x_0 \in X_0$. The problem we tackle is to find a causal controller $u = K(y)$ that will solve the following problem.

Problem $\mathcal{P}$.

$$\min_{K} \max_{\substack{v \in \Omega_v \\ x_0 \in X_0}} J_{x_0,t_0}(K(y), v).$$

The solution to this problem as given in [7] involves, for each $t_1 \in [t_0, T]$, a family of *auxiliary problems* $Q^{t_1}(u, y)$ parametrized, beyond t_1, by the past control and observation histories $u[t_0, t_1]$ and $y[t_0, t_1]$, with payoff function

$$G^{t_1}(x_0, u[t_0, t_1], v[t_0, t_1]) = V(x(t_1), t_1) + \int_{t_0}^{t_1} L(x(t), u(t), v(t), t)\, dt + N(x_0) \tag{5}$$

and that on the other hand, if for some t_1^* the auxiliary problems have an infinite supremum for all (u, y), then the original problem had an infinite supremum for all causal controllers K.

The proof is indeed very simple, and is based upon showing that the return function

$$W^{t_1}(u, y) = \max_{x_0 \in X_0} \max_{v \in \Omega_v^{t_1}} G^{t_1}(x_0, u[t_0, t_1], v[t_0, t_1])$$

is decreasing with time t_1 as soon as u is chosen according to (6). This also proves that this controller is strongly optimal in that sense that it not only solves problem $\mathcal{P}$, but moreover, if one uses any control until some intermediary t_1, and then (6) from then on, it guarantees the best possible value to J given the information up to time t_1. (In [7] we called this property time consistency, but it seems to be a bad choice.)

1.2. The sampled data problem.

It is straightforward to see that the principle of [7] extends to the set up where the available measurements occur at a sequence of time instants $\{\tau_1, \tau_2, \ldots, \tau_N\}$, where $t_0 \le \tau_1 < \tau_2 < \cdots < \tau_N < T$. We must now split the perturbation v into (v, w) where v is a continuous part and w a discrete sequence (or impulsive part) $\{w_k\}$. Let (4) be replaced by

$$y_k = h_k(x(\tau_k), w_k). \tag{7}$$

We must also include a term in w in the cost, replacing (2) by

$$J_{x_0, t_0}(u, v, w) = M(x(T)) + \int_{t_0}^{T} L(x(t), u(t), v(t), t)\, dt + \sum_{k=1}^{N} \tilde{L}_k(x(\tau_k), w_k) + N(x_0). \tag{8}$$

To make things simple, we shall assume that, $\forall x$,

$$\tilde{L}^k(x, 0) = 0, \qquad \tilde{L}^k(x, w) < 0, \quad \forall w \neq 0. \tag{9}$$

We are given the following linear system in $\mathbb{R}^n$ over the time interval $[t_0, T]$, in which a sequence $\{\tau_k\}$ is given, $t_0 \leq \tau_1 < \tau_2 \cdots < \tau_N < T$.

$$\begin{aligned} \dot{x} &= F(t)x + G(t)u + E(t)v, \qquad x(t_0) = x_0, \\ y_k &= H_k x(\tau_k) + J_k w_k, \\ z &= C(t)x + D(t)u. \end{aligned}$$

We set the following notations :

$$\begin{pmatrix} C' \\ D' \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix} = \begin{pmatrix} Q & S \\ S' & R \end{pmatrix}, \qquad J_k J_k' = N_k. \tag{10}$$

A causal controller

$$u(t) = K(y_1, y_2, \dots, y_i)(t), \quad \text{where} \quad \tau_i < t \leq \tau_{i+1}$$

is said to have an attenuation level γ if it guarantees,

$$\forall (x_0, v, w) \in X_0 \times \Omega_u \times \Omega_w, \qquad \|z\|^2 + \|x(T)\|_A^2 \leq \gamma^2 \left(\|v\|^2 + \|w\|^2 + \|x_0\|_{B^{-1}}^2 \right).$$

The norms have to be understood with respect to the appropriate spaces : L^2 spaces for the continuous variables z and v, $\mathbb{R}^{nN}$ for w, and $\mathbb{R}^n$, with weigting positive definite matrices for x_0 and $x(T)$. The H^∞ control problem consists in characterizing the infimum γ^* of all possible attenuation levels, and for any $\gamma > \gamma^*$ in finding a controller that acheives that attenuation level.

2.2. Applying the general theory.

It is now classical to associate to the above problem the criterion

$$
\begin{aligned}
u(t) &= -R^{-1}(G'P + S')x(t), \\
v(t) &= \gamma^{-2}E'Px(t),
\end{aligned}
\tag{13}
$$

and the value function is $V(x,t) = \|x\|^2_{P(t)}$.

To express the auxiliary problem, we turn to an operator form. Let $t_1 \in (\tau_i, \tau_{i+1}]$. We write u, v, z for $u[t_0, t_1]$, $v[t_0, t_1]$, and $z[t_0, t_1]$, that belong to appropriate L^2 spaces, and y and w for $\{y_k\}_{k \le i}$ and $\{w_k\}_{k \le i}$ in appropriate euclidean spaces. The system restricted to $[t_0, t_1]$ may be written in the form

$$
\begin{aligned}
y &= \mathcal{A}u + \mathcal{B}v + \mathcal{J}w + \eta x_0, \\
z &= \mathcal{C}u + \mathcal{D}v + \zeta x_0, \\
x(t_1) &= \lambda u + \mu v + \nu x_0.
\end{aligned}
\tag{14}
$$

The lemma in [7] extends to the following one :

Lemma. The system

$$
\begin{aligned}
\alpha &= \mathcal{A}^* p + \mathcal{C}^* q + \lambda^* \xi_1, \\
\beta &= \mathcal{B}^* p + \mathcal{D}^* q + \mu^* \xi_1, \\
\rho &= \mathcal{J}^* p, \\
\xi(t_0) &= \eta^* p + \zeta^* q + \nu^* \xi_1,
\end{aligned}
\tag{15}
$$

admits the following internal representation :

$$
\begin{aligned}
-\dot{\xi} &= F'\xi + C'q, \quad \xi(t_1) = \xi_1, \quad \xi(\tau_k^-) = \xi(\tau_k^+) + H_k' p_k, \\
\alpha &= G'\xi + D'q, \\
\beta &= E'\xi, \\
\rho_k &= J_k' p_k.
\end{aligned}
\tag{16}
$$

Proof. Compute

Form the Lagrangian with a multiplier $2p$, differentiate with respect to v, w, and x_0 to obtain :

$$\mathcal{B}^* p + \mathcal{D}^* \hat{z} + \mu^* P_1 \hat{x}(t_1) = \gamma^2 \hat{v},$$

$$\mathcal{J}^* p = \gamma^2 \hat{w},$$

$$\eta^* p + \zeta^* \hat{z} + \nu^* P_1 \hat{x}(t_1) = \gamma^2 B^{-1} \hat{x}_0,$$

where $\hat{z} = \mathcal{C}u + \mathcal{D}\hat{v} + \zeta \hat{x}_0$, and $\hat{x}(t_1) = \lambda u + \mu \hat{v} + \nu \hat{x}_0$.

As in [7], we use the representation (15)(16) to express these conditions more concretely. We easily get $\hat{v} = \gamma^{-2} E' \xi$, $\hat{w}_k = \gamma^{-2} J_k' p_k$, with $p_k = \gamma^2 N_k^{-1}(y_k - H_k \hat{x}(\tau_k))$. Overall, the necessary conditions read :

$$\dot{\hat{x}} = F\hat{x} + \gamma^{-2} EE'\xi + Gu, \quad \hat{x}_0 = \gamma^{-2} B\xi(t_0), \tag{17}$$

$$\dot{\xi} = -Q\hat{x} - F'\xi - C'Du, \quad \xi(t_1) = P_1 \hat{x}(t_1), \tag{18}$$

$$\xi(\tau_k^-) = \xi(\tau_k^+) + \gamma^2 H_k' N_k^{-1} (y_k - H_k \hat{x}(\tau_k)). \tag{19}$$

If we can find a solution of this boundary value problem, and if the auxiliary problem is concave, then the optimal controller we propose is $u(t_1) = -R^{-1}(G'P + S')\hat{x}(t_1)$ for each t_1, or more explicitly, recalling that $\hat{x}$ in the above equations stands for $\hat{x}^{t_1}$,

$$u(t) = -R^{-1}(G'P + S')\hat{x}^t(t). \tag{20}$$

2.2. Recursive formulas.

The apparent difficulty to obtain recursive formulas similar to those in [7] comes from the fact that the jumps in ξ are in terms of $\hat{x}^{t_1}(\tau_k)$, therefore depending on t_1. The way around this difficulty will be to introduce jumps in Σ so as to get jumps in terms of the recursive variable $\check{x}$. Let us therefore introduce the matrix $\Sigma(t)$ defined, when it exists, by

$$\dot{\Sigma} = F\Sigma + \Sigma F' + \gamma^{-2}\Sigma Q \Sigma + EE', \qquad \Sigma(t_0) = B, \tag{21a}$$

But now, these equations do not depend on t_1. Therefore they give a recursive formula for $\check{x}^{t_1}(t_1)$, from which we recover $\hat{x}^{t_1}(t_1)$ using the final condition of (18) (expressed in terms of t instead of t_1) :

$$\hat{x}^t(t) = \left[I - \gamma^{-2}\Sigma(t)P(t)\right]^{-1}\check{x}(t). \tag{23}$$

We can now state the theorem.

Theorem. *Let γ^* be the optimal attenuation level for the sampled data output feedback H^∞ control problem. Then a necessary condition for $\gamma \geq \gamma^*$ is that the Riccati equations (12) and (21) have a solution over $[t_0, T]$, satisfying $\rho(\Sigma(t)P(t)) \leq \gamma^2$, $t_1 \in [t_0, T]$. If this last inequality is strengthened to a strict one, then $\gamma > \gamma^*$, and a strongly optimal controller is given by equations (12), (21),(22),(23), and (20) to be placed back into (22).*

Proof. The proof goes exactly as in [7], with some obvious modifications that we shall indicate here. We still look at the homogeneous problem, where $(u, y) = (0, 0)$. The generating matrices of the extremals are now defined by

$$\dot{\Phi} = F\Phi + EE'\Psi, \qquad \Phi(t_0) = B,$$
$$\dot{\Psi} = -\gamma^{-2}Q\Phi - F'\Psi, \quad \Psi(t_0) = I,$$
$$\Psi(\tau_k^+) = \Psi(\tau_k^-) + H_k'N_k^{-1}H_k\Phi(\tau_k).$$

As previously, we see that $\Sigma = \Phi\Psi^{-1}$. The fundamental identity of the extremals $x(t) = \Phi(t)\mu$, $\xi(t) = \gamma^2\Psi(t)\mu$, now reads

$$\xi'(t_1)x(t_1) - \xi'(t_0)x(t_0) + \int_{t_0}^{t_1} (\|x\|_Q^2 - \gamma^2\|\hat{v}\|^2)\, dt - \gamma^2 \sum_{k=1}^{i} \|\hat{w}_k\|^2 = 0,$$

so that on an extremal, we have

$$G^{t_1} = \|x(t_1)\|_{P_1}^2 - \xi'(t_1)x(t_1) = \|x(t_1)\|^2_{P(t_1)-\gamma^2\Sigma^{-1}(t_1)},$$

$$G(x_0, v, w) = M(x(t_1)) + \int_{t_0}^{t_1} L(x, v)\, dt + \sum_{k=1}^{i} \tilde{L}(x(\tau_k), w_k) + N(x_0).$$

We are interested in the problem of maximizing G with respect to x_0, v, and w, subject to the constraint that for a sequence of time instants $\{\tau_k\}$,

$$h_k(x(\tau_k), w_k) = 0.$$

Assume there exists a function $W(x,t)$, and an admissible feedback $\hat{v} = \hat{\psi}(x,t)$ satisfying the following equations

$$\forall x \in X_0, \qquad W(x, t_0) = -N(x_0),$$

$$\forall t \neq \tau_k, \forall x \in I\!R^n, \qquad \frac{\partial W}{\partial t} + \frac{\partial W}{\partial x} f(x, v) + L(x, v) \leq 0,$$

with equality for $v = \hat{v}$,

$$\text{for } k = 1, \ldots, i, \quad \forall x \in I\!R^n, \qquad W(x, \tau_k^+) = W(x, \tau_k^-) - \hat{L}_k(x)$$

where $\hat{L}$, assumed to exist, is defined by

$$\hat{L}_k(x) = \max_w \tilde{L}_k(x, w) \quad \text{under the constraint that} \quad h_k(x, w) = 0,$$

the max being reached at $w = \hat{w}_k(x)$,
then, for all admissible x_0, v, w,

$$G(x_0, v, w) \leq M(x(t_1)) - W(x(t_1), t_1)$$

the equality being reached for $v = \hat{v}$ and $w = \hat{w}$.

It suffices to integrate by part along a trajectory the partial differential equation, noticing that the jump condition on W at τ_k can be written

Conclusion.

This short developpement shows the power of the min-max certainty equivalence principle in solving various forms of the four block H^∞ control problem. Clearly, many particular cases of this deserve closer examination. One is the case where the measurements are produced by integrating devices, that is, of the form $y_k = y(\tau_k) + w_k$, with

$$\dot{y} = Hx + Jv, \qquad y(t_0) = y_0.$$

This is actually a particular case of the above, taking $(x', y')'$ as the state. Either y_0 is assumed to be unknown. Then one should add a cost on it in the payoff, presumably with a very large weight B_y^{-1}, hence a very small B_y, to mean that it has to be small, or it is assumed to be zero. A slight extension of the theory shows that this is obtained by making the corresponding B_y zero.

Many other classical problems could be solved along the same lines, including delayed measurement, mixed sampled data and continuous measurements, etc.

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