Boundary Value Problems and Dynamical Games

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Abstract. We discuss earlier papers concerning two-point boundary value problems arising in the solution of dynamical games, with emphasis on one coming from a Stackelberg game, for which we offer a simple solution.

1. Introduction.

In references [1] and [2], Abou-Kandil and Bertrand have applied the so called eigenvalue technique of [4] to the solution of two-point boundary value Riccati equations arising in a Cournot-Nash and in a Stackelberg linear quadratic dynamical game. We point out here that, as was shown by Caratheodory [3] and popularized by Kalman, the Riccati equation is intimately related to the underlining linear system, whose diagonalization provides an explicit solution so that it also provides the solution of the Riccati equation.

However, in the quoted references, Abou-Kandil and Bertrand also pointed to a Hamiltonian structure in the systems. We shall look more closely at the system of ref [1], whose hamiltonian structure hides another such structure, which yields a simpler solution.

Let us first review quickly the linear two-point boundary value problem, as it relates to a Riccati equation.

Let $A(t)$ be a piecewise continuous time varying square matrix, of dimension N. Let S_0 and S_1 be two matrices, of type $p_0 \times N$ and $p_1 \times N$ respectively, and of rank p_0 and p_1 respectively. We shall assume that $p_0 + p_1 \leq N$. Let finally t_0 and t_1 be two different real numbers. We consider the following boundary value problem in $I\!R^N$:

$$\dot{x} = A(t)x, \tag{1}$$

$$S_0 x(t_0) = s_0, \tag{2}$$

$$S_1 x(t_1) = s_1. \tag{3}$$

We can solve this problem by the following classical *partial transition matrix* technique. By subtracting a solution of (1) satisfying one of the boundary conditions, we may instantly reduce this problem to one with the corresponding $s_i = 0$. Pick therefore one of the boundaries, preferably with the largest p_i, and make the corresponding s_i null. We shall assume that this is done for t_1. Let then X_1 be a *full rank* matrix of type $N \times N - p_1$, satisfying $S_1 X_1 = 0$. Solve the $N - p_1$ differential equations of dimension N

$$\dot{X} = A(t)X, \quad X(t_1) = X_1. \tag{4}$$

When r ranges over $I\!R^{N-p_1}$, the vector functions $x(t) = X(t)r$ provide all the solutions of (1)(3). It therefore remains to choose r such that

$$S_0 X(t_0) r = s_0, \tag{5}$$

to get a solution of (1),(2) and (3), which exists if and only if (5) has a solution. (If $p_0 + p_1 < N$, this solution may not be unique.)

In the square case, where $N = 2n$, and $p_0 = p_1 = n$, let

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

be a partition of A in $n \times n$ blocks,

$$x = \begin{pmatrix} y \\ z \end{pmatrix}$$

and

$$X = \begin{pmatrix} Y \\ Z \end{pmatrix}$$

be the corresponding partitioning of x and X. It is a classical fact that if Y_1, say, is chosen invertible, the matrix

$$K = ZY^{-1} \tag{6}$$

satisfies the Riccati equation

$$\dot{K} = D + EK - KB - KCK, \quad K(t_1) = Z_1 Y_1^{-1} \tag{7}$$

whose interval of definition is precisely the interval over which $Y(t)$ remains invertible. Moreover, if A is a Hamiltonian matrix, i.e. if $E = -B'$ and D and C are symmetric, and if furthermore the terminal value in (7) is symmetric, then K is a symmetric matrix.

Therefore, if the matrix A happens to be *constant* and diagonalizable, the diagonalization of A provides an explicit solution of (4), thus an explicit solution of (7) via (6).

Once K is known, the solution of (1),(2),(3) is obtained by integrating the differential equation of size n, obtained by letting $z = Ky$:

$$\dot{y} = (B + CK)y, \quad S_0 \begin{pmatrix} I \\ K(t_0) \end{pmatrix} y(t_0) = s_0.$$

In reference to the papers quoted, notice that this provides a solution of the two-point boundary value problem in [2], even in the absence of the assumption that $Q_2 = \alpha Q_1$.

2. The Stackelberg game.

We consider the Stackelberg game given by the system in $\mathbb{R}^n$,

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(t_0) = x_0,$$

and the performance indices

$$J_i = x_1' S_i x_1 + \int_{t_0}^{t_1} (x' Q_i x + u_1' R_{i1} u_1 + u_2' R_{i2} u_2)\, dt$$

with player 2, say, as the leader. (Notice that as a result, R_{12} plays no role in the problem). All the matrices involved are piecewise continuous time varying. The weighting matrices in the performance indices are symmetric positive semidefinite, the R_{ij}'s being positive definite.

It is known that its *open loop* solution is given by

$$u_i = -R_{ii}^{-1} B_i' \lambda_i$$

where the vector functions $\lambda_i(t)$, $i = 1, 2$, are given together with the optimal trajectory and an auxilliary sensitivity function $\gamma(t)$ by the following two-point boundary value problem.

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} A & -M_1 & -M_2 & 0 \\ -Q_1 & -A' & 0 & 0 \\ -Q_2 & 0 & -A' & Q_1 \\ 0 & -N & M_1 & A \end{pmatrix} \begin{pmatrix} x \\ \lambda_1 \\ \lambda_2 \\ \gamma \end{pmatrix}, \tag{8}$$

$$\begin{aligned} x(t_0) &= x_0, && (9a) \\ \lambda_1(t_1) &= S_1 x(t_1), && (9b) \\ \lambda_2(t_1) &= S_2 x(t_1) - S_1 \gamma(t_1), && (9c) \\ \gamma(t_0) &= 0. && (9d) \end{aligned}$$

Here, we have set $M_i = B_i R_{ii}^{-1} B_i'$, and $N = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1'$.

One can regroup the above equations otherwise, e.g. in $y = \binom{x}{\gamma}$ and $z = \binom{\lambda_1}{\lambda_2}$, and apply the above theory, with $Y(t_1) = I$ and

$$Z(t_1) = \begin{pmatrix} S_1 & 0 \\ S_2 & -S_1 \end{pmatrix}.$$

Using the partial transition matrix technique, this gives a solution in $8n^2$ scalar linear differential equations. One should pick $r \in \mathbb{R}^{2n}$ such that

$$Y(t_0) r = \begin{pmatrix} x_0 \\ 0 \end{pmatrix},$$

and the solution is given by $\lambda(t) = Z(t) r$.

If the solution exists for all x_0, it can be obtained by the Riccati equation technique. But as the system in this form is not hamiltonian, the corresponding K matrix is not square. This requires therefore $4n^2$ scalar nonlinear differential equations, plus $2n$ linear differential equations to be integrated forward.

3. The Hamiltonian structure.

The special ordering of the equations in (8) was suggested by Abou-Kandil and Bertrand [1], who rightly pointed out that then the system is Hamiltonian.

Notice first that by introducing square $n \times n$ matrices X, Λ_1, Λ_2, and Γ instead of the corresponding vector variables in (8), the matrices $K_1 = \Lambda_1 X^{-1}$, $K_2 = \Lambda_2 X^{-1}$, and $P = \Gamma X^{-1}$, satisfy the Riccati equations

$$\dot K_1 = -A'K_1 - K_1A - Q_1 + K_1M_1K_1 + K_1M_2K_2, \quad K_1(t_1) = S_1, \tag{10}$$
$$\dot K_2 = -A'K_2 - K_2A - Q_2 + Q_1P + K_2M_1K_1 + K_2M_2K_2, \quad K_2(t_1) = S_2 - S_1P, \tag{11}$$
$$\dot P = AP - PA + PM_1K_1 + PM_2K_2 - NK_1 + M_1K_2, \quad P(t_0) = 0, \tag{12}$$

which appear, (with the same numbering), in [1]. However, these are not a direct means to solve the boundary value problem (8),(9), since it is again a two-point boundary value problem, because P is initialize at t_0.

We may still exploit the hamiltonian character of (8) using the above partial transition matrices technique. Let now $y = \binom{x}{\lambda_1}$ and $z = \binom{\lambda_2}{\gamma}$. Introduce the corresponding $2n \times 2n$ matrices Y and Z, initialized by

$$Y(t_1) = \begin{pmatrix} I & 0 \\ S_1 & 0 \end{pmatrix}$$

$$Z(t_1) = \begin{pmatrix} S_2 & -S_1 \\ 0 & I \end{pmatrix}$$

Now, the standard method will do, chosing the constant vector r such that

$$Y_{11}(t_0)r_1 + Y_{12}(t_0)r_2 = x_0,$$
$$Z_{21}(t_0)r_1 + Z_{22}(t_0)r_2 = 0.$$

This system has hamiltonian features, like the fact that $Z'Y$ will be a symmetric matrix. If furthermore, S_2 *is invertible*, then we have

$$Y(t_1)Z^{-1}(t_1) = \begin{pmatrix} I \\ S_1 \end{pmatrix} S_2^{-1} \begin{pmatrix} I & S_1 \end{pmatrix} = L_1, \tag{13}$$

which is a symmetric matrix. Therefore, the matrix $L = YZ^{-1}$ will satisfy the symmetric Riccati equation

$$\dot L = \mathcal{H}L + L\mathcal{H}' + L\mathcal{N}L - \mathcal{G}, \tag{14}$$

initialized at t_1 by (13). We have obviously written the system (8) as

$$\begin{pmatrix} \dot y \\ \dot z \end{pmatrix} = \begin{pmatrix} \mathcal{H} & -\mathcal{G} \\ -\mathcal{N} & -\mathcal{H}' \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Since L is now symmetric, its computation involves only $2n^2+n$ scalar nonlinear differential equations. To get the solution of the game, one must still integrate $2n$ forward differential equations, with a rather large amount of nonlinear computations, since the substitution to make is now to be derived from

$$\begin{pmatrix} x \\ \lambda_1 \end{pmatrix} = L \begin{pmatrix} \lambda_2 \\ \gamma \end{pmatrix}$$

which yields

$$\lambda_1 = L_{21}L_{11}^{-1}x + (L_{22} - L_{21}L_{11}^{-1}L_{12})\gamma,$$
$$\lambda_2 = L_{11}^{-1}x - L_{11}^{-1}L_{12}\gamma.$$

The matrices K_1, K_2, and P can also be recovered, but we shall detail how in the next section only.

Let us finally recall that if the matrices of the problem are constant, a diagonalization of the linear system yields an explicit solution for Y and Z, and therefore for L and the K_i's and P. This is the essence of the treatment of [1], although it does not appear clearly.

4. The second hamiltonian structure.

A striking feature of the above system is that its diagonal matrix $\mathcal{H}$ is itself hamiltonian. We do no know how to use this fact. But the simple fact that the overall system be hamiltonian implies that the system obtained by reordering the variables as $(-\gamma, x, \lambda_1, \lambda_2)$ is again hamiltonian. As a matter of fact, let

$$J = \begin{pmatrix} 0 & -I_{2n} \\ I_{2n} & 0 \end{pmatrix}$$

be the $4n \times 4n$ simplectic operator. The fact that the $4n \times 4n$ matrix $\mathbf{H}$ be hamiltonian may be characterized as $J\mathbf{H}$ *is symmetric*. But introduce then the matrix

$$\tilde{J} = \begin{pmatrix} 0 & -I_n \\ I_{3n} & 0 \end{pmatrix}$$

and notice that it is again orthogonal: $\tilde{J}\tilde{J}' = I$, but moreover that $\tilde{J}^2 = J$. As a consequence, we have that $J\tilde{J}\mathbf{H}\tilde{J}' = \tilde{J}J\mathbf{H}\tilde{J}'$ is again symmetric, thus as claimed, $\tilde{J}\mathbf{H}\tilde{J}'$ is hamiltonian.

In the present case we get

$$\begin{pmatrix} -\dot{\gamma} \\ \dot{x} \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} A & 0 & N & -M_1 \\ 0 & A & -M_1 & -M_2 \\ 0 & -Q_1 & -A' & 0 \\ -Q_1 & -Q_2 & 0 & -A' \end{pmatrix} \begin{pmatrix} -\gamma \\ x \\ \lambda_1 \\ \lambda_2 \end{pmatrix}. \tag{15}$$

Set therefore $y = \binom{-\gamma}{x}$ and $\lambda = \binom{\lambda_1}{\lambda_2}$, and the initial theory applies trivially. Let

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} -N & M_1 \\ M_1 & M_2 \end{pmatrix}, \quad \mathcal{Q} = \begin{pmatrix} 0 & Q_1 \\ Q_1 & Q_2 \end{pmatrix}.$$

Introduce the $2n \times 2n$ matrices Y and Λ satisfying

$$\begin{pmatrix} \dot{Y} \\ \dot{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & -\mathcal{M} \\ -\mathcal{Q} & -\mathcal{A}' \end{pmatrix} \begin{pmatrix} Y \\ \Lambda \end{pmatrix}, \qquad \begin{pmatrix} Y(t_1) \\ \Lambda(t_1) \end{pmatrix} = \begin{pmatrix} I \\ \mathcal{S} \end{pmatrix}$$

with

$$\mathcal{S} = \begin{pmatrix} 0 & S_1 \\ S_1 & S_2 \end{pmatrix}.$$

The solution of the initial two point boundary value problem is given by $\lambda(t) = \Lambda(t)r$ where, as usual, r is determined by $Y(t_0)r = \binom{0}{x_0}$. The solution via the Riccati equation is now elementary. Set $K = \Lambda Y^{-1}$. It satisfies

$$\dot{K} = -K\mathcal{A} - \mathcal{A}'K + K\mathcal{M}K - \mathcal{Q}, \quad K(t_1) = \mathcal{S}. \tag{16}$$

It is symmetric, and therefore yields the solution of the two-point boundary value problem (8),(9), or equivalently (15),(9), in $4n^2 + 3n$ scalar differential equations, without the numerous matrix inversions of the previous method, and without any invertibility assumption on S_2.

Although the matrices K_i and P are of litle interest by themselves, notice that they can be recovered in the following way. Introduce the $n \times n$ matrices X and Γ defined by

$$\begin{pmatrix} -\dot{\Gamma} \\ \dot{X} \end{pmatrix} = (\mathcal{A} - \mathcal{M}K) \begin{pmatrix} -\Gamma \\ X \end{pmatrix}, \quad \begin{pmatrix} -\Gamma(t_0) \\ X(t_0) \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix}.$$

It is a simple matter to check that we have

$$P = \Gamma X^{-1}, \qquad K_i = K_{i2} - K_{i1}P, \quad i = 1, 2.$$

5. Conclusion.

Since a diagonalization argument always allows one to solve explicitely a linear two-point boundary value problem, by giving explicit formulas for the partial transition matrices, it also provides an explicit solution of the associated Riccati equation. However, the same theories, without the explicit exponential formulas, also yield the solution of the two-point boundary value problem in the nonstationary case. In the case of the problem arising from the linear quadratic Stackelberg problem, two distinct hamiltonian structures can be exhibited. One of them yields a simple solution to the problem.

References.

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