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ON SINGULAR IMPLICIT LINEAR DYNAMICAL SYSTEMS*

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Abstract. We investigate properties of existence, unicity, representation, of the (causal) solutions of implicit linear systems (or "generalized systems") when the underlying matrix pencil is singular. We relate the geometric and the algebraic approaches. The main conclusion is that if the underlying matrix pencil is "column singular" (i.e., has a nonempty set of column minimal indices) the causal solutions, when they exist, can exactly be represented as the output of a classical two-player dynamical system, where the second player accounts for the nonunicity. Properties of the equivalent system are related to those of the singular matrix pencils made with the given matrices.

1. Introduction.

1.1. Problems considered. We study systems given in one of the following two forms, respectively, discrete and continuous:

$$Ey(t+1)=Fy(t)+Gu(t), \tag{$*$}$$

$$E\frac{dy}{dt}(t)=Fy(t)+Gu(t), \tag{$**$}$$

with the following definitions:

$y(t)\in\mathbb{R}^m$ is the (fundamental) output of the system,

$u(t)\in\mathbb{R}^p$ is the input.

E and F are $r\times m$ constant matrices, G is a $r\times p$ constant matrix. r is called the *rank* of the system. It may be larger than, equal to or lower than m.

The questions of existence and unicity we shall investigate arise only if E is not invertible (in case $r=m$). We shall also consider problems of representation and canonical forms. We are mainly interested in *singular* systems, where the solution is nonunique. (See Definition 3 and Theorem 2 for a precise statement.)

DEFINITION 1. If $r=m$, the system is called *square.*

PROPOSITION 1. *A system* (E, F, G) *is always equivalent*:

(i) *to a system with rank equal to the rank of the composite matrix* $[E\ \ F\ \ G]$;

(ii) *if this rank is lower than m to a square system.*

Proof. (i) If the lines of the composite matrix $[E\ \ F\ \ G]$ are not independent, we can always delete redundant equations in $(*)$ or $(**)$.

(ii) If $r<m$ we can add lines of zeros to them.

HYPOTHESIS. *Because of property* (i) *above, we shall always assume that* rank $[E\ \ F\ \ G]=r$.

1.2. Motivation. (i) P.I.D. control. Systems of the form $(**)$ naturally arise when applying output derivative feedback to an ordinary system. There the resulting implicit system is square. The interesting question is its limit behavior when the E matrix is "close" to be singular. A prerequisite to a complete understanding of the resulting "infinite frequency" modes (see [20]) is the present analysis.

(ii) Systems with a linear state or state-control constraint. An equation of the form

$$0=Cy+Du$$

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may be added to a standard system as an extra set of equations resulting in a matrix E made of the identity and lines of zeros. There $r > m$.

(iii) Interconnected systems. The natural statement of the equations of sets of interconnected systems may lead to equations of the type (ii).

(iv) Econometric systems. Econometric systems are almost always of the form (*) (or a more complex one with nonlinear r.h.s.). Most famous among them are Leontief's models, and ARMA models with noninvertible leading coefficient.

(v) Perturbed systems. The perturbed system

$$\dot{x} = Ax + Bu + Cv$$

is equivalent to the implicit system

$$E\dot{x} = EAx + EBu,$$

where E is a matrix of maximum rank such that $EC = 0$. As a matter of fact, for any pair of measurable input functions $(u(\cdot), v(\cdot))$, the solution of the first, explicit, differential equation satisfies the second, implicit, one. Conversely, to any pair of an absolutely continuous $x(\cdot)$ and a measurable $u(\cdot)$ satisfying the second one, corresponds a measurable $v(\cdot)$ such that $x(\cdot)$ satisfies the first one with inputs $u(\cdot)$ and $v(\cdot)$.

(vi) Time reversibility in discrete time systems. Backward projection for a standard discrete system

$$x_{k+1} = Fx_k + Gu_k$$

leads to the study of the backward system

$$F\bar{x}_{k+1} = \bar{x}_k - G\bar{u}_k,$$

where $\bar{x}_{k+1} = x_{-(k+1)}$ and $\bar{u}_k = u_{-k-1}$.

(vii) Operator splitting numerical methods. Solution of the equation

$$Ay = f$$

can be pursued using a recursion of form (*) with $A = E - F$ and $Gu = f = \text{constant}$ (or $Gu_k \to f$).

(viii) Implicit differential equations. The representation results obtained here may be of some interest for their own sake in the study of implicit linear differential equations.

1.3. Originality. More than ten years ago, Rosenbrock's theory was explicitly devised to address implicit systems of a more complicated type since higher derivatives were allowed as well as derivatives of the control. See a rather complete account in Rosenbrock [14]. Since then the precise type of systems we study have been investigated by Luenberger and coworkers [12], [13], [15]. Beyond problems of existence and unicity, they have considered optimization problems. More recently, papers by Verghese, Kailath and coworkers have dealt with the infinite frequency aspects of these systems [16], [17]. Systems of the form (*) also appear in connection with linear programming, see, for instance, [5].

All the above references deal with the "regular case", i.e., square systems with $\det(zE - F) \not\equiv 0$. In that case, as we shall see, existence implies unicity. Our main emphasis is on the singular case, and the representation of nonunicity. Some works on that topic are due to Campbell. While [4] again deals only with the regular case, [3] considers a very particular instance of the singular case. It is a subcase of our "static nonunicity". Moreover, his application to linear systems is further restricted to the regular case.

While this paper was being typed, the author became aware (through D. Gabay, of Inria) of the work of Wilkinson [21]. It deals with the general singular case but lacks the necessary tools of control theory to give a complete description of the nonunicity via invariants. It essentially covers the method of our paragraph 5.4 without the references to the geometric and transfer function theories.

After this paper was first submitted for publication, several articles appeared on that topic[1], covering both the regular case, see [18], [19] and [20] (which is a more complete account of an earlier publication in the 1979 IEEE CDC, held in 1980), and, more importantly to us, the singular case. See [10] and [11], which rely heavily on an analysis very similar to that of our paragraph 5.4. Reference [1] is also an approach of system theory without unicity.

While [18], and to a smaller extent [17], use some geometrical concepts, the literature has been in most part algebraic in nature. We believe, however, that our § 2 shows that the geometric approach allows a completely elementary treatment of both the regular and singular cases.

1.4. Outline. In § 2 we develop the (elementary) geometric theory of strictly causal discrete systems (*). In the very short § 3, we check that all the results but a minor one carry over to the continuous case. In § 4 we investigate the geometric theory of the causal (but not strictly causal) case. Section 5 is devoted to the algebraic theory, invariants, transfer functions and canonical forms.

2. Discrete time systems, the strictly causal case.

2.1. Causality. We quickly review here what *causality*, or *strict causality*, means for a dynamical system with possibly nonunique solutions. We deal with the discrete system (*), the extension to (**) is straightforward, provided, in the definition of causality, "$\forall t$" be replaced by "for almost all t". As a consequence, the difference between causality and strict causality, as given here, vanishes. Strict causality, in the continuous case, will carry an added requirement. See § 3.

Let Ω be the set of admissible control functions, i.e., applications from $[t_0, t_1]$ into $\mathbb{R}^m$. (Usually, $t_1 = +\infty$.) A *correspondence of solutions* is a set-valued function S from Ω into the set of trajectories, which to each $u(\cdot)$ in Ω associates a set $S(u(\cdot))$ of trajectories $y(\cdot)$ satisfying (*). Let $S_\tau(u(\cdot))$ be the set of the restrictions to $[t_0, \tau]$ of the elements of $S(u(\cdot))$. We recall the following.

DEFINITION. The correspondence S is called *strictly causal* if given $u_1(\cdot)$ and $u_2(\cdot)$ in Ω

$$\text{if } u_1(t) = u_2(t)\ \forall t < \tau \quad \text{then } S_\tau(u_1(\cdot)) = S_\tau(u_2(\cdot)).$$

S is said *causal* if the conclusion holds provided $u_1(t) = u_2(t)$, for all $t \leqq \tau$. (In all the sequel, "strictly causal" may correspondingly be replaced by "causal".) The set of strictly causal solutions of the system is the maximal strictly causal correspondence of solutions, i.e., the union $\bar{S}$ of all of them. Given $u(\cdot)$ in Ω, a trajectory $y(\cdot)$ is called a strictly causal solution if it belongs to $\bar{S}(u(\cdot))$.

A characteristic property of a strictly causal solution is that, in addition to satisfying (*) for all t, it is such that, for all τ in (t_0, t_1), the system (*) initialized at $y(\tau)$ has strictly causal solutions for every sequence $\{u(t), t \geqq \tau\}$. The reader may easily check that this inductive characterization is indeed necessary and sufficient. It will be used hereafter in the proofs.

[1] Some were pointed out to us by a reviewer whom we thank here.

Remark. Restricting oneself to the causal case, as we shall do, amounts precisely to ignoring the "impulsive modes" of the theory as developed in [17], [20]. As a matter of fact, we want to focus here on the nonunicity, not on impulsive modes.

2.2. Existence. We write $\mathscr{E}=\mathscr{R}(E)$ and $\mathscr{G}=\mathscr{R}(G)$, the respective ranges of E and G, as subspaces of $Y=\mathbb{R}^r$. Consider the following relation for a linear subspace $\mathscr{V}$ of Y:

$$F\mathscr{V}\subset E\mathscr{V}. \tag{1}$$

DEFINITION 1. We call *characteristic subspace* of the pair (E, F) the largest subspace $\mathscr{V}^*$ satisfying (1).

PROPOSITION 1. *This subspace exists since* $\{0\}$ *satisfies* (1), *and this equation being stable under addition of subspaces,* $\mathscr{V}^*$ *is the sum of all subspaces that satisfy it.* (*However,* $\mathscr{V}^*$ *may be trivial.*)

This space is clearly related to the solution of

$$E\dot{x}=Fx \quad \text{or} \quad x(t+1)=Fx(t)$$

and can be considered a dynamic invariant. It will be seen further that it contains the "generalized eigenvectors" of this sytem.

In the special case where we are given a two input system

$$x(t+1)=Ax(t)+Bu(t)+Cv(t)$$

and where as in motivation (v) (*) is obtained by taking an injective matrix E such that $\operatorname{Ker} E=\mathscr{R}(C)$:

$$Ex(t+1)=EAx(t)+EBu(t),$$

then (1) translates in

$$EA\mathscr{V}\subset E\mathscr{V},$$

which is equivalent to

$$A\mathscr{V}\subset \mathscr{V}+\mathscr{R}(C).$$

Therefore $\mathscr{V}$ is A invariant mod C and $\mathscr{V}^*$ is then the largest A invariant mod C subspace, i.e., $\mathbb{R}^n$. The similarity with (A, C) invariance, which was suggested by a reviewer, is further displayed in the algebraic theory. See Remark 7.

THEOREM 1. *The system* (*) *has a strictly causal solution over an interval of arbitrary length, for any control sequence* $u(\cdot)$, *if and only if*

$$\mathscr{G}\subset E\mathscr{V}^*, \tag{2}$$

$$y(0)\in\mathscr{V}^*. \tag{3}$$

Proof. (i) *Necessity.* Let t be given. In order for $y(t+1)$ to exist, it is necessary that

$$Fy(t)+Gu(t)\in\mathscr{E},$$

and since this must be true for all $u(t)\in\mathbb{R}^p$, this implies

$$\mathscr{G}\subset\mathscr{E} \quad \text{and} \quad y(t)\in\mathscr{V}^0=F^{-1}(\mathscr{E}).$$

In order for the last relation to hold for every $u(t-1)$, we need

$$\mathscr{G}\subset E\mathscr{V}^0, \qquad y(t-1)\in\mathscr{V}^1=F^{-1}(E\mathscr{V}^0).$$

Continuing this process, we construct the sequence $\mathcal{V}^k$ by

(4) $$\mathcal{V}^{k+1}=F^{-1}(E\mathcal{V}^k),$$

and we must have for all k,

$$\mathcal{G}\subset E\mathcal{V}^k, \qquad y(t-k)\in\mathcal{V}^k.$$

Necessity follows from the following fact.

PROPOSITION 2. *The sequence $\mathcal{V}^k$ is decreasing and converges to $\mathcal{V}^*$ in no more than m steps.*

Proof. Clearly, $F^{-1}(E\mathcal{V}^0)\subset F^{-1}(\mathscr{E})$, and thus, $\mathcal{V}^1\subset\mathcal{V}^0$, and so on by induction. However, subspaces can decrease only by losing one dimension, which cannot occur more than m times in $\mathbb{R}^m$. Let k be the first index such that $\mathcal{V}^{k+1}=\mathcal{V}^k$. The sequence $\mathcal{V}^k$ becomes stationary from this point on, and (4) shows that $\mathcal{V}^k$ satisfies (1). Therefore, $\mathcal{V}^k\subset\mathcal{V}^*$. This establishes the necessity of (2), (3), but not the proposition, which states that $\mathcal{V}^k=\mathcal{V}^*$. This can easily be proved directly but follows also from the sufficiency of (2), (3) that we now establish.

(ii) *Sufficiency*. Let V be a rectangular injective (full column rank) matrix such that $\mathcal{R}(V)=\mathcal{V}^*$ (let $\dim\mathcal{V}^*=n^*$, $V\colon m\times n^*$). Relations (1) and (2) imply

(5) $$\exists\bar{A}:\quad FV=EV\bar{A},$$

(6) $$\exists\bar{B}:\quad G=EV\bar{B},$$

where $\bar{A}$ is a $n^*\times n^*$ matrix and $\bar{B}$ is $n^*\times p$. We also have that $y(t)\in\mathcal{V}^*$ is equivalent to

(7) $$\exists\xi(t)\in\mathbb{R}^{n^*}:\quad y(t)=V\xi(t).$$

Now, $(*)$ is equivalent to

(8) $$EV\xi(t+1)=EV(\bar{A}\xi(t)+\bar{B}u(t)),$$

which together with (3) has the obvious solution

(9) $$\xi(t+1)=\bar{A}\xi(t)+\bar{B}u(t), \qquad y(0)=V\xi(0). \qquad \square$$

Remark 1. When (2) is not satisfied, we may restrict u to belong to $\mathcal{U}_{\mathrm{ad}}=G^{-1}(E\mathcal{V}^*)$. In the sequel, condition (2) may always be understood to mean that this reduction has been performed and will always be assumed to hold.

2.3. Unicity.

DEFINITION 2. We call *characteristic kernel* of the pair (E,F) the subspace $\mathcal{N}$ defined by

(10) $$\mathcal{N}=\operatorname{Ker}E\cap\mathcal{V}^*.$$

Let $\dim\mathcal{N}=q$.

DEFINITION 3. The pair (E,F) is said *C-regular* (or more accurately *column regular*) if $q=0$:

(11) $$\mathcal{N}=\{0\}.$$

THEOREM 2. *Under conditions* (2) *and* (3), *the solution to equation* $(*)$ *is unique, for any* $u(\cdot)$, *if and only if the system* (*the pair* E, F) *is C-regular. Otherwise, the nonunicity is described by the arbitrary choice of the sequence* $v(\cdot)$ *in equation* (14), *and* (14), (15) *constitute a representation of all solutions of* $(*)$.

Proof. (i) *Unicity.* Equation (8) implies (9) only modulo the kernel of EV, which reduces to $\{0\}$ under and only under condition (11).

(ii) *Nonunicity.* If $\mathcal{N} \neq \{0\}$, let us choose a decomposition of $\mathcal{V}^*$ of the form

$$\mathcal{V}^* = \mathcal{M} \oplus \mathcal{N}. \tag{12}$$

To this decomposition we may associate a partition of V of the form

$$V = [M \quad N], \qquad EV = [EM \quad 0]. \tag{13}$$

Let us partition accordingly ξ, $\bar{A}$ and $\bar{B}$ in the following way:

$$\xi = \begin{pmatrix} x \\ v \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} A & C \\ \tilde{A} & \tilde{C} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} B \\ \tilde{B} \end{pmatrix}.$$

By definition EM is injective so that (8) is equivalent to

$$x(t+1) = Ax(t) + Bu(t) + Cv(t), \tag{14}$$

$$y(t) = Mx(t) + Nv(t). \tag{15}$$

□

The nonunicity is therefore described as the effect of an extra input in a classical linear system. We may apply to it the tools of two-player control systems. In that respect it is worthwhile to notice that V being injective knowledge of y is equivalent to the knowledge of both x and v. (This is important, for instance, in discrete capturability theory [2].)

Remark 2. The matrix C may of course be of less than full column rank. If this is the case, by a proper choice of basis we can write

$$C = [C_1 \quad 0],$$

accordingly partitioning v in $v' = (v_1' v_2')$. Then v_1 must be considered as parametrizing a *dynamic nonunicity* since its effect propagates forward in time through the dynamics, while v_2 parametrizes a *static nonunicity* since it appears only in the output equation (15) (Recall that N is injective.)

The triple (A, B, C) is clearly nonunique. It may be altered through a change of basis within $\mathcal{V}^*$. This leads to the following fact.

PROPOSITION 3. *The pair* (A, C) *is uniquely defined up to a transformation of Brunovsky's feedback group* (see Kalman [9]).

Proof. A change of basis within $\mathcal{V}^*$ can be described as

(i) a change of basis within $\mathcal{N}$, i.e., on v;

(ii) a change of choice of $\mathcal{M}$ within $\mathcal{V}^*$. Let $\tilde{M}$ generate an alternate $\tilde{\mathcal{M}}$:

$$y = Mx + Nv = \tilde{M}\tilde{x} + N\tilde{v}.$$

The difference $v - \tilde{v}$ depends linearly on y and is null when $y \in \mathcal{N}$, i.e., when $x = 0$. Therefore it depends linearly on x alone:

$$\tilde{v} = Px + v.$$

Using the fact that M is injective, this gives

$$\tilde{x} = Qx,$$

where Q can be calculated as a function of M, $\tilde{M}$, N and P. Therefore, this is equivalent to a state feedback superimposed on v and a change of basis on x.

(iii) a change of basis on x alone (which can, of course, undo the previous one). □

We shall study further the invariants of (A, C). However, an interesting geometric one at this point will be provided by the following definition.

DEFINITION 4. We call the *neutral subspace* of the pair (E, F) the smallest subspace $\mathcal{V}_*$ that satisfies (1) and contains $\mathcal{N}$.

PROPOSITION 4. *Such a subspace exists as a consequence of Theorem 5 below, that is, by applying it with* $G = 0$.

THEOREM 3. *Two solutions of* $(*)$ *corresponding to the same initial point and same sequence* $u(\cdot)$ *are equal modulo* $\mathcal{V}_*$. $\mathcal{V}_*$ *is image by* V *of the reachable space of the pair* (A, C) *in* (14).

Proof. By subtraction, two solutions of $(*)$ corresponding to the same initial point and the same sequence $u(\cdot)$ have their differences δy that satisfy

$$\delta y(t) = V\delta\xi(t) = M\delta x(t) + N\delta v(t), \qquad \delta\xi(0) = 0,$$
$$\delta x(t+1) = A\delta x(t) + C\delta v(t), \qquad \delta x(0) = 0.$$

Therefore, $\delta x(t)$ belongs to the reachable space of the pair (A, C). Conversely, any solution of this system remains strictly causal and satisfies

$$E\delta y(t+1) = F\delta y(t), \qquad \delta y(0) = 0 \tag{16}$$

and can therefore be added to a solution of $(*)$ and still remain a solution.

The fact that $\mathcal{V}_*$ is exactly the (image of) reachable subspace of the pair (A, C) will be a corollary of Theorem 5 below. □

2.3. Minimality.

DEFINITION 5. We call the *maximum subspace* of the triple (E, F, G) the largest subspace $\mathcal{W}^*$ satisfying

$$F\mathcal{W}^* + \mathcal{G} = E\mathcal{W}^*. \tag{17}$$

PROPOSITION 5. *The subspace* $\mathcal{W}^*$ *exists; it is a subspace of* $\mathcal{V}^*$ *and is the limit, attained in no more than m steps of the sequence* $\mathcal{W}^k$ *defined by*

$$\mathcal{W}^0 = \mathcal{V}^*, \qquad \mathcal{W}^{k+1} = E^{-1}(F\mathcal{W}^k + \mathcal{G}) \cap \mathcal{V}^*. \tag{18}$$

Proof. Notice that since $F\mathcal{V}^* + \mathcal{G} \subset E\mathcal{V}^*$ we have

$$E\mathcal{W}^1 = F\mathcal{W}^0 + \mathcal{G} \quad \text{and} \quad \mathcal{W}^1 \subset \mathcal{V}^*. \tag{19}$$

It follows easily that property (19) holds at every step of the algorithm, shifting the indices of $\mathcal{W}$ by an equal number and also that the sequence is decreasing. It therefore has a limit which satisfies (17), of which it is easy to check that it is the largest solution of (17) (which is stable by addition of subspaces).

THEOREM 4. $\mathcal{W}^*$ *is the largest subspace traversed by the asymptotic regime of* $(*)$, *i.e., for all* $k \geqq n$; *the application* $(y(0), u(\cdot)) \mapsto y(k)$ *is surjective over* $\mathcal{W}^*$, *which is exactly its range.*

Proof. By construction, (3) implies $y(1) \in \mathcal{W}^1$ and, by induction, $y(t) \in \mathcal{W}^t$ with surjectivity. This, with the proposition, proves the theorem. □

While this result characterizes in some sense the reachable subspace of $(*)$, it is not the most interesting one. As a matter of fact, classical system theory teaches us that the reachable subspace of interest is that which is reachable from the state zero. We therefore proceed with the following.

DEFINITION 6. The minimal subspace of the triple (E, F, G) is the smallest subspace $\mathcal{W}_*$ satisfying (1) and (2) and containing $\mathcal{N}$.

Remark 3. In the case where $E=I$, this is a classical characterization of the reachable space of (F, G).

THEOREM 5. *$\mathscr{W}_*$ exists and is the limit* (*in m steps or less*) *of the same recurrence as in* (18) *but initialized with $\mathscr{W}_0=\{0\}$. It is the image by V of the reachable space of the system* (14), *where both u and v are taken as controls.*

Proof. Notice first that property (1) is not stable under intersection and, therefore, the existence of a smallest subspace satisfying it and other conditions is not obvious. Consider the recurrence (18) initialized with $\mathscr{W}_0=\{0\}$;

$$\mathscr{W}_1=E^{-1}(\mathscr{G})\cap \mathscr{V}^*.$$

Because of (2),

$$E\mathscr{W}_1=\mathscr{G}.$$

The sequence $\mathscr{W}_k$ is clearly increasing and, by induction, satisfies the same sequence of equalities of the form (19) as $\mathscr{W}^k$. By construction, $\mathscr{N}\subset\mathscr{W}_k$ for all k. Therefore, it has a limit $\mathscr{W}_*$ that satisfies (1), (2) and contains $\mathscr{N}$.

According to Theorem 2, the image by V of the reachable space of (14) is exactly the reachable space for $y(t)$ from zero. By construction it is the limit of the above recurrence.

That $\mathscr{W}_*$ be the smallest subspace satisfying (1), (2) and containing $\mathscr{N}$ follows from the following lemma.

LEMMA 1. *Any subspace $\mathscr{W}$ satisfying* (1), (2) *and containing $\mathscr{N}$ contains the reachable space of* (*) *from zero.*

Proof. $\mathscr{W}$ is a subspace of $\mathscr{V}^*$ since it satisfies (1). Let us assume that the matrix V has been chosen in such a way that a submatrix W generates $\mathscr{W}$:

$$V=[L \quad W].$$

Since $\mathscr{N}\subset\mathscr{W}$ we may choose W such that N be a submatrix of it.

We may therefore partition V further in $W=(\tilde{M}, N)$ and therefore

$$V=[L \quad \tilde{M} \quad N],$$

with $M=[L \quad \tilde{M}]$. Now (5) and (6) give

$$FM=EMA, \quad FN=EMC, \quad G=EMB, \tag{20}$$

which, further partitioned according the above partition of M, gives

$$F\tilde{M}=ELA_{12}+E\tilde{M}A_{22}, \qquad G=ELB_1+E\tilde{M}B_2. \tag{21}$$

Now, by hypothesis, $\mathscr{W}$ satisfies (1) and (2) so that there exist $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{B}$ such that

$$F\tilde{M}=E\tilde{M}\tilde{A}_1, \quad FN=E\tilde{M}\tilde{A}_2, \quad G=E\tilde{M}\tilde{B}. \tag{22}$$

If we remember that $[EL \ \ E\tilde{M}]=EM$ is injective, comparison of (21) and (22) yield

$$A_{12}=0, \qquad B_1=0.$$

This is the standard form for a system whose reachable space is contained in $\mathscr{W}$. □

COROLLARY 1. *The neutral space $\mathscr{V}_*$ exists and is the reachable space of the pair* (A, C).

Proof. Apply Theorem 5 with $G=0$. □

Let now W be a submatrix of V generating $\mathcal{W}_*$, and let

$$W = [\hat{M} \quad N]. \tag{23}$$

As before, there exists $\hat{A}$, $\hat{B}$ and $\hat{C}$ such that

$$F\hat{M} = E\hat{M}\hat{A}, \quad FN = E\hat{M}\hat{C}, \quad G = E\hat{M}\hat{B}. \tag{24}$$

If the system $(*)$ is initialized at $y(0) \in \mathcal{W}_*$, we can always represent its solution as

$$\hat{x}(t+1) = \hat{A}\hat{x}(t) + \hat{B}u(t) + \hat{C}v(t), \tag{25}$$

$$y(t) = \hat{M}\hat{x}(t) + Nv(t), \tag{26}$$

and this constitutes a minimal representation of system (14) (15) (possibly with a feedback on v if we have changed of choice for $\mathcal{M}$). It can therefore be considered as a *minimal representation of* $(*)$. It is unique up to a change of basis and a feedback on $v(\cdot)$.

3. Continuous time system. This short section is aimed at checking that all previous results, except Theorem 4 which is not important in the theory, carry over to the continuous case. We keep same notations and same numbers to the theorems.

Strict causality is taken to mean causality plus the fact that to a measurable input corresponds an absolutely continuous output.

3.1. Existence.

Proof of Theorem 1. (i) *Necessity*. Let $\mathcal{V}$ be the subspace generated by those y's that can be reached by the system. Necessarily, $\dot{y} \in \mathcal{V}$, therefore $\mathcal{V}$ must satisfy (1) and, thus, be included in $\mathcal{V}^*$ and (2).

(ii) *Sufficiency*. Perform exactly as in § 2.2 to end up with

$$\dot{\xi}(t) = \bar{A}\xi(t) + \bar{B}u(t).$$

3.2. Unicity.

Proof of Theorem 2. Unchanged, except for the substitution of an arbitrary measurable time function $v(\cdot)$ to the arbitrary sequence.

Proof of Theorem 3. The proof that two solutions corresponding to the same initial condition and the same control function $u(\cdot)$ differ at each time instant of an element of the image by V of the reachable space of the pair (A, C) is unchanged. The rest of the theorem relies on the next paragraph.

3.3 Minimality. Theorem 4 does not carry over in a simple way. One can prove that

$$y(t) \in y_0 + ty_1 + \cdots + \frac{t^{K-1}}{(K-1)!} y_{K-1} + \mathcal{W}^*,$$

where K is the smallest integer such that $\mathcal{W}^{K+1} = \mathcal{W}^k = \mathcal{W}^*$ and y_k is a sequence satisfying the homogeneous discrete system (16). The proof is a direct consequence of the remark that

$$\dot{y} \in \mathcal{W}^1 \Rightarrow y(t) \in y_0 + \mathcal{W}^1 \Rightarrow \dot{x} \in E^{-1}(F(\mathcal{W}^1 + x_0) + \mathcal{G}) \cap \mathcal{V}^* = \mathcal{W}^2 + x_1$$

and then iterating.

Proof of Theorem 5. Defining $\mathcal{W}_*$ as previously, the algebraic constructions of § 2 remain the same. Moreover, classical system theory teaches us that given a pair $(A, [B \quad C])$ the reachable space is the same for the discrete time system and the continuous time system. Therefore $\mathcal{W}_*$ is still the reachable space of the system. □

Notice that without the parallel between continuous time and discrete time systems, Theorem 5 would be far less trivial in the continuous time case, since the identification of the limit of the recurrence $\mathcal{W}_k$ with the reachable space relies on a direct study of the system $(*)$.

The corollary carries over unchanged.

4. The nonstrictly causal case. We investigate here existence, unicity and representation of the solution of $(*)$ when $y(t)$ is allowed to depend on past $u(s)$ and on $u(t)$. As for system $(**)$, the same (algebraic) results hold if causality is defined via the existence of a proper transfer function (see § 5) since $y(\cdot)$ may now be nondifferentiable. Deriving the results of § 2 from those of this section is obviously possible; however, this would hide the elementary character of § 2 and make the introduction of $\mathcal{V}^*$ very artificial.

4.1. Existence.

THEOREM 5. *There exists a causal solution to* $(*)$ *over any time interval for any sequence* $u(\cdot)$ *if and only if*

$$\mathcal{G} \subset E\mathcal{V}^* + F \operatorname{Ker} E = E\mathcal{M} + F \operatorname{Ker} E, \tag{27}$$

$$y_0 \in \mathcal{V}^* + \operatorname{Ker} E = \mathcal{M} + \operatorname{Ker} E. \tag{28}$$

Proof. (i) *Necessity.* Let us arbitrarily write

$$y(t) = z(t) + \varepsilon(t),$$

where

$$\varepsilon(t) \in \operatorname{Ker} E, \qquad z(t) \in \mathcal{Z},$$

and $\mathcal{Z}$ is a subspace that we shall choose later on. By an appropriate restriction, we can manage to have $\mathcal{Z} \cap \operatorname{Ker} E = \{0\}$ so that the above decomposition of y is unique. Equation $(*)$ yields

$$Ez(t+1) = Fz(t) + F\varepsilon(t) + Gu(t), \tag{29}$$

so that, given $y(t)$ and $u(t)$, $Ez(t+1)$ is uniquely determined and also z once we restrict $\mathcal{Z}$ to have no intersection with Ker E.

By the same type of induction as in paragraph 2.1, we readily see that we must have

$$F\mathcal{Z} \subset E\mathcal{Z} + F \operatorname{Ker} E, \tag{30}$$

$$\mathcal{G} \subset E\mathcal{Z} + F \operatorname{Ker} E, \tag{31}$$

in order for (29) to have a solution $(z(t+1), \varepsilon(t))$ once $z(t)$, which depends upon the past, and $u(t)$ are given. The result then follows from the following fact.

LEMMA 2. *The largest subspace satisfying* (30) *is* $\mathcal{V}^* + \operatorname{Ker} E$.

Proof. Notice first that $\mathcal{V}^* + \operatorname{Ker} E$ satisfies (30). Now let $\mathcal{Z}$ satisfy (30) and contain Ker E (since the maximal one does). Write

$$\mathcal{Z} = \mathcal{V} + \operatorname{Ker} E,$$

and

$$F(\mathcal{V} + \operatorname{Ker} E) \subset E\mathcal{V} + F \operatorname{Ker} E.$$

This implies that

$$\forall a \in \mathcal{V}, \quad \exists \tilde{a} \in \mathcal{V} \quad \text{and } b \in \operatorname{Ker} E \text{ such that} \quad Fa = E\tilde{a} + Fb.$$

Clearly, $\tilde{a}$ and b can be chosen depending linearly on a. Let therefore K generate Ker E. There exists a matrix of appropriate type, such that, for every $a \in \mathscr{V}$

$$Fa = E\tilde{a} + FKPa,$$

thus,

$$F(I-KP)a = E\tilde{a} = E(I-KP)\tilde{a}.$$

Let therefore

$$\bar{\mathscr{V}} = (I-KP)\mathscr{V}.$$

Clearly,

$$\bar{\mathscr{V}} + \text{Ker } E = \mathscr{V} + \text{Ker } E = \mathscr{X},$$

but also

$$F\bar{\mathscr{V}} \subset E\bar{\mathscr{V}},$$

so that

$$\bar{\mathscr{V}} \subset \mathscr{V}^*, \qquad \mathscr{X} \subset \mathscr{V}^* + \text{Ker } E.$$

This proves the lemma. Notice that to get the unicity of $z(t)$ we must choose $\mathscr{X} = \mathscr{M}$, a complement of Ker E in $\mathscr{V}^*$.

(ii) *Sufficiency*. Let $\bar{K}$ be a matrix whose columns span Ker E and M be as in § 2. Let

$$y(t) = Mx(t) + \bar{K}w(t). \tag{32}$$

Condition (27) implies that there exist matrices $\bar{B}$ and $\bar{P}$ such that

$$G = EM\bar{B} + F\bar{K}\bar{P}.$$

Now equation (*) can be written equivalently

$$EMx(t+1) = EMAx(t) + F\bar{K}w(t) + EM\bar{B}u(t) + F\bar{K}\bar{P}u(t) \tag{33}$$

so that one possible solution of (*) is, using again (32),

$$x(t+1) = Ax(t) + \bar{B}u(t), \tag{34}$$

$$y(t) = Mx(t) - \bar{K}\bar{P}u(t). \tag{35}$$

(Notice that A is defined using only E and F, as in § 2. However, since the requirement on G has been changed, one should not look for a relation between the matrices G, $\bar{B}$, $\bar{P}$ of this section and G, $\bar{B}$ in the previous ones.) (34) and (35) together provide a causal solution and end the proof. □

4.2. Unicity.

THEOREM 7. *The causal solution of* (*) *under conditions* (27), (28) *is unique for each sequence* $u(\cdot)$ *if and only if the pair* (E, F) *is column regular.*

Proof. We want to find under what conditions (33) has a unique solution $x(t+1)$, $w(t)$, once $x(t)$ and $u(t)$ are given. As a matter of fact, if this is true, since $x(0)$ is uniquely determined by $y(0)$, $x(1)$ and $w(0)$ will be unique and all succeeding y's will be by induction.

By taking the difference $\delta x(t+1)$, $\delta w(t)$ between two solutions, we are led to the investigation of the nonzero solutions of

$$EM\delta x(t+1) = FK\delta w(t).$$

The only solution is zero if and only if

$$\operatorname{Ker} F \cap \operatorname{Ker} E = \{0\} \tag{36}$$

and

$$E\mathcal{M} \cap F \operatorname{Ker} E = \{0\}. \tag{37}$$

This is so because EM is injective. Therefore, for a nonzero solution, either both sides are zero (but then (36) does not hold) or there is a nonzero element in $E\mathcal{M} \cap F \operatorname{Ker} E$.

Notice that $\mathcal{V}^* = F^{-1}(E\mathcal{M})$, so that

$$\mathcal{N} = F^{-1}(E\mathcal{M}) \cap \operatorname{Ker} E. \tag{38}$$

Now, it can easily be checked that for two subspaces $\mathcal{A}$ and $\mathcal{B}$ and an arbitrary linear operator F, one has

$$F\mathcal{A} \cap F\mathcal{B} = F[(\mathcal{A} + \operatorname{Ker} F) \cap \mathcal{B}].$$

Apply this to (38), noticing that $F^{-1}(E\mathcal{M}) \subset \operatorname{Ker} F$; it becomes

$$F\mathcal{N} = E\mathcal{M} \cap F \operatorname{Ker} E. \tag{39}$$

Notice also that $\operatorname{Ker} F \subset \mathcal{V}^*$, so that

$$\mathcal{N} \supset \operatorname{Ker} E \cap \operatorname{Ker} F. \tag{40}$$

From (39) and (40) we conclude that if (37) or (36) is violated, $\mathcal{N}$ is nontrivial, i.e., the system is not C-regular.

Conversely, if $\mathcal{N}$ is nontrivial and if, moreover, (36) holds, then since $\mathcal{N} \subset \operatorname{Ker} E$, (36) implies

$$\mathcal{N} \cap \operatorname{Ker} F = \{0\},$$

and therefore, $F\mathcal{N}$ has same dimension as $\mathcal{N}$ and (39) shows that (37) is violated. □

Remark 4. We may again make a distinction between two types of nonunicity as in Remark 2. In the case (37) holds (but not (36)), the nonunicity in y involves only $w(t)$ and does not propagate in time. The sequence $x(\cdot)$ is unique. The nonunicity may be called "static". The dynamic nonunicity is induced by nonzero elements in $E\mathcal{M} \cap F \operatorname{Ker} E$.

The fact that the unicity condition is the same as in the strictly causal case will be more fully explained by the algebraic theory. It is not a trivial consequence of the fact that it is in both cases a study of nonzero solutions of (16) since y ranges over a larger subspace here.

4.3. Representation. Let us be more precise in representation (32), putting

$$\bar{K} = [N \quad K]$$

(and with w having now a different meaning)

$$y(t) = Mx(t) + Nv(t) + Kw(t).$$

We also have (recalling that $F\mathcal{N} \subset E\mathcal{M}$)

$$G = EMB + FKP$$

so that (33) can now be written

$$EMx(t+1) = EM(Ax(t) + Bu(t) + Cv(t)) + FK(Pu(t) + w(t)).$$

But now,

$$\mathcal{R}(EM) \cap \mathcal{R}(FK) = \{0\},$$

since any part of Ker E whose image by F is in $E\mathcal{M}$ belongs to $\mathcal{V}^*$, i.e., to $\mathcal{N}$, and moreover, since clearly Ker $F \subset \mathcal{V}^*$, FK is, as well as EM, injective. Therefore, the only solution is

$$w(t) = -Pu(t)$$

or defining $-KP = D$,

$$y(t) = Mx(t) + Du(t) + Nv(t), \qquad x(t+1) = Ax(t) + Bu(t) + Cv(t).$$

These equations will be summarized further ((48) to (52)). Notice that those for the strictly causal case are identical to these where we set $D = 0$. Notice also that the same analysis applies to a representation of system (**).

5. Algebraic theory.

5.1. Generalized spectrum and regularity.

DEFINITION 7. We call a *generalized eigenvalue* of the pair (E, F) and associated *generalized eigenvector* a complex number z and a nonzero complex vector ξ of $\mathbb{C}^m$ such that

$$(zE - F)\xi = 0. \tag{41}$$

LEMMA 3. *Both the real part and imaginary part of a generalized eigenvector of* (E, F) *belong to* $\mathcal{V}^*$. *Under condition* (2) *this is also true of the first component* (*in* $\mathbb{R}^m$) *of a generalized eigenvector of the pair* $([E \quad 0], [F \quad G])$.

Proof. Let

$$z = \sigma + i\omega, \qquad \xi = \eta + i\zeta \tag{42}$$

be a generalized eigenvalue and eigenvector of (E, F). Then (41) yields

$$F[\eta \quad \zeta] = E[\eta \quad \zeta]\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}.$$

Calling $\mathcal{X}$ the subspace generated by $[\eta \quad \zeta]$, this reads

$$F\mathcal{X} \subset E\mathcal{X},$$

and according to Proposition 1, this implies $\mathcal{X} \subset \mathcal{V}^*$, hence, the first claim. Keeping the notation (42), let $\varphi \in \mathbb{C}^p$:

$$\varphi = \chi + i\psi$$

constitute with ξ a generalized eigenvector of $([E \quad 0], [F \quad G])$:

$$(zE - F)\xi - G\varphi = 0.$$

Using (6) and separating again real and imaginary parts, we get

$$F[\eta \quad \zeta] = E[\eta \quad \zeta]\begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix} - EV\bar{B}[\chi \quad \psi],$$

which gives

$$F\mathcal{X} \subset E\mathcal{X} + E\mathcal{V}^*.$$

Adding $F\mathcal{V}^*$ to the left and using (1), this gives according to Proposition 1 $\mathcal{X} + \mathcal{V}^* \subset \mathcal{V}^*$, and thus, $\mathcal{X} \subset \mathcal{V}^*$. □

THEOREM 8. *The generalized spectrum of the pair (E, F) is finite if and only if this pair is C-regular (Definition 3). Otherwise, the generalized spectrum is the whole set $\mathbb{C}$ and the rank defect of $zE-F$ is at least q for all z (where $q=\dim \mathcal{N}$).*

Proof. Equations (5) and (6) yield

$$EV(zI_{n^*}-\bar{A})=(zE-F)V. \tag{43}$$

Assume (E, F) is C-regular. Then EV is injective. Let ξ be a generalized eigenvector; we know according to Lemma 3 that there exists a vector ν of $\mathbb{C}^{n^*}$ such that $\xi=V\nu$. Placing this in (41) and using (43) gives

$$EV(zI-\bar{A})\nu=0, \tag{44}$$

and since EV is injective, z has to be an eigenvalue of $\bar{A}$, of which there are at most n^*.

To the contrary, assume now the (E, F) is not C-regular. Then (43) yields, partitioning V and $\bar{A}$ as in § 2.3,

$$(zE-F)[M \quad N]=EM[zI_n-A \quad -C]. \tag{45}$$

The matrix $[zI-A \quad -C]$ has q fewer lines than columns. Therefore, it has for all z's a kernel of dimension at least q. V being injective, it gives rise to a kernel of dimension at least q for $(zE-F)$. □

This theorem is the justification for Definition 3. As a matter of fact, a *pencil of matrices* $(zE-F)$ is said to be column singular if its columns are not independent as polynomials in $\mathbb{R}^n[z]$, a characterization that coincides with Theorem 8. The degrees of the vectors of a polynomial minimal basis [6] of its kernel are called the Kronecker minimal column indices of the pencil. They are invariant under pencil similarity [7]. It also justifies the following definition.

DEFINITION 8. We call an *essential eigenvalue* of the pair (E, F) a complex number z such that

$$\operatorname{rank}(zE-F)<m-q.$$

(It is a root of an invariant factor of the pencil (E, F).)

As a corollary of Lemma 3 and Theorem 8, we have:

COROLLARY 1. *q is the column rank defect of the matrix pencil $(zE-F)$, equal to m minus the size of the largest nonidentically null determinant in this matrix.*

- *If $r<m$, $\mathcal{V}^*$ is never trivial, the system never C-regular, $q\geqq m-r$.*
- *If $r=m$, $\mathcal{V}^*$ is never trivial, the system is C-regular if and only if* $\det(zE-F)\not\equiv 0$.
- *If $r>m$, $\mathcal{V}^*$ is nontrivial if and only if the matrix $(zE-F)$ is reducible, i.e., all $m\times m$ determinants have a common root (for this value of z, the columns of $(zE-F)$ are not independent in $\mathbb{C}^m$). The system is C-regular if and only if one of the $m\times m$ determinants is not identically zero.*

Proof. According to Lemma 3, if there exists a generalized eigenvector, $\mathcal{V}^*$ is nontrivial. Conversely, if $\mathcal{V}^*$ is nontrivial, (43) shows that (E, F) has generalized eigenvalues: those of $\bar{A}$ at least. Now a generalized eigenvalue is clearly a complex number z such that the columns of $(zE-F)$ are not independent in $\mathbb{C}^m$, i.e., no $m\times m$ determinant is different from zero. And if the generalized spectrum of (E, F) is $\mathbb{C}$, all $m\times m$ determinants are null for all z's, i.e., identically zero. □

5.2. Invariants. We first recall a fact of system theory:

PROPOSITION 6. *Let (A, C) be a (noncompletely controllable) system. A complete* *set of invariants under the feedback group is given by*:

(i) *the control invariants of the controllable part*;

(ii) *the invariant factors of the uncontrollable part.*

THEOREM 9. *Given a pair* (E, F), *the corresponding system* (A, C) *is entirely characterized by*:

(i) *the control invariants of the controllable part of* (A, C), *which coincide with the Kronecker minimal column indices of the pencil* $(zE-F)$.

(ii) *the invariant factors of the uncontrollable part of* A, *which coincide with the finite invariant factors of the pencil* $(zE-F)$.

Proof. Because of Propositions 3 and 6, the elements quoted for the pair (A, C) are indeed a complete set of invariants. There only remains to relate them to the corresponding quantities of the pencil $(zE-F)$.

(i) From Kalman [9] we know that the control invariants of the pair (A_{11}, C_1) are the minimal column indices of the pencil $[zI-A_{11} \ \ -C_1]$. From (45) it follows that they are the same as the minimal column indices of the pencil $[zI-A \ \ -C]$. As a matter of fact, let

$$\nu(z)=\begin{pmatrix}\nu_1(z)\\ \nu_2(z)\\ \mu(z)\end{pmatrix}$$

be a polynomial vector in Ker $[zI-A \ \ -C]$; this is equivalent to

$$(zI-A_{11})\nu_1(z)-A_{12}\nu_2(z)-C_1\mu(z)=0, \qquad (zI-A_{22})\nu_2(z)=0.$$

However, $zI-A_{22}$ is a regular pencil, and therefore, $\nu_2(z)$ is identically null (since it is a polynomial, null for all z that are not in the spectrum of A_{22}). Thus, $[\nu_1'(z) \ \ \mu'(z)]'$ is in the kernel of $[zI-A_{11} \ \ -C_1]$. According to Lemma 3, all generalized eigenvectors, and therefore the basis vectors of Ker $(zE-F)$, can be written as

$$\xi(z)=V\nu(z).$$

Therefore, using (44) we see that to each $\xi(z)$ in Ker $(zE-F)$ corresponds a $\nu(z)$ in Ker $[zI-A \ \ -C]$ and conversely. Moreover, V being injective, $\xi(z)$ and $\nu(z)$ are of same degree.

(ii) We now show that essential eigenvalues of (E, F) are eigenvalues of A_{22}, with the rank defect of A_{22} equal to that of $(zE-F)$, minus q. Let λ be an essential eigenvalue of (E, F) with a corresponding kernel of dimension $q+k$. According to Lemma 3 and (44), $[\lambda I-A \ \ -C]$ has a kernel of dimension $q+k$ in $\mathbb{R}^{n^*}$, with $n^*=n+q$. Therefore, only $n-k$ of its lines are independent, and this is a fortiori true for $(\lambda I-A)$. Thus, λ is an eigenvalue of A, with an associated eigensubspace of dimension at least k. Now this property is independent of the particular choice of basis within $\mathcal{V}^*$, and thus, according to Proposition 3, invariant under feedback. Therefore, this eigenvalue and eigensubspace are associated to the uncontrollable part of A.

Conversely, considering the form (45) of (A, C), we have seen that polynomial vectors in Ker $[zI-A \ \ -C]$ have a zero block in the uncontrollable part of the state space. Thus, to an eigenvalue of A_{22}, with an eigensubspace of dimension k, correspond k generalized eigenvectors (that we shall choose with zero blocks in the first and third parts), independent of each other and of any vector in Ker $(zE-F)$. Therefore, this complex number is an essential eigenvalue with a column rank defect at least $q+k$.

At this stage, we know that essential eigenvalues of (E, F) are eigenvalues of A_{22} and that the number of Jordan blocks associated to it coincide. There remains to prove that they are identical in dimension. The technique is the same, using Jordan chains, and only heavier. We shall not go into too much detail. To a Jordan block of $(\lambda E-F)$ corresponds a Jordan chain $\xi_1, \xi_2, \cdots, \xi_p$ satisfying

$$\begin{aligned}(\lambda E-F)\xi_1&=0,\\(\lambda E-F)\xi_2&=E\xi_1,\\&\vdots\\(\lambda E-F)\xi_p&=E\xi_{p-1}.\end{aligned}$$

Here p is the size of the Jordan block. There remains to check that all the ξ_i's are in $\mathcal{V}^*$ and can be chosen independent of the vectors of Ker $(zE-F)$ at $z=\lambda$. Hence, there are $q+1$ independent solutions to each of the above equations, and consequently, using a linear combination with total weight one, we can find one with a zero component in $\mathcal{N}$. Consequently, there corresponds to it a Jordan chain of $\lambda I-A$. Independence modulo Ker $(zE-F)|_{z=\lambda}$ in $\mathbb{R}^m$ corresponds to independence in $\mathbb{R}^n$. Therefore, elementary divisors of $(zE-F)$ are elementary divisors of $(zI-A)$ fixed under feedback and, thus, according to Rosenbrock's feedback theorem, elementary divisors of $zI-A_{22}$. The converse proof goes exactly as above. □

The particularization of the above results to the fact that the eigenvalues of A_{22} coincide with the essential eigenvalues of (E, F) leads to the following definition and corollary.

DEFINITION 9. The system (*) or (**), satisfying (2) or (27), is called *stable* if for every bounded input function $u(\cdot)$, there exists a bounded (causal) output function $y(\cdot)$ from any initial condition.

COROLLARY 1. *The implicit system is stable if and only if the essential eigenvalues of the pair* (E, F) *are stable* (*i.e., of modulus less than one or simple and of modulus unity in case* (*) *and of negative real part or simple imaginary in case* (**)).

Proof. If the condition of the corollary is met, the equivalent system is stabilizable with v with a linear feedback (or, equivalently, can be chosen stable). Therefore, there exists bounded solutions $x(\cdot)$ from any initial condition, with a choice of a bounded function $v(\cdot)$ (zero if the system is chosen stable). Therefore, $y(\cdot)$ as given by (15) or (49) remains bounded for these solutions.

To the contrary, if the condition is not met, there is a mode, uncontrollable with v, which is unstable. Therefore, except for a strict subspace of initial conditions, the solution $x(\cdot)$ will diverge for all choices of $v(\cdot)$. And since the matrix M is injective and has a range $\mathcal{M}$ in direct sum with the range $\mathcal{N}$ of N, $y(\cdot)$ as given by (15), or (49) recalling that $u(\cdot)$ is assumed bounded, will diverge as well for all (causal) solutions. □

Remark 5. It is impossible to request, for singular systems, that all solutions be bounded in view of Theorem 3.

Remark 6. One may, of course, define in the same way asymptotically stable implicit systems.

Finally, one can clearly define the feedback group for systems (*) or (**) exactly in the same way as for an ordinary system. It clearly preserves existence of a strictly causal solution.

DEFINITION 10. The implicit system is *minimal* if the minimal subspace $\mathcal{W}_*$ coincides with the characteristic subspace $\mathcal{V}^*$. Then (15) (16) is completely controllable. We have:

THEOREM 10. *Under condition* (2), *if the implicit system is minimal, a complete set of invariants under the feedback group is provided by the Kronecker minimal indices* *of the matrix pencil* $[zE-F \quad -G]$, *and they coincide with the control invariants of the system* $(A, [B \quad C])$.

Proof. The proof is in two steps. First check that the feedback group on the implicit system, combined with the nonunicity pointed out in Proposition 3, translates exactly in the classical feedback group for $(A, [B \quad C])$ and that, conversely, the latter generates the former. This in an easy consequence of the fact that V is injective. We leave it to the reader to check. Then using the fact that $(A, [B \quad C])$ is by hypothesis completely controllable and Kalman's theorem, we have that its control invariants are a complete set of invariants for the implicit system.

The second step is to identify the control invariants of $(A, [B \quad C])$, i.e., according to Kalman [9], the column indices of $[zI-A \quad -B \quad -C]$ with the column indices of $[zE-F \quad -G]$. This is done in the same fashion as in Theorem 9 (i), using the second claim of Lemma 3 and

$$[(zE-F)[M \quad N] \quad -G] = EM[zI-A \quad -C \quad -B]. \qquad \square$$

5.3. Transfer functions.

THEOREM 11. *There exists a (strictly) causal solution to the system* $(*)$ *or* $(**)$ *if and only if there exists a (strictly) proper rational matrix* $K(z)$ *such that*

$$(zE-F)K(z) = G. \tag{46}$$

Let also $L(z)$ *be a proper (not strictly) rational matrix of maximum rank, such that*

$$(zE-F)L(z) = 0. \tag{47}$$

Then all solutions of the implicit system are given by

$$Y(z) = K(z)U(z) + L(z)V(z),$$

where $Y(z)$ *and* $U(z)$ *are the* z*-transforms of* $y(\cdot)$ *and* $u(\cdot)$, *respectively, and* $V(z)$ *is an aribitrary power series of* z^{-1} *of appropriate dimension.*

Proof. Notice first that there exist complex (column) vectors $l_i(z)$ satisfying (47) if and only if the pair (E, F) is not C-regular. It is easy to see (see [7]) that they can be chosen polynomial or, dividing each such column by the highest power of z present in it (since (47) is homogeneous), rational proper. If these degrees are chosen as small as possible, they are the column minimal indices or Kronecker indices of the pencil.

(i) *Necessity.* We know that, if a strictly causal solution exists, it is represented by (14), (15) or in the nonstrictly causal case by the following set (that coincides with the former if we set $D=0$):

$$x(t+1) = Ax(t) + Bu(t) + Cv(t), \tag{48}$$

$$y(t) = Mx(t) + Du(t) + Nv(t) \tag{49}$$

with the definitions of the matrices A, B, C, D, M and N as

$$F[M \quad N] = [EMA \quad EMC], \tag{50}$$

$$G + FD = EMB, \tag{51}$$

$$ED = 0, \qquad EN = 0. \tag{52}$$

Hence, the formula of the theorem for $Y(z)$ with

$$K(z) = D + M(zI-A)^{-1}B, \tag{53}$$

$$L(z) = N + M(zI-A)^{-1}C. \tag{54}$$

We can calculate

$$(zE-F)K(z)=(zE-F)M(zI-A)^{-1}B-FD. \tag{55}$$

Now (43) still holds with $V=[MN]$. Taking the first blocks in both sides, it comes

$$(zE-F)M(zI-A)^{-1}=EM, \tag{56}$$

and therefore, (55) with (51) yield (46). Similarly, we have with the second block in (43) (or with 50)

$$(zE-F)N=EMC$$

and this together with (56) yields (47).

(ii) *Sufficiency*. Assume the two proper rational matrices $K(z)$ and $L(z)$ exist, satisfying (46) and (47). Consider the rational matrix

$$H(z)=[K(z) \quad L(z)]. \tag{57}$$

It can be realized according to standard realization theory, and we partition the last matrix according to the partition of H. There exist therefore matrices A, B, C, D, M and N such that

$$H(z)=[D \quad N]+M(zI-A)^{-1}[B \quad C], \tag{58}$$

and we may choose M, A, B and C such that the system $(M, A, [B \ \ C])$ be minimal (i.e., completely controllable and observable). Take equality (46), which holds by hypothesis:

$$(zE-F)(D+M(zI-A)^{-1}B)=G.$$

Expand $(zI-A)^{-1}$ in a series in z^{-1} and equate like powers on both sides. It becomes

$$\begin{aligned}
&\text{power } 1: && ED=0,\\
&\text{power } 0: && EMB-FD=G,\\
&\text{power } -k: && (EMA-FM)A^{k-1}B=0, \qquad k=1,\cdots.
\end{aligned}$$

We do the same with (47). It becomes

$$\begin{aligned}
&\text{power } 1: && EN=0,\\
&\text{power } 0: && EMC-FN=0,\\
&\text{power } -k: && (EMA-FM)A^{k-1}C=0, \qquad k=1,\cdots.
\end{aligned}$$

The "power 1" relations yield (52), "power 0" (51) and the second block of (50). The two "power-k" together can be written

$$(EMA-FM)[[B \quad C] \quad A[B \quad C] \quad \cdots \quad A^{n-1}[B \quad C]]=0.$$

Since $(A, [B \ \ C])$ is taken completely controllable, the right matrix in this equality is surjective, and therefore, we get the first part of (50). Straightforward calculation shows that the solutions (48) (49), subject to (50) (51) (52), satisfy (*) and similarly for the continuous case.

The strictly causal case is a specialization of this one with $D=0$. □

Notice also that the theorem yields

$$(zE-F)Y(z)=GU(z), \tag{59}$$

which is the direct z transform of $(*)$ or Laplace transform of $(**)$.

The rational matrix $H(z)$ of (57) can be considered as the generalized transfer function of the implicit (or generalized) system.

5.4. Canonical form. A change of coordinates on y amounts to a right multiplication by an invertible $m\times m$ matrix Q of both E and F. (In case one is interested in an output $Hy(t)$, H should be multiplied to the right by Q also.) The system is not changed either if we replace some or all of the r equations $(*)$ or $(**)$ by independent linear combinations of them, i.e., if we multiply to the left E, F and G by an invertible $r\times r$ matrix P.

Therefore, two implicit systems (H, E, F, G) and (H_1, E_1, F_1, G_1), where H is an output matrix, will be said to be strictly equivalent if there exist two invertible matrices P and Q of appropriate dimension such that

$$\begin{aligned} &H_1 = HQ,\\ &E_1 = PEQ, \qquad F_1 = PFQ,\\ &G_1 = PG. \end{aligned} \tag{60}$$

Relations (60) *are precisely the definition of equivalence of the pencils* $(zE-F)$ and (zE_1-F_1). We know, therefore, that by a proper choice of matrices P and Q, $(zE-F)$ can be brought into the canonical form described; e.g., in [7].

Let $\alpha_1(z)$ be a polynomial vector of minimum degree, say ε_1, such that

$$(zE-F)\alpha_1(z)=0.$$

Let then $\alpha_2(z)$ be a polynomial independent of $\alpha_1(z)$ satisfying the same equality, and so on. The numbers $\varepsilon_1, \cdots, \varepsilon_q$ are the column minimal indices. Performing similarly for E' and F', we get the line minimal indices, say, $\eta_1, \cdots, \eta_l$. The canonical form of $(zE-F)$ is block diagonal, made of four types of blocks.

(i) *Blocks* L_{ε_i}: To each ε_i, corresponds a block $\varepsilon_i \times \varepsilon_i+1$ of the form

$$L_{\varepsilon_i} = \begin{pmatrix} z & -1 & 0 & \cdots & 0 & 0\\ 0 & z & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & z & -1 \end{pmatrix}.$$

In this basis we obviously have

$$\alpha_i(z) = \begin{pmatrix} 1\\ z\\ \vdots\\ z^{\varepsilon i} \end{pmatrix}.$$

We make correspond to it the column $l_i(z)$ of $L(z)$:

$$l_i(z) = \begin{pmatrix} z^{-\varepsilon_i}\\ z^{-\varepsilon_i+1}\\ \vdots\\ 1 \end{pmatrix}.$$

This makes up the matrix $L(z)$ of (47).

Writing equations $(*)$ with this special form for E and F, we immediately see that each such block involves ε_i+1 coordinates of y. They always have a solution whatever the coefficients of G in the same lines, and the last coordinate of this subvector of y is free. It corresponds to a coordinate in $\mathcal{N}$, the ε_i first corresponding to coordinates in $\mathcal{V}_*$.

As a matter of fact, L_{ε_i} has the rational strictly proper right inverse

$$L_{\varepsilon_i}^r = \begin{pmatrix} z^{-1} & z^{-2} & \cdots & z^{-\varepsilon_i} \\ 0 & z^{-1} & \cdots & z^{-\varepsilon_i+1} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & z^{-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

so that whatever the corresponding lines of G, (46) will have a strictly proper solution for this block, obtained by multiplying these lines of G par $L_{\varepsilon_i}^r$ to the left.

(ii) *Blocks* L_{η_i}. To the row indices η_j correspond blocks L_{η_j} of type $\eta_j+1\times\eta_j$ having the form of the transpose of a block L_ε.

Writing equations (*) with this block, we see that it involves η_j coordinates of y but that the last line amounts to a recurrence relation between the elements of the sequence $u(\cdot)$. It can be satisfied for all sequences only if the corresponding lines of G are all zero, but then all these coordinates must be and remain zero. They correspond to coordinates in a complement of $\mathcal{V}^*$ in $\mathbb{R}^m$, and the requirement on G is (part of) condition (2).

Correspondingly, it is a simple task to see, thanks to the triangular form of L_{η_j}, that (46) can be satisfied with a strictly proper block in $K(z)$ if and only if the corresponding lines of G are null, the solution being then zero.

(iii) *Blocks* L_{μ_k}. These are square blocks of type $\mu_k\times\mu_k$ corresponding to the infinite invariant factors of the pencil $(zE-F)$. They are of the form

$$L_{\mu_k} = \begin{pmatrix} -1 & z & 0 & \cdots \\ 0 & -1 & z & \\ \vdots & & \ddots & \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Again, writing the equations (*) for this block, we see that they involve μ_k coordinates of y, but depending in an anticausal way on the sequence $u(\cdot)$. Therefore, these coordinates also correspond to a complement of $\mathcal{V}^*$ in $\mathbb{R}^m$, and the corresponding rows of G must be zero for a strictly causal solution to exist.

However, the dependence of y on $u(\cdot)$ is anticausal but not strictly. Therefore, a causal but not strictly causal solution may exist where the first coordinate of the corresponding subvector of y is nonzero but all others zero. The same row in G may be nonzero. This corresponds to the fact that E has a column of zeros in the first column of L_{μ_k}, and the corresponding coordinate of y is therefore in Ker E but not in $\mathcal{N}$. We recover conditions (28) and (27).

A complete information is given again looking at (46). As a matter of fact, L_{μ_k} is invertible:

$$L_{\mu_k}^{-1} = \begin{pmatrix} -1 & -z & \cdots & -z^{\mu_k-1} \\ 0 & -1 & \cdots & -z^{\mu_k-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & -1 \end{pmatrix}$$

so that the only solution of (46) for this block is $L_{\mu_k}^{-1}G_\mu$, which is anticipatory of μ_k-1

steps unless G_μ has some zero rows. If its first row is the only nonzero one, then the corresponding block of $K(z)$ is proper but not strictly.

(iv) *Blocks* L_{λ_l}. These are square blocks that together constitute a characteristic matrix

$$zI - A_\lambda,$$

where A_λ is in Jordan form, for example. This clearly corresponds to coordinates of y for which there is a unique strictly causal solution. They are therefore in $\mathcal{V}^*$ but in a complement of $\mathcal{V}_*$. The corresponding block of $K(z)$ is $(zI - A_\lambda)^{-1}$. The corresponding eigenvalues are the essential eigenvalues of the pair (E, F).

Remark 7. This kind of link between geometrical concepts and the system pencil was shown for standard systems in Jaffe and Karcanias [8]. Using their characterization our space $\mathcal{V}^*$ appears as a generalization of (A, C) invariant subspaces since it is characterized by the fact that $(zE - F)V$ has only column minimal indices and finite invariant factors. This also clearly shows how to investigate the impulsive behavior of our system (**), or noncausal behavior of (*), by looking at the infinite invariant factors and the associated subspaces.

6. Conclusion. We have a simple theory of singular implicit systems whether they are square, or over- or underdetermined. It should be noted that overdetermination may go along with nonunicity of the solution in a nontrivial way.

The recurrences defining the various subspaces $\mathcal{V}^*$, $\mathcal{W}^*$, $\mathcal{W}_*$, $\mathcal{V}_*$, provide the basis for finite algorithms, unfortunately rather ill-behaved in terms of robustness in their native form. They involve finding zero determinants and computing right or left inverses, numerically difficult operations. Standard techniques could be applied to improve them (like computing the rank of AA^*, or A^*A, instead of A).

The stage seems to be set to extend a significant part of Rosenbrock's theory to these systems and of its modern developments, in the spirit of Wolovich or Fuhrman. Also, the study of impulsive (or noncausal) behavior seems to be straightforward, using the literature on that topic.

A domain of interest is naturally the use of tools of two-player control systems theory to study the property of implicity systems: making an output sequence unique (decoupling $v(\cdot)$ through feedback), ensuring that all trajectories meet a given subspace at a given instant (capturing the state), or that some do (controllability through v), insuring that all trajectories will do better than a given amount with respect to some criterion (dynamical games), etc.

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