# Exact Controllability of Perturbed Continuous-Time Linear Systems

P. BERNHARD

***Abstract*—We derive a necessary and a sufficient condition for the exact controllability of a linear system to a linear subspace in the framework of square integrable controls and causal information structure. We also give some results in the framework of absolutely integrable controls, relating the latter to the former.**

## Introduction

The aim of this paper is to investigate the output controllability in fixed time of a continuous-time finite-dimensional linear system, in the presence of unpredictable perturbations. Since the perturbation can as well be thought of as an opponent, this work belongs also to the theory of (qualitative) differential games and therefore bears relationships we shall comment upon with [6]. In contrast to previous works on related topics [9], [10], but in keeping with classical system theory, we allow unbounded controls and perturbations, and seek conditions on the algebraic data (the matrices) of the system. We furthermore have to specify admissible classes of control functions as a means of prescribing how the controls are allowed to increase in the vicinity of final time. (As usual when dealing with nonstationary systems, the problem really makes sense over a prescribed time interval $[t_0, t_1]$, but if $t_1$ were free, one could look for, say, the smallest such instant such that the sufficient conditions be met.) Again keeping with traditional control theory, and in contrast to [4], [5], we use classical causal information to control the system. We shall nevertheless provide some comparison with [5].

In [1] we studied the same question for discrete-time systems. In addition to strong controllability (noncausal information) for which we pointed out that the criterion is the same for discrete-time and continuous-time systems, we had to distinguish between capturability, where the controller plays "after" nature, i.e., knows the perturbation at current time, and ideal capturability where he does not. We shall see that while the capturing strategies we propose do not make use of such information, and the class of strategies we allow is more of the ideal capturability type, the criteria we get resemble those of capturability for discrete-time systems.

In Section I we state the problem and some preliminaries. In Section II we deal with $L^2$ capturability and in Section III we deal with $L^1$ capturability. In Section IV we give some counterexamples and discuss open questions.

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The author is with CEREMADE, University of Paris 9, Paris, France, and the Centre d'Automatique-Informatique, Ecole Nationale Superieure des Mines de Paris, Fontainebleau, France.

## I. Preliminaries

### A. The Problem

We consider a linear system whose state $x$ ranges over $\mathbb{R}^n$:

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) + E(t)v(t) \tag{1}$$

$$y(t_1) = Hx(t_1). \tag{2}$$

Initial and final time, $t_0$ and $t_1$, are given. The control $u$ ranges over $\mathbb{R}^p$ and the perturbation $v$ over $\mathbb{R}^q$. The output $y$, considered only at final time, ranges over $R^l$. The admissible control functions and perturbations are specified by

$$u(\cdot) \in \Omega_u, \quad v(\cdot) \in \Omega_v. \tag{3}$$

The matrix functions $F(\cdot)$, $G(\cdot)$, $E(\cdot)$ are of appropriate type, piecewise continuous and bounded, chosen right continuous everywhere and left continuous at $t_1$. The matrix $H$ has full rank (i.e., $l$).

The controller wants to control $y(t_1)$, which, by the superposition principle, he can do if the unperturbed system is output controllable, and if he can impose, in the perturbed system the *capture condition*,

$$x(t_1) \in \mathcal{M} = \ker H. \tag{4}$$

We pose $\dim \mathcal{M} = n - l = m$.

In forming his control, the controller is allowed to make use of past information on the perturbation (which he can get through proper processing of the measurement of the state, for instance). Precisely, this means that the control function will be given by a strategy $\varphi$: $\mathbb{R}^n \times \mathbb{R} \times \Omega_v \to \Omega_u$,

$$u(t) = \varphi(x_0, t_0; v(\cdot))(t). \tag{5}$$

The set $\Phi$ of admissible strategies is that of all such functions that satisfy the following conditions.

1) $\varphi$ is causal, i.e., if $v_1(\cdot), v_2(\cdot) \in \Omega_v$ and $v_1(t) = v_2(t)$ almost all $t \leqslant \tau$, then $\varphi(x_0, t_0; v_1(\cdot))(\tau) = \varphi(x_0, t_0; v_2(\cdot))(\tau)$.

2) $\varphi$ is compatible (with the system), i.e., for all admissible perturbations $v(\cdot)$, the differential equation

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$$\dot{x}(t)=F(t)x(t)+G(t)\varphi(x_0,t_0;v(\cdot))(t)+E(t)v(t) \tag{6}$$

has at least one solution.

The sufficient conditions here after will dwell on exhibiting a capturing strategy which is a linear pure state feedback, and thus satisfy these conditions. The necessary condition, however, will hold for a class of strategies slightly less general than that stated here, which we consider a weakness of the present theory. We shall require admissible strategies to satisfy.

3) $\varphi$ is a compatible pure state feedback or is feedback-compatible, i.e., for every compatible pure state feedback $v(t)=\psi(x(t),t)$, (that satisfy the equivalent of condition 2) above, but with the roles of $u$ and $v$ reversed), the differential equation

$$\dot{x}(t)=F(t)x(t)+G(t)\varphi(x_0,t_0;\psi(x(\cdot),\cdot))(t)+E(t)\psi(x(t),t) \tag{7}$$

has at least one solution. [We can notice that this condition is in fact somewhat stronger than necessary. We only need it to hold for the very particular class of feedbacks $\psi$ specified by (19)].

We now state the definitions that specify the object of investigation.

*Definition 1:* The initial state $x_0$ [respectively the initial phase $(x_0,t_0)$] is *capturable* modulo $\mathfrak{M}$ over $[t_0,t_1]$ (respectively at $t_1$) if there exists a strategy $\varphi$ in $\Phi$ ensuring capture for any admissible perturbation $v(\cdot)$ and for all trajectories generated by $(\varphi,v(\cdot))$ form $(x_0,t_0)$.

*Remark 1:* If the origin is capturable modulo $\mathfrak{M}$ over $[t_0,t_1]$, then so is any initial state which is $u$-controllable (i.e., controllable in the perturbation free system) modulo $\mathfrak{M}$ over this same interval. As a matter of fact, let $\varphi^0(v(\cdot))$ be the capturing strategy for the origin and $\hat{u}(x_0)(\cdot)$ an open-loop control $u$-controlling $x_0$ to $\mathfrak{M}$; then the strategy $\varphi(x_0;v(\cdot))=\varphi^0(v(\cdot))+\hat{u}(x_0)$ captures $x_0$ to $\mathfrak{M}$, as direct calculation shows. Furthermore, an initial state which is not $u$-controllable modulo $\mathfrak{M}$ is *a fortiori* not capturable. Hence, we have the following definition.

*Definition 2:* The system $(H,F(\cdot),G(\cdot),E(\cdot))$ is *capturable* if every initial phase which is $u$-controllable modulo $\mathfrak{M}$ at $t_1$ is also capturable modulo $\mathfrak{M}$ at $t_1$. The system is completely capturable if it is completely $u$-output controllable and capturable.

*Remark 2:* When both $\Omega_u$ and $\Omega_v$ are $L^2$ (respectively $L^1$) (although they are different spaces since the image spaces are not the same), we shall write $L^2$ capturability (respectively $L^1$ capturability).

## II. $L^2$ Capturability

### A. Sufficient Condition

The condition we shall give makes use of the controllability matrices

$$C(t)=H\Big(\int_t^{t_1}\Phi(t_1,s)G(s)G'(s)\Phi'(t_1,s)\,ds\Big)H' \tag{8a}$$

$$D(t)=H\Big(\int_t^{t_1}\Phi(t_1,s)E(s)E'(s)\Phi'(t_1,s)\,ds\Big)H' \tag{8b}$$

where $\Phi(\cdot,\cdot)$ is the transition matrix associated with $F(\cdot)$. (Here and in the rest of the text, the accent means "transposed.")

For $\epsilon$ a given (positive) number, we use the notation

$$X_\epsilon(t)=C(t)-\epsilon D(t). \tag{9}$$

The following two conditions are equivalent:

$$\exists\epsilon>0:\quad \forall t\in(t_0,t_1),\quad X_\epsilon(t)\geqslant 0 \tag{10a}$$

$$\exists\epsilon>0:\quad \forall\xi\in\mathbb{R}^l,\quad \forall t\in(t_0,t_1),\quad \xi'X_\epsilon(t)\xi\geqslant 0. \tag{10b}$$

We prove the following result.

*Theorem 1:* Under the condition (10) [where $X_\epsilon$ is defined by (8) and (9)], the system $(H,F(\cdot),G(\cdot),E(\cdot))$ is $L^2$-capturable modulo $\mathfrak{M}$ at $t_1$. There exists a pure state feedback capturing strategy $u(t)=\varphi_\epsilon(x(t),t)$, which, for a differentially $u$-output controllable system, can be chosen as

$$\varphi_\epsilon(x,t)=-G'(t)\Phi'(t_1,t)H'X_\epsilon^{-1}(t)H\Phi(t_1,t)x. \tag{11}$$

*Proof:* To simplify the notations, to the phase $(x,t)$ we associate the $(l+1)$-dimensional phase $(\tilde{x},t)$ defined by

$$\tilde{x}=H\Phi(t_1,t)x. \tag{12}$$

It is a classical fact that $\tilde{x}$ is governed by

$$\dot{\tilde{x}}(t)=\tilde{G}(t)u(t)+\tilde{E}(t)v(t) \tag{13}$$

where

$$\tilde{G}(t)=H\Phi(t_1,t)G(t),\quad \tilde{E}(t)=H\Phi(t_1,t)E(t). \tag{14}$$

Moreover, capture is defined by

$$\tilde{x}(t_1)=0. \tag{15}$$

Since, as we have seen, only $u$-controllable modulo $\mathfrak{M}$ phases matter, we can reduce the state space of (13) to $R(C(t_0))$, the range space of $C(t_0)$, so that then, $C(t_0)>0$. We also know that $R(C(t))$ is a piecewise constant subspace, decreasing at isolated instants of time $\tau_i$. Therefore, $C(t)$ is positive definite over $[t_0,\tau_1)$. Now, in order to capture the state, it is necessary to ensure that

$$\tilde{x}(\tau_1)\in\mathfrak{M}_1=R(C(\tau_1)).$$

We therefore have to solve a capturability problem over $[t_0,\tau_1]$ for $\tilde{x}$, the relevant $H_1$ matrix being, for instance, the orthogonal projection from $R^l$ onto the orthogonal complement of $\mathfrak{M}_1$, suitably endowed with a basis. The corresponding $X_\epsilon$ matrix is now

$$\tilde{X}_\epsilon(t)=H_1X_\epsilon(t)H_1'=H_1C(t)H_1'-\epsilon H_1D(t)H_1'.$$

Since it is positive semidefinite, with $C(t)$ positive definite, $\epsilon$ can be chosen small enough to make it positive definite. It is null at $t=\tau_1$.

From $\tau_1$ on, the state space can be further reduced to $\mathfrak{M}_1$, and the same analysis applies to each time interval where $R(C(t))$ is constant, down to the last one $[t_*,t_1]$. We have therefore brought the capturability problem to a sequence of capturability problems for differentially $u$-output controllable systems, with the $X_\epsilon$ matrix positive. We continue the analysis for the last one. It clearly holds for all.

Consider the following time derivative, taken along a trajectory generated by any pair of controls $u$ and $v$:

$$\frac{d}{dt}(\tilde{x}(t)X_\epsilon^{-1}\tilde{x}(t))=(u'\tilde{G}'+v'\tilde{E}')X_\epsilon^{-1}\tilde{x}+\tilde{x}'X_\epsilon^{-1}(\tilde{G}u+\tilde{E}v)+\tilde{x}'X_\epsilon^{-1}(\tilde{G}\tilde{G}'-\epsilon\tilde{E}\tilde{E}')X_\epsilon^{-1}\tilde{x}.$$

Through a classical "completion of the squares," this may be written as

$$\frac{d}{dt}(\tilde{x}'X_\epsilon^{-1}\tilde{x})=\|u+G'X_\epsilon^{-1}\tilde{x}\|^2-\|u\|^2-\frac{1}{\epsilon}\|v-\epsilon E'X_\epsilon^{-1}\tilde{x}\|^2+\frac{1}{\epsilon}\|v\|^2.$$

Integrating this equality from $t_*$ to $t$, we get (with $x(t_*)=x_*$)

$$\int_{t_*}^t\Big(\|u(s)\|^2-\frac{1}{\epsilon}\|v(s)\|^2\Big)ds=\tilde{x}_*X_\epsilon^{-1}(t_*)\tilde{x}_*-\tilde{x}'(t)X_\epsilon^{-1}(t)\tilde{x}(t)+\int_{t_*}^t\|u(s)+\tilde{G}'(s)X_\epsilon^{-1}(s)\tilde{x}(s)\|^2ds-\frac{1}{\epsilon}\int_{t_*}^t\|v(s)-\epsilon\tilde{E}'(s)X_\epsilon^{-1}(s)\tilde{x}(s)\|^2ds.$$

Assume the controller uses the strategy (11), which was chosen such as to make the first integrand on the right-hand side (RHS) vanish. We find then

$$\int_{t_*}^t\|u(s)\|^2ds=\frac{1}{\epsilon}\int_{t_*}^t\|v(s)\|^2ds+\tilde{x}_*'X_\epsilon^{-1}(t_*)\tilde{x}_*-\tilde{x}'(t)X_\epsilon^{-1}(t)\tilde{x}(t)-\frac{1}{\epsilon}\int_{t_*}^t\|v(s)-\epsilon\tilde{E}'(s)X_\epsilon^{-1}(s)\tilde{x}(s)\|^2ds. \tag{16}$$

From this relation, we derive first that, because of (10),

$$\int_{t_*}^{t}\|u(s)\|^2 ds \le \frac{1}{\epsilon}\int_{t_*}^{t}\|v(s)\|^2 ds + \tilde{x}_*' X^{-1}(t_*)\tilde{x}_*.$$

Since $v(\cdot)$ is by hypothesis square integrable over $[t_*, t_1]$, so is $u(\cdot)$ generated by this feedback, which is thus an admissible strategy. Further, since the left-hand side (LHS) of (16) is positive, and the first two terms of its RHS are positive and bounded, we conclude that each of the last two terms, which are both negative, is bounded as $t$ goes to $t_1$. In particular, there exists a positive number $a$ such that

$$\tilde{x}'(t)X_\epsilon^{-1}(t)\tilde{x}(t) = \|X_\epsilon^{-1/2}(t)\tilde{x}(t)\|^2 \le a^2$$

and thus

$$\|\tilde{x}(t)\| = \|X_\epsilon^{1/2}(t)X_\epsilon^{-1/2}(t)\tilde{x}(t)\|$$
$$\le \|X_\epsilon^{1/2}(t)\|\,\|X_\epsilon^{-1/2}(t)\tilde{x}(t)\| \le \|X_\epsilon^{1/2}(t)\|a.$$

But $X_\epsilon(t)$, and thus its square root, goes to zero as $t$ goes to $t_1$, and thus the result follows, in view of the characterization (15) of capture.

*Remark 3:* It can be shown [2] that if the system is not differentially $u$-output controllable, one get a capturing strategy, obtained by suitably concatenating the above construction, by replacing the central term $H'X_\epsilon^{-1}H$ in (11) by

$$H^\dagger H\left[(I-H^\dagger H)+\int_t^{t_1}\Phi(t_1,s)(G(s)G'(s)-\epsilon E(s)E'(s))\Phi'(t_1,s)\,ds H^\dagger H\right]^\dagger.$$

*Remark 4:* It is very easy to check that the control

$$u(t) = -G'\Phi'(t_1,t)H'X_\epsilon^{-1}(t)H(\Phi(t_1,t)x(t)-x_1)$$

for $x_1 \in R^n$ fixed, ensures capture to the affine set $\mathfrak{M}+x_1$.

Now, under condition (10), the system (1) can be driven by $u$ to $\mathfrak{M}$ "under worst perturbation." However, it is very possible (for instance in the case $E=G$) that there exist a strategy $\psi_\eta$ for $v$ that drives the system to $\mathfrak{M}+x_1$, $x_1\neq 0$, "under worst controls." The paradox is only apparent. This only proves that under the pair of strategies $(\varphi_\epsilon, \psi_\eta)$, none of the two controls generated (if they exist) is admissible.

*Remark 5:* Theorem 1 bears a strong relationship to the early paper by Ho *et al.* [6]. However, two important differences are that on the one hand, instead of their heuristic limit approach, we have a precise statement and a rigorous proof, and on the other hand, by varying $\epsilon$, which stands for their $S^{-1}$, we do not have to impose *a priori* constraints on the players available energy, nor do we have to worry about conjugate points.

### B. Necessary Condition

We prove the following result.

*Theorem 2:* A necessary condition for the system $(H,F,G,E)$ to be $L^2$ capturable at $t_1$ with feedback compatible strategies, or with compatible pure state feedbacks, is that the following condition be fulfilled:

$$\forall \xi \neq 0 \in \mathbb{R}^l,\ \exists \epsilon>0:\ \forall t\in(t_0,t_1),\ \xi' X_\epsilon(t)\xi \ge 0. \qquad (17)$$

*Proof:* The proof rests on two lemmas, the first of which seems rather fundamental (although it is there that the restriction to feedback compatible strategies rests).

*Lemma 1:* If the system is $L^2$-capturable mod $\mathfrak{M}$ at $t_1$, for every square integrable perturbation $v(\cdot)$, there exists a square integrable control $u(\cdot)$ such that [see notations (14)]

$$\forall \xi\in\mathbb{R}^l,\ \forall t\in(t_0,t_1),\ \xi'\int_t^{t_1}\tilde{G}(s)u(s)\,ds \ge \int_t^{t_1}|\xi'\tilde{E}(s)v(s)|\,ds. \qquad (18)$$

*Proof of Lemma 1:* Let $v(\cdot)$ be fixed, and $\varphi$ be a capturing strategy for some initial phase $(x_0,t_0)$. Consider the feedback for $v$ given by

$$\psi(x,t) = v(t)\operatorname{sgn}(\xi'\tilde{x}(t))\operatorname{sgn}(\xi'\tilde{E}(t)v(t)). \qquad (19)$$

This feedback is compatible. Moreover, we have the following proposition.

*Proposition:* A compatible pure state feedback $\varphi$ is also $\psi$-compatible.

*Proof of the Proposition:* Check that at a point where $\xi'\tilde{x}(t)\neq 0$, (19) locally behaves as an open-loop control. At a point where $\xi'\tilde{x}(t)=0$, if $(\varphi,\psi)$ does not generate locally, a trajectory $(\varphi,0)$ does not either, since we would have, for any sequence $x_n\to x$, with $\xi'\tilde{x}_n$ of constant sign,

$$\lim_{n\to\infty}[\xi' G(t_n)\varphi(\tilde{x}_n,t_n)+|\xi' E(t_n)v(t_n)|]<0 \qquad \text{if } \xi'\tilde{x}_n>0$$
$$\lim_{n\to\infty}[\xi' G(t_n)\varphi(\tilde{x}_n,t_n)-|\xi' E(t_n)v(t_n)|]>0 \qquad \text{if } \xi'\tilde{x}_n<0$$

and this would be *a fortiori* true with $v=0$.

Therefore, the pair $(\varphi,\psi)$ generates trajectories over $(t_0,t_1)$. Moreover, the controls $\hat{v}(\cdot)$ generated are, by construction, square integrable, since $\|\hat{v}\|^2=\|v\|^2$. Thus, these trajectories, which are also those generated by $(\varphi,\hat{v})$, induce capture. We have, using the system (12)–(15),

$$(\xi'\tilde{x})^{\cdot} = \xi'\tilde{G}\varphi + |\xi' Ev|\operatorname{sgn}(\xi'\tilde{x})$$

and thus, multiplying by $\operatorname{sgn}(\xi'\tilde{x})$

$$|\xi'\tilde{x}|^{\cdot} = \xi'\tilde{G}\varphi\operatorname{sgn}(\xi'\tilde{x}) + |\xi' Ev|.$$

Integrate between $t$ and $t_1$ taking into account that $x(t_1)=0$, and posing, on the trajectory chosen, $\varphi\operatorname{sgn}(\xi'\tilde{x})=-u$. It becomes

$$-|\xi' x(t)| = -\int_t^{t_1}\xi'\tilde{G}u\,ds + \int_t^{t_1}|\xi'\tilde{E}v|\,ds \le 0$$

and hence (18).

We now establish the second lemma, which is technical and whose proof will only be sketched.

*Lemma 2:* Let $\{a_k\}$ and $\{d_k\}$ be two sequences of real positive numbers, such that the series $\Sigma a_k$ and $\Sigma d_k$ converge. There exists a sequence of positive numbers $\{b_k\}$ such that

$$\sum_{k=0}^{\infty} b_k^2 d_k \text{ converges}$$

and for an infinite sequence of integers $m_i$,

$$\forall i,\quad \sum_{k=m_i}^{\infty} b_k d_k \ge \left(a_{m_i}\sum_{k=m_i}^{\infty} d_k\right)^{1/2}.$$

*Proof:* Consider the sequence

$$c_k = a_k^{1/2}\left(\sum_{i=k}^{\infty} d_i\right)^{-1/2}.$$

If it is bounded, pick $b_k=b=\sup c_k$, and take $m_i=i$. If not,

$$b_k = \sup_{i\le k} c_i$$

and let $m_i$ be such that $b_{m_i}=c_{m_i}$. It readily follows that

$$\sum_{k=m_i}^{m_{i+1}-1} b_k^2 d_k = c_{m_i}^2 \sum_{k=m_i}^{m_{i+1}-1} d_k \le c_{m_i}^2 \sum_{k=m_i}^{\infty} d_k = a_{m_i},$$

hence the first property, and $b_{m_i}$, being a nondecreasing sequence, that

$$\sum_{k=m_i}^{\infty} b_k d_k \ge b_{m_i}\sum_{k=m_i}^{\infty} d_k = \left(a_{m_i}\sum_{k=m_i}^{\infty} d_k\right)^{1/2}.$$

This proves the lemma.

Now, assume that condition (17) of the theorem is not met, but that the system is $L^2$ capturable. Let $\xi$ be a vector for which (17) does not hold. Choose a sequence of positive numbers $\epsilon_k$ such that $\Sigma\epsilon_k$ converges. There exists a sequence $\{t_k\}$ such that

$$\forall k,\quad \xi' C(t_k)\xi < \epsilon_k \xi' D(t_k)\xi. \qquad (20)$$

Furthermore, $D(t_k)$ is bounded; therefore, the RHS goes to zero as $k$

goes to infinity. Two cases may arise. First, for some $k$,

$$\xi' C(t_k)\xi = 0 < \xi' D(t_k)\xi.$$

This implies that

$$\forall t > t_k, \quad \xi' \tilde{G}(t) = 0,$$

while this is not true for $\tilde{E}(t)$. Placing this in (13), we see that $v$ could ensure that $\xi'\tilde{x}(t_1) \neq 0$, and therefore $\tilde{x}(t_1) \neq 0$. [This says that a necessary condition, but not sufficient, is that $\mathcal{R}(D(t)) \subset \mathcal{R}(C(t))$, which can easily be shown to be equivalent to $\mathcal{R}(\tilde{E}(t)) \subset \mathcal{R}(C(t))$.]

Or second, $\xi' C(t_k)\xi \neq 0$, but, by continuity, any accumulation point $\tau$ of the sequence $t_k$ satisfies $\xi' C(\tau)\xi = 0$. Furthermore, since $\mathcal{R}(C(t))$ is continuous from the right, we can assume that we have extracted an increasing subsequence $t_k$. Finally, as we have already seen that all time instants where $\mathcal{R}(C(t))$ decreases play the same role, we shall, for simplicity, assume that $t_k \to t_1$. (This is the only possible case if the system is differential $u$-output controllable at $t_1$.) Let $a_k = n\epsilon_k$. Let $e(t)$ be the value of the entry of maximum absolute value of $\xi'\tilde{E}(t)$. One has

$$\xi' D(t)\xi = \int_t^{t_1} \xi' \tilde{E}(s)\tilde{E}'(s)\xi\, ds \geq \int_t^{t_1} e^2(s)\, ds \geq \frac{1}{n}\xi' D(t)\xi. \tag{21}$$

Let

$$d_k = \int_{t_k}^{t_{k+1}} e^2(s)\, ds.$$

To these sequences $\{a_k\}$ and $\{d_k\}$ apply Lemma 2. Then choose for control $v(t)$ a vector whose only nonzero coordinate has the same rank as the entry of maximum absolute value in $\xi'\tilde{E}$, this coordinate being equal to

$$v_i(t) = e(t) b_k \qquad \text{for } t_k \leq t \leq t_{k+1}.$$

This control is square integrable, since

$$\int_{t_0}^{t_1} \|v\|^2 dt = \sum_{k=0}^{\infty} b_k^2 d_k.$$

Moreover,

$$\int_{t_{m_i}}^{t_1} |\xi' E v|\, dt = \sum_{k=m_i}^{\infty} b_k d_k \geq \left(a_{m_i} \sum_{k=m_i}^{\infty} d_k\right)^{1/2}$$

Recalling the definition of $a_{m_i}$, and (20) and (21),

$$\left(\int_{t_{m_i}}^{t_1} |\xi' \tilde{E} v|\, dt\right)^2 \geq \epsilon_{m_i} \xi' D(t_{m_i})\xi \geq \xi' C(t_{m_i})\xi. \tag{22}$$

Now, assume that (18) holds with $u(\cdot)$ square integrable. We may take the square of each side, since both are positive, and majorize the LHS using Cauchy–Schwarz' inequality. Using also (22) for the RHS, we obtain

$$\forall i, \quad \int_{t_{m_i}}^{t_1} \xi' \tilde{G}(t)\tilde{G}'(t)\xi\, dt \int_{t_{m_i}}^{t_1} \|u(t)\|^2 dt > \int_{t_{m_i}}^{t_1} \xi' \tilde{G}(t)\tilde{G}'(t)\xi\, dt.$$

Since $t_{m_i} \to t_1$, this constitutes a contradiction, and the theorem is proved.

*Remark 6:* The form (10b) of condition (10) was provided to point out the difference with (17). We give in Section IV an example of a system that satisfies (17) but not (10). However, these two conditions are equivalent if $l = 1$. We thus have the following result.

*Corollary 1:* If $l = 1$ (i.e., the dimension $m$ of the capture set is $n-1$), a necessary and sufficient condition for $L^2$ capturability (within the class of strategies specified in Theorem 2) is that the ratio $D(t)/C(t)$ be bounded over $(t_0, t_1)$

*Remark 7:* Conditions (10) or (17) are formally the same as the capturability condition of the discrete case (see [1] or [2]), provided we substitute the standard discrete controllability matrices $C$ and $D$ to (8). In that case, because a finite number of instants $t_k$ have to be considered, there is no difference between (10) and (17). It is interesting to recall that in the discrete case, we defined capturability allowing the current control $u(t)$ to depend on the current perturbation $v(t)$ (but not on future $v$'s).

This similarity will be further pursued in the constant case we investigate now.

### C. The Time-Invariant Case

We now specialize our results to the case where the system is autonomous: $F$, $G$, $E$, in addition to $H$, are constant matrices. We need the following notations:

$$\mathfrak{M} = \ker H, \quad \mathcal{G} = \mathcal{R}(G), \quad \mathcal{E} = \mathcal{R}(E), \tag{23}$$

$$C^d(k) = H \sum_{i=0}^{k-1} F^i G G' F'^i H', \quad D^d(k) = H \sum_{i=0}^{k-1} F^i E E' F'^i H'. \tag{24}$$

$C^d$ and $D^d$ are the discrete output controllability matrices.

The following conditions can easily be shown to be equivalent (see [1] or [2]):

$$\forall k \in \mathbb{N}, \quad F^k \mathcal{E} \subset F^k \mathcal{G} + F^{k-1}\mathcal{G} + \cdots + \mathcal{G} + \mathfrak{M} \tag{25a}$$

$$\forall k \in \mathbb{N}, \quad F^k \mathcal{E} + F^{k-1}\mathcal{E} + \cdots + \mathcal{E} + \mathfrak{M} \subset F^k + F^{k-1}\mathcal{G} + \cdots + \mathcal{G} + \mathfrak{M} \tag{25b}$$

$$\forall k \in \mathbb{N}, \quad \exists \epsilon_k: C^d(k) - \epsilon_k F^{k-1} E E' F'^{k-1} \geq 0 \tag{25c}$$

$$\forall k \in \mathbb{N}, \quad \exists \epsilon_k: C^d(k) - \epsilon_k D^d(k) \geq 0. \tag{25d}$$

Moreover, from the Cayley–Hamilton theorem, it suffices to check each of them for $0 \leq k \leq n-1$.

We can now state the theorem of this section.

*Theorem 3:* Condition (25) is necessary and sufficient for the system $(H, F, G, E)$ to be $L^2$ capturable within the class of strategies specified in Theorem 2.

*Proof:* We still write $\Phi(t_1 - t)$ for the transition matrix $e^{F(t_1 - t)}$. We know from Theorem 2 that for every nonzero $\xi$, there must exist $\epsilon$ such that $\xi' X_\epsilon(t)\xi$ is positive in a left neighborhood of $t_1$. Since $X_\epsilon(t_1) = 0$, this is equivalent to the fact that there must exist $\epsilon$ such that the first nonzero derivative of $\xi' X_\epsilon^{-1}(t)\xi$ at $t_1$ is negative. It is a simple task to check that the $k$th derivative of $X_\epsilon(t)$ at $t_1$ is given by

$$x_\xi^{(k)}(t_1) = (-1)^k \sum_{i=0}^{k-1} \binom{k-i}{i} H F^i (GG' - \epsilon EE') F'^{k-1-i} H'.$$

Therefore, a first necessary condition is that

$$\forall \xi, \quad \exists \epsilon > 0: \xi' HGG'H'\xi - \epsilon \xi' HEE'H'\xi \geq 0.$$

This is equivalent to the fact that $\xi' HE = 0$ whenever $\xi' HG = 0$, i.e., that

$$(\mathcal{R}(HG))^\perp \subset (\mathcal{R}(HE))^\perp \Leftrightarrow \mathcal{R}(HE) \subset \mathcal{R}(HG) \tag{26}$$

which in turn implies

$$\mathcal{E} \subset \mathcal{G} + \mathfrak{M}$$

which is condition (25a) for $k = 0$.

We, moreover, notice that there exists an $\epsilon_0$ such that

$$\forall \xi: \xi' HG \neq 0, \quad \xi' HGG'H'\xi - \epsilon_0 \xi' HEE'H'\xi > 0. \tag{27}$$

It is further necessary that for all $\xi$ such that $\xi' HG = 0$, the second derivative of $\xi' X_\epsilon \xi$ be nonpositive for some $\epsilon$. But we have

$$\xi' \ddot{X}_\epsilon(t_1)\xi = \xi' HF(GG' - \epsilon EE')H'\xi + \xi' H(GG' - \epsilon EE')FH\xi$$

and if (26) is satisfied, this is zero for the $\xi$'s considered. We therefore turn to the third derivative,

$$-\xi' X_\epsilon^{(3)}(t_1)\xi = \xi' HF^2(GG' - \epsilon EE')H'\xi + 2\xi' HF(GG' - \epsilon EE')FH\xi + \xi' H(GG' - \epsilon EE')F^2 H'\xi.$$

Again, for $\xi$ in $(\mathcal{R}(HG))^{\perp}$, only the middle term is nonzero, and there must therefore exist $\epsilon$ such that

$$\forall \xi \in (\mathcal{R}(HG))^{\perp}, \quad \xi' HFGG'F'H'\xi - \epsilon \xi' HFEE'F'H'\xi \geq 0.$$

As previously, this is equivalent to

$$(\mathcal{R}(HG))^{\perp} \cap (\mathcal{R}(HFG))^{\perp} \subset (\mathcal{R}(HFE))^{\perp}$$

or equivalently to

$$\mathcal{R}(HG) + \mathcal{R}(HFG) \supset \mathcal{R}(HFE)$$

or

$$F\mathcal{E} \subset F\mathcal{G} + \mathcal{G} + \mathcal{M}$$

which is condition (25a) for $k=1$.

We must again notice that there exists $\epsilon_1$ such that

$$\forall \xi: \ \xi' HFG \neq 0, \quad \xi' HFGG'F'H'\xi - \epsilon_1 \xi' HFEE'F'H'\xi > 0. \tag{28}$$

Going on this way we see that only odd derivatives will come into play, with only the middle term nonzero on the subspace considered. We end up with condition (25a). (Notice that with $k = n-1$, it is trivially satisfied if the system is completely $u$-output controllable.)

Conversely, assume condition (25a) is satisfied. Looking at the relations (27), (28), and similar for higher orders, we see that setting

$$\epsilon = \min_{0 < k < n-1} \{\epsilon_k\}$$

we get that the first nonzero derivative of $\xi' X_\epsilon(t)\xi$ at $t_1$ will be negative for all nonzero $\xi$ in $R^l$; thus, our sufficient condition is satisfied in a left neighborhood of $t_1$, which is sufficient, since in the autonomous case $\mathcal{R}(C(t))$ and $\mathcal{R}(D(t))$ are constant. The theorem is proved.

*Remark 8:* Condition (25) is identically the capturability condition of the discrete system

$$x(k+1) = Fx(k) + Gu(k) + Ev(k).$$
$$y(k) = Hx(k)$$

as can be seen in [1], or more explicitly in [2]. It should therefore be satisfied whenever Wonham's modified perturbation decoupling problem [12] has a solution, since then there exists a strategy $u(t) = \varphi(x(t), v(t))$ that makes $y$ independent of $v$ for all times. As a matter of fact, it is known [12] that the corresponding condition is

$$\exists \mathcal{V} \subset \mathcal{M}: F\mathcal{V} \subset \mathcal{V} + \mathcal{G}, \quad \mathcal{E} \subset \mathcal{V} + \mathcal{G}. \tag{29}$$

This clearly implies $\mathcal{E} \subset \mathcal{G} + \mathcal{M}$ and

$$F\mathcal{E} \subset F\mathcal{V} + F\mathcal{G} \subset F\mathcal{G} + \mathcal{G} + \mathcal{V} \subset F\mathcal{G} + \mathcal{G} + \mathcal{M}$$

and by induction (25a).

*Remark 9:* Although Heymann *et al.* [5] consider noncausal information structures, some similarity exists at this level. While their Theorem 2.1 [5] is a modification if [1, Theorem I.1] (see also the remark after [1, Theorem II.1]) and in a very essential way "noncausal," the condition of their Proposition 2.3 [5] is Wonham's condition (29) above. In that situation, our theory shows that the system is capturable to $\mathcal{V}$ at any later time $t$, and this is *a fortiori* true with noncausal information on $v$. This is essentially their Theorem 2.4 [5].

*Remark 10:* We provided the forms (25c) and (25d) of condition (25) as being probably the most algebraic, and therefore most amenable to numerical computation.

## III. $L^1$ CAPTURABILITY

### A. Sufficient Condition

We establish the following result, whose main interest lies in the fact that it uses the strategy $\varphi_0$ given by classical least-square control theory.

*Theorem 4:* If

$$\operatorname{rank} \tilde{G}(t_1) + m = n \tag{30}$$

the system is $L^1$ capturable over $[t_0, t_1]$, and a capturing strategy is $\varphi_0$ given by setting $\epsilon = 0$ in (11):

$$u(t) = -G'(t)\Phi'(t_1,t)H'C^{-1}(t)H\Phi(t_1,t)x(t). \tag{31}$$

*Proof:* The proof uses the following lemma.

*Lemma 3:* Under condition (30), one has

$$\exists a > 0 \in R: \forall t \in (t_0, t_1) \quad \|C(t)\|\,\|C^{-1}(t)\| \leq a < \infty. \tag{32}$$

*Proof of the Lemma:* If (30) holds, in a left neighborhood of $t_1$ we have

$$\Gamma(s) = \tilde{G}(s)\tilde{G}'(s) \geq \gamma I.$$

Therefore, if $\nu(A)$ denotes the smallest eigenvalue of $A$, there exists $\nu(\Gamma(s)) > \gamma$, and in this same neighborhood

$$\nu(C(t)) = \inf_{\|\xi\|=1} \xi' \int_t^{t_1} \Gamma(s)\, ds\, \xi \geq \int_t^{t_1} \nu(\Gamma(s))\, ds \geq \gamma(t_1 - t).$$

We therefore have

$$\|C^{-1}(t)\| = (\nu(C(t)))^{-1} \leq \frac{1}{\gamma(t_1 - t)}.$$

But we also have, in that neighborhood, for some $\Gamma = \sup_s \|\Gamma(s)\|$

$$\|C(t)\| \leq \int_t^{t_1} \|\Gamma(s)\|\, ds \leq \Gamma(t_1 - t).$$

Hence the claim (32) of the lemma.

Turning back to the proof of the theorem, we first notice that the strategy (31) gives for the system (13)

$$\dot{\tilde{x}} = -\tilde{G}\tilde{G}'C^{-1}\tilde{x} + \tilde{E}v$$

whose explicit solution is

$$\tilde{x}(t) = C(t)C^{-1}(t_0)\tilde{x}_0 + C(t)\int_{t_0}^{t} C^{-1}(s)\tilde{E}(s)v(s)\, ds. \tag{33}$$

We must therefore prove that for every absolutely integrable $v(\cdot)$, the second term of the RHS goes to zero as $t$ goes to $t_1$. Since the first one obviously does, this will establish (15).

The positive matrix $C^{-1}(t)$ is increasing with time for the ordering of positive definite matrices, thus so is its norm. We therefore have

$$\left\| \int_{t_0}^{t} C^{-1}(s)w(s)\, ds \right\| \leq \int_{t_0}^{t} \|C^{-1}(s)\|\, \|w(s)\|\, ds$$
$$\leq \|C^{-1}(t)\| \int_{t_0}^{t} \|w(s)\|\, ds. \tag{34}$$

Choose a positive number $\eta$ and a positive number $\epsilon$ such that for $t$ in a left neighborhood of $t_1$, and with $a$ as in Lemma 3,

$$a\int_{t-\epsilon}^{t} \|\tilde{E}(s)v(s)\| < \eta. \tag{35}$$

Using (34) between $t_0$ and $t-\epsilon$, and between $t-\epsilon$ and $t_1$ it becomes

$$\|\tilde{x}(t)\| \leq \|C(t)\|\,\|C^{-1}(t_0)\tilde{x}_0\| + \|C(t)\|\,\|C^{-1}(t-\epsilon)\|\,\|\tilde{E}v\|_{L^1}$$
$$+ \|C(t)\|\,\|C^{-1}(t)\| \int_{t-\epsilon}^{t} \|\tilde{E}(s)v(s)\|\, ds.$$

The first two terms on the RHS go to zero as $t$ goes to $t_1$, (with $\epsilon$ fixed) and using (32) and (35), we have in the limit

$$\|x(t_1)\| \leq \eta.$$

Since $\eta$ was chosen arbitrary, (15) follows.

We must now prove that $u(\cdot)$ thus generated is absolutely integrable. Notice that in a neighborhood of $t_1$, $\tilde{G}$ is of rank $l$ and that

$$\tilde{x}(t_1) = 0 = \tilde{x}(t_0) + \int_{t_0}^{t_1} \tilde{G}(t)u(t)\, dt + \int_{t_0}^{t_1} \tilde{E}(t)v(t)\, dt,$$

so that for any $L^1$ perturbation $v(\cdot)$, the control $u(\cdot)$ generated is integrable. Now, from (31) and (33), it follows that

$$u(t) = -\tilde{G}'(t)C^{-1}(t_0)\tilde{x}_0 - \tilde{G}'(t)\int_{t_0}^{t} C^{-1}(s)\tilde{E}(s)v(s)\,ds. \tag{36}$$

The first term on the RHS is bounded and creates no problem. In the neighborhood of $t_1$, where $u$ may be unbounded, $G$ is of full rank. Thus, the problem is equivalent to showing that

$$w(t) = \int_{t_0}^{t} C^{-1}(s)\tilde{E}(s)v(s)\,ds \tag{37}$$

is absolutely integrable, knowing it is integrable for all absolutely integrable $v(\cdot)$. Let $\hat{v}(\cdot)$ be a given $L^1$ perturbation, and $\hat{w}$ the corresponding $w$ by (37). Pick a coordinate $\hat{w}_i$ of $\hat{w}$. It is a continuous function, and thus nonzero on a union, at most countable, of intervals $(\sigma_k,\tau_k)$, with $\hat{w}_i(\sigma_k) = \hat{w}_i(\tau_k) = 0$. To each such interval, associate the perturbation defined over $[t_0,t_1]$,

$$\begin{cases} \tilde{v}^k(t) = \hat{v}(t)\operatorname{sgn}(\hat{w}_i(t)) & \text{if } t\in(\sigma_k,\tau_k), \\ \tilde{v}^k(t) = 0 & \text{otherwise.} \end{cases}$$

Notice that since $\hat{w}_i(\sigma_k) = 0$,

$$\forall t\in(\sigma_k,\tau_k),\quad \hat{w}_i(t) = \int_{\sigma_k}^{t} (C^{-1}(s)\tilde{E}(s)v(s))_i\,ds$$

and that $\tilde{v}^k$ generates via (37) a

$$\tilde{w}_i^k = \int_{\sigma_k}^{t} (C^{-1}(s)\tilde{E}(s)v(s))_i\,ds \operatorname{sgn}\hat{w}_i(t) = |\hat{w}_i(t)|$$

if $t\in(\sigma_k,\tau_k)$, and $\tilde{w}_i^k(t) = 0$ otherwise (since $w_i(\tau_k)=0$). Now, the perturbation

$$\tilde{v}(t) = \sum_{k=0}^{\infty} \tilde{v}^k(t)$$

is absolutely integrable, since $\|v(t)\| = \|\hat{v}(t)\|$ for all $t$. By linearity of (37) it generates a $\tilde{w}_i$ given by

$$\tilde{w}_i(t) = \sum_{k=0}^{\infty} \tilde{w}_i^k(t) = |\hat{w}_i(t)|.$$

Therefore $|\hat{w}_i(t)|$ is integrable. This being true for all coordinates of $\hat{w}$, the result follows, and the theorem is proved.

Notice that the relation (32) trivially holds if $C$ is scalar, i.e., $l=1$. We therefore have the following corollary.

*Corollary 2:* If $l=1$, $\varphi_0$ induces capture against any $L^1$ perturbation. However, the control $u(\cdot)$ generated may not be $L^1$ if $C(t_1)=0$.

*Remark 11:* Theorem 3 and Corollary 2 make no use of $E(t)$. Therefore, they ensure capturability even for completely unknown perturbations, i.e., $E=I$, provided the perturbations remain absolutely integrable. However, this is also a sign that condition (30) is probably stronger than necessary.

### B. Necessary Condition

We first prove the following result.

*Theorem 5:* A necessary condition for the system $(H,F,G,E)$ to be $L^1$-capturable modulo $\mathfrak{M}$ at $t_1$ with feedback compatible (causal) strategies, or with compatible pure state feedbacks, is that the following condition be fulfilled:

$$\forall\xi\neq 0\in\mathbb{R}^l,\quad \exists\epsilon\colon \xi'(\tilde{G}(t_1)\tilde{G}'(t_1) - \epsilon\tilde{E}(t_1)\tilde{E}'(t_1))\xi \geq 0. \tag{38}$$

*Proof:* Notice that Lemma 1 holds for any class of controls $\Omega_u$ and $\Omega_v$ which are closed under product by $-1$ and concatenation. We can therefore use it with $L^1$ controls. Assume that $\xi'\tilde{G}(t_1)=0$ and $\xi'\tilde{E}(t_1)\neq 0$ (and remember we picked $G$ and $E$ left continuous at $t_1$). Then it is a simple matter to check that there exists an $L^1$ perturbation $v(\cdot)$ such that

$$\int_t^{t_1} |\xi'\tilde{E}(s)v(s)|\,ds \geq \sup_{s>t}\|\xi'\tilde{G}(s)\|$$

since the RHS goes to zero as $t\to t_1$. Then using (18) we obtain

$$\sup_{s>t}\|\xi'G(s)\|\int_t^{t_1}\|u(s)\|\,ds \geq \int_t^{t_1}\xi'\tilde{G}(s)u(s)\,ds$$
$$\geq \int_t^{t_1}|\xi'\tilde{E}(s)v(s)|\,ds \geq \sup_{s>t}\|\xi'\tilde{G}(s)\|$$

and we have the same contradiction as in the proof of Theorem 2.

*Remark 12:* This result is not very sharp, since we did not attempt to compare the possible rates of decrease of $\xi'\tilde{G}(t)$ and $\xi'\tilde{E}(t)$ to zero. Moreover, we can show that the gap between our necessary and our sufficient conditions for $L^1$ capturability is wider than for $L^2$ capturability. As a matter of fact, the estimation

$$\nu(C(t)) \geq \gamma(t_1 - t)$$

proved in Lemma 3, and the obvious estimation

$$\|D(t)\| \leq \Delta(t_1 - t)$$

that one can get as in the same lemma, shows that under condition (30) of Lemma 3, the sufficient condition (10) of Theorem 1 is met.

As for necessity, if the necessary condition (38) of Theorem 5 is not met, i.e., if there exists $\xi$ such that

$$\xi'\tilde{G}(t_1) = 0 \neq \xi'\tilde{E}(t_1)$$

then, for any $\epsilon$, there exists a $\delta$ so small that

$$\forall t\in(t_1-\delta,t_1),\quad \|\xi'\tilde{G}(t)\| < \frac{1}{2}\sqrt{\epsilon}\,\|\xi'\tilde{E}(t_1)\|$$

$$\forall t\in(t_1-\delta,t_1),\quad \|\xi'\tilde{E}(t)\| > \frac{1}{2}\|\xi'\tilde{E}(t_1)\|$$

so that, integrating between $t_1-\delta$ and $t_1$, we see that the necessary condition (17) of Theorem 2 is not met either. We therefore have

$$\text{sufficient } L^1 \Rightarrow \text{sufficient } L^2 \Rightarrow \text{necessary } L^2 \Rightarrow \text{necessary } L^1.$$

This only points out the various gaps that remain to be filled. It should be noticed that although $L^2\subset L^1$, there is no obvious relationship between $L^1$ capturability and $L^2$ capturability, since going from one to the other places more constraints on both players.

## IV. Examples and Conjecture

### A. $L^2$ Capturability: Necessity and Sufficiency

Consider the following two matrices with $0\leq\tau\leq 1$:

$$C(t_1-\tau) = e^{-1/\tau}\begin{pmatrix} \tau^2 & -\tau\sqrt{1-\tau^2} \\ -\tau\sqrt{1-\tau^2} & 1 \end{pmatrix}$$

$$D(t_1-\tau) = e^{-1/\tau}\begin{pmatrix} \tau^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to check that they are positive definite, with negative definite $t$-derivatives $(t=t_1-\tau)$. The square roots of these derivatives may be taken as being $\tilde{G}$ and $\tilde{E}$. Consider also the vector $\xi$ of unit norm

$$\xi = \begin{pmatrix} \sqrt{1-\tau^2} \\ \tau \end{pmatrix}.$$

We have

$$\frac{\xi'(t)D(t)\xi(t)}{\xi'(t)C(t)\xi(t)} = \frac{2}{(t_1-t)^2} - 1 \to \infty \qquad \text{as } t\to t_1.$$

Therefore, the sufficient condition (10) is not satisfied. However, let now $\xi$ be a constant vector $(\xi_1,\xi_2)'$. If $\xi_2=0$, then $\xi'D(t)\xi = \xi'C(t)\xi$ for all $t$. If $\xi_2\neq 0$, then

$$\frac{\xi'D(t)\xi}{\xi'C(t)\xi} = \frac{\xi_2^2 + (t_1-t)^2\xi_1^2}{\xi_2^2 + (t_1-t)\xi_1\left[(t_1-t)\xi_1 - \xi_2\sqrt{1-(t_1-t)^2}\right]} \to 1 \qquad \text{as } t\to t_1.$$

Therefore the necessary condition (17) is satisfied. At this point, we do not know whether the corresponding system is $L^2$ capturable or not.

*B. The Strategy* $\varphi_0$

This strategy [given by (31)] is important in that it is the one classically synthesized to drive a linear system to a linear target. We first show that it may not be capturing, even against a finite perturbation. We again pick a two-dimensional system (because of Corollary 2) of the form (13), with $t_1=0$, $t<0$, and

$$\tilde{G}(t)=\begin{pmatrix}1\\t^2\end{pmatrix},\quad \tilde{E}(t)=\begin{pmatrix}1\\t\end{pmatrix}.$$

The condition $\mathcal{R}(\tilde{E})\subset\mathcal{R}(C)$ is satisfied, since the system is differentially $u$-controllable at $t_1$. However, for $x_0=0$ and $v(t)=1$ for all $t$, we get, with $\varphi_0$

$$x_1(t)=\frac{5}{2}+\epsilon_1(t),\qquad \epsilon_1(t)\to 0 \text{ as } t\to 0.$$

($x_2(t)$ goes to zero.) Therefore, $\varphi_0$ does not generate capture.

Notice however that if a system satisfies (10) for a given $\epsilon$, to which there corresponds through (11) a capturing strategy $\varphi_\epsilon$, then (10) is also satisfied with any positive $\epsilon'<\epsilon$. Therefore, the corresponding strategies $\varphi_{\epsilon'}$ are all $L^2$ capturing. It can be shown that for a given $L^2$ perturbation $v(\cdot)$, the $L^2$ norm of the $u(\cdot)$ generated increases when $\epsilon$ decreases. But we always meet (15). We suggest that a limiting process could prove the following result.

*Conjecture:* If a system satisfies (10), then $\varphi_0$ is capturing against any square integrable perturbation, although the control $u(\cdot)$ generated may not be square integrable.

As for the character $L^2$ of the control generated, formula (29) together with classical results on Hilbert–Schmidt operators [10] yield the following fact.

*Proposition:* The control generated by $\varphi_0$ is square integrable for any square integrable perturbation if and only if

$$\iint_{t_0<s<t<t_1}\tilde{G}'(t)C^{-1}(s)\tilde{E}(s)\tilde{E}'(s)C^{-1}(s)\tilde{G}(t)\,ds\,dt<\infty. \tag{39}$$

(Notice that it suffices to look at the trace of this operator.)

While we pointed out in the Remark 11 that the Theorem 4 allows one to take $E=I$, it can be shown [3] that then (31) is never satisfied. We can further show that, for scalar systems ($n=1$), this proposition is a weaker result than the Theorem 1 [or (39) stronger than (10)].

*Theorem 6:* If $n=1$, then (39) implies (10), and $\varphi_0$ is $L^2$ capturing.

*Proof:* Assume

$$\forall t\in(t_0,t_1),\quad |\tilde{E}(t)|\geq\alpha|\tilde{G}(t)|\qquad \text{for some } \alpha>0.$$

Then, we get ($T(\tau)$ is the triangle $t_0\leq s\leq t\leq\tau$)

$$\iint_{T(\tau)}\frac{\tilde{G}^2(t)\tilde{E}^2(s)}{C^2(s)}\,ds\,dt\geq\alpha^2\iint_{T(\tau)}\frac{\tilde{G}^2(t)\tilde{G}^2(s)}{C^2(s)}\,ds\,dt$$
$$=\alpha^2(\log C(t_0)-\log C(\tau)-1)$$

and the last term diverges to $+\infty$ as $\tau\to t_1$. Therefore, if (39) is satisfied, $\tilde{E}(t)/\tilde{G}(t)$ goes to zero as $t\to t_1$ this proves the first claim. As we know, from the proposition, that $\varphi_0$ generates an $L^2$ control, we also infer that

$$\tilde{E}(t)C^{-1}(t)x(t)=\frac{-\tilde{E}(t)}{\tilde{G}(t)}\varphi_0(x(t),t)$$

is square integrable. Then, we make a calculation completely analogous to that of Section II-A, but with $C$ in place of $X_\epsilon$, and it comes the analogous to (16),

$$\int_{t_0}^{t}u^2(s)\,ds=\int_{t_0}^{t}(u+\tilde{G}'C^{-1}x)^2\,ds+2\int_{t_0}^{t}v\tilde{E}C^{-1}x\,ds$$
$$+x_0'C^{-1}(t_0)x_0-x'(t)C^{-1}(t)x(t).$$

Placing $u(t)=\varphi_0(x(t),t)$ annihilates the first term in the RHS. The LHS is bounded by the proposition, and the second integral in the RHS is bounded as the integral of the product of two square integrable functions. Therefore, the last term is bounded and we conclude as in Section II-A.

*C. The Autonomous System*

We first want to show that for the autonomous system, our condition (25) is weaker than (29), the condition for the modified perturbation decoupling problem to have a solution. As a matter of fact consider the following system:

$$F=\begin{pmatrix}0&0&0\\0&0&0\\1&1&0\end{pmatrix},\quad G=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad E=\begin{pmatrix}1\\0\\0\end{pmatrix},$$

$$H=\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix},\quad \text{i.e.,}\quad \mathfrak{M}=\left\{\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}\middle|\,\alpha\in R\right\}.$$

Since $\mathfrak{M}$ itself is not $(F,G)$ invariant, the only such subspace of $\mathfrak{M}$ is $\{0\}$. However, $\mathcal{E}\not\subset\mathcal{G}+\{0\}$. Therefore, (29) is not satisfied. However, we easily see that

$$\mathcal{E}=\mathfrak{M}\subset\mathcal{G}+\mathfrak{M},\quad F\mathcal{E}\subset F\mathcal{G}+\mathcal{G}+\mathfrak{M}=R^3,\quad F^2\mathcal{E}=\{0\}.$$

Therefore, condition (25) is satisfied. As a matter of fact, a simple calculation shows that

$$X_\epsilon(t)=\begin{bmatrix}t_1-t & \frac{1}{2}(t_1-t)^2\\ \frac{1}{2}(t_1-t)^2 & \frac{1-\epsilon}{3}(t_1-t)^3\end{bmatrix}$$

which is positive definite if $\epsilon<1/4$.

The fact that condition (25) coincides with the capturability condition of the discrete system, where we allow $u(t)$ to depend on $v(t)$, leads us to the following.

*Conjecture:* Our necessary conditions of $L^2$ capturability also hold if $u(t)$ is allowed to depend explicitly on $v(t)$.

However, this is not clear from our proof, because such a strategy might not be compatible with $\psi$. We can only make the following weak statement.

*Proposition:* In the autonomous system, if $u(t)$ is allowed to depend linearly on $v(t)$, with constant coefficients, (25) is still the necessary condition of $L^2$ capturability.

*Proof:* Using $u(t)=\varphi(v(\cdot))+Kv(t)$ only changes $E$ to $E+GK$, and this leaves condition (25) unchanged.

## Conclusion

We believe our definition of capturability is the most natural extension to perturbed systems of the concept of controllability. It is unfortunate that at this point there seems to be differing conditions depending on the classes of admissible perturbations and controls. However, as one might expect, the $L^2$ case gives rise to a rather nice theory, and in particular to a necessary and sufficient condition for the autonomous system that turns out to be the same as that for the discrete-time system, that we had derived in our Ph.D. dissertation in 1970 (see [1]). The curious fact is that in the continuous case we do not make use of the current value of the perturbation, while we do in the discrete case. In some sense, allowing the gain in the feedback to grow to infinity has the same effect as knowing the perturbation instantaneously. This might be interesting to pursue further.

We also see as a challenge the gap remaining between our necessary and our sufficient conditions in the nonstationary nonscalar case. It would also be interesting to remove the restriction on the admissible strategies we have in Theorem 2. A possible path might be to generalize the solution concept of a differential equation as in [8]. We also offer two conjectures mainly related to $L^2$ capturability. It might be interesting to investigate them.

Finally, while we did not attempt to construct a very refined theory for $L^1$ capturability, the few results we have do give some indications. The interest of this case lies in the fact that $L^1$ controls is a natural set up for linear differential equations. We did not mention some $L^\infty$ results that appear in [2]. But there clearly is much more to be done in these areas.

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