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Linear–Quadratic, Two-Person, Zero-Sum Differential Games: Necessary and Sufficient Conditions[1]

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Abstract. We consider linear–quadratic, two-person, zero-sum perfect information differential games, possibly with a linear target set. We show a necessary and sufficient condition for the existence of a saddle point, within a wide class of causal strategies (including, but not restricted to, pure state feedbacks). The main result is that, when they exist, the optimal strategies are pure feedbacks, given by the classical formulas suitably extended, and that existence may be obtained even in the presence of a conjugate point within the time interval, provided it is of a special type that we call *even*.

Key Words. Linear differential games, conjugate points, saddle points.

1. Introduction

It has long been known that, for the two-person, zero-sum differential game with linear dynamics, quadratic payoff, fixed end-time, and free end-state (*standard LQ game*), the existence of a solution to a Riccati equation is a sufficient condition for the existence of a saddle point within the class of instantaneous state feedback strategies (Refs. 1–2), and therefore within any wider class (Ref. 3).

In the simpler case of optimal control theory (*one-player game*), it is also known that this constitutes a necessary condition for the existence of a minimum (Refs. 4–6), and this result can be extended to the case where the

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final state is constrained to lie in a given linear subspace, although this raises the problem of abnormal trajectories (Refs. 4 and 7).

Up to now, however, the problem remained unsettled for the standard LQ game. Two main questions were posed: Is the absence of conjugate points a necessary condition for the existence of a pure feedback solution? Is the answer any different if one allows greater use of past information in the elaboration of the controls? The aim of this article is to provide a rather complete answer to these questions. The answer to both questions is negative. And we shall provide the necessary and sufficient condition, within a mild positivity hypothesis on some of the data (hypothesis that corresponds to the standard situation of worst-case design).

It turns out that, to investigate the problem, even with free end-state, we need to study the linearly constrained end-state game. We thus start out with that more general situation. As in the one-player case, this obliges us to look into normality questions. These questions, however, happen to be more complicated than in the previous case, and we are obliged to introduce a further distinction between normalizable and unnormalizable problems.

In Section 2, we state the problem and the hypotheses. In Section 3, we introduce the necessary concepts to state the main theorem, including normalizability. Section 4 is devoted to the proof of the theorem in the absence of conjugate points, and Section 5 is devoted to the study of conjugate points. Many details will be skipped in the proofs; they may be found in Ref. 3.

2. Problem

We consider a *linear system*

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) + E(t)v(t), \tag{1}$$

$$x(t_0) = x_0, \tag{2}$$

where $t \in [t_0, t_1] \subset R$ is the time, t_0 and t_1 are prescribed, $x(t) \in R^n$ is the state, the dot means time derivative and $u(t) \in R^m$ and $v(t) \in R^p$ are the pursuer's and evader's controls.

The sets Ω_u and Ω_v of admissible control functions are made of all square-integrable functions $u(\cdot)$ and $v(\cdot)$ from $[t_0, t_1]$ to R^m and R^p, respectively. $F(\cdot)$, $G(\cdot)$, $E(\cdot)$ are matrix time functions of appropriate type, piecewise continuous, chosen right continuous everywhere and left continuous at t_1.

A linear *target set* $\mathcal{M}$ of dimension m is given in R^n, as the range of a full rank $m \times n$ matrix M:

$$\mathcal{M} = \mathcal{R}(M), \qquad \dim \mathcal{M} = m = n - l.$$

When necessary, we shall assume that an orthogonal basis has been chosen in $\mathscr{L}=\mathscr{M}^{\perp}$, and use the orthogonal projection from R^n onto $\mathscr{L}$, whose matrix Π is of type $l\times n$ and satisfies[3]

$$\Pi'\Pi = I - MM^{\dagger}, \qquad \Pi\Pi' = I_l. \tag{3}$$

A *payoff* or *criterion* is given by

$$\begin{aligned} J(x_0, t_0; u(\cdot), v(\cdot)) = x'(t_1)Ax(t_1) + \int_{t_0}^{t_1} [&x'(t)Q(t)x(t) + x'(t)S(t)u(t) \\ &+ u'(t)S'(t)x(t) + x'(t)T(t)v(t) + v'(t)T'(t)x(t) \\ &+ u'(t)R(t)u(t) - v'(t)B(t)v(t)]\, dt \end{aligned} \tag{4}$$

if

$$x(t_1) \in \mathscr{M} \qquad \text{or equivalently } \Pi x(t_1) = 0, \tag{5}$$

and

$$J(x_0, t_0; u(\cdot), v(\cdot)) = +\infty \qquad \text{if } x(t_1) \notin \mathscr{M} \text{ or } \Pi x(t_1) \neq 0, \tag{6}$$

that the pursuer $\mathscr{U}$ tries to minimize and the evader $\mathscr{V}$ to maximize. Hence, the final constraint (4) is under the pursuer's responsibility.

In (4), A, $Q(t)$, $R(t)$, $B(t)$ are symmetric matrices, and $S(t)$, $T(t)$ are any matrices of appropriate type, the last five matrices with the same regularity as F, G, E as functions of time.

We further make the following positivity assumptions:

$$\forall t \in [t_0, t_1], \qquad R(t) > 0, \qquad B(t) > 0, \tag{7}$$

$$\forall t \in [t_0, t_1], \qquad \begin{bmatrix} Q(t) & S(t) \\ S'(t) & R(t) \end{bmatrix} \geq 0, \qquad A \geq 0. \tag{8}$$

Notice that specifications (6) and (8) are asymmetrical, in that they give different roles to the two players. We would have a completely analogous treatment by reversing both, using in (8) the matrix made with Q, T, $-B$.

It is assumed that both players have perfect instantaneous state measurement, perfect recall, and (if necessary) perfect knowledge of their opponent's past control function. Thus, strategies are functions φ and ψ of $R^n \times R \times \Omega_v$ into Ω_u and $R^n \times R \times \Omega_u$ into Ω_v, respectively, giving the players' controls through

$$u(t) = \varphi(x_0, t_0; v(\cdot))(t), \qquad v(t) = \psi(x_0, t_0; u(\cdot))(t), \tag{9}$$

[3] We use the prime notation for transpose, and † for pseudoinverse.

with the casuality property: for $v_1(\cdot)$, $v_2(\cdot)$ in Ω_v, if $v_1(t) = v_2(t)$ for almost all $t \in [t_0, \tau]$, $\tau \le t_1$, then

$$\varphi(x_0, t_0; v_1)(\tau) = \varphi(x_0, t_0; v_2)(\tau),$$

and conversely for ψ.

To each initial phase (x_0, t_0), we associate two sets Φ and Ψ of admissible strategies, chosen such that:

(i) they contain open-loop control functions in Ω_u or Ω_v;
(ii) they are closed under concatenation;
(iii) any $(\varphi, \psi) \in \Phi \times \Psi$ generates through (1)–(2) at least one trajectory, generating control functions in $\Omega_u \times \Omega_v$.

Our main theorem holds tor any such pair of admissible strategy sets that contains φ^* and ψ^* given below.

Remark 2.1. The sets Φ and Ψ are chosen a priori for each initial phase, so that the game always takes place over a product set of strategies. We omitted writing explicitly the dependence of Φ and Ψ on (x_0, t_0). We shall make use repeatedly of the possibility to *concatenate games*, using the value of the game over $[t_2, t_1]$ as the final cost of a game on $[t_0, t_2]$. For the legitimacy of this, see for instance Ref. 3. For consistency, we must assume that, for any $t_2 \in [t_0, t_1]$, the restriction to $[t_2, t_1]$ of a strategy pair $(\varphi, \phi) \in \Phi(x_0, t_0) \times \Psi(x_0, t_0)$ belongs to $\Phi(x_2, t_2) \times \Psi(x_2, t_2)$, where x_2 is the state at time t_2 on any trajectory generated by (φ, ψ) from (x_0, t_0). The converse is implied by (ii) above: one is allowed to concatenate a strategy pair of $\Phi(x_2, t_2) \times \Psi(x_2, t_2)$ to any trajectory from (x_0, t_0) to (x_2, t_2).

We are obliged to use this set-up, because we want to allow state feedbacks with a gain that can be unbounded in the neighborhood of a time τ, such that at that time trajectories exist only through some special states of interest. In that case, the meaning of (1) is that it must be satisfied for almost all t. If τ is the initial time, we state the following definitions.

Definition 2.1. A trajectory generated from $x(\tau) = \xi$ is an absolutely continuous function $x(\cdot)$, satisfying (1) for all $t > \tau$ in a right neighborhood of τ, and such that $x(t) \to \xi$ when $t \downarrow \tau$.

Definition 2.2. Admissible strategies sets are sets Φ and Ψ satisfying (i)–(iii) above, and such that all admissible strategies are locally bounded, except at most in the neighborhood of finitely many instants of time.

Definition 2.3. A solution of the differential game is a set of admissible strategies (φ^*, ψ^*) such that, $\forall(\varphi, \psi) \in \Phi \times \Psi$,

$$J(x_0, t_0; \varphi^*, \psi) \leq J(x_0, t_0; \varphi^*, \psi^*) = V(x_0, t_0) \leq J(x_0, t_0; \varphi, \psi^*), \quad (10)$$

or equivalently (see Ref. 3 or Ref. 8),

$$\forall(u(\cdot), v(\cdot)) \in \Omega_u \times \Omega_v,$$

$$J(x_0, t_0; \varphi^*, v(\cdot)) \leq J(x_0, t_0; \varphi^*, \psi^*) = V(x_0, t_0) \leq J(x_0, t_0; u(\cdot), \psi^*), \quad (11)$$

with a transparent abuse of notations for the arguments of J. If a pair of admissible strategies generates several trajectories, the inequalities must hold for J evaluated on any of them, and this, in turn, ensures the unicity of the value $J(\varphi^*, \psi^*)$.

3. Basic Equations

'We introduce the following *canonical equations*, involving two square $n \times n$ matrix functions of time $X(\cdot)$ and $\Lambda(\cdot)$:

$$\dot{X} = (F - GR^{-1}S' + EB^{-1}T')X - (GR^{-1}G' - EB^{-1}E')\Lambda,$$
$$X(t_1) = MM^\dagger, \quad (12)$$

$$\dot{\Lambda} = -(Q - SR^{-1}S' + TB^{-1}T')X - (F' - SR^{-1}G' + TB^{-1}E')\Lambda,$$
$$\Lambda(t_1) = AMM^\dagger + I - MM^\dagger, \quad (13)$$

and the definition

$$P(t) = \Lambda(t)X^\dagger(t). \quad (14)$$

We have the following classical lemma (see Ref. 3 or Ref. 7).

Lemma 3.1. On any interval of time where $X(t)$ is invertible, $P(\cdot)$ satisfies the following *Riccati equation*:

$$\dot{P} + PF + F'P - (PG + S)R^{-1}(G'P + S') + (PE + T)B^{-1}(E'P + T') + Q = 0; \quad (15)$$

conversely, if $X(\tau)$ is invertible, $X(t)$ is invertible on any interval $[\tau_1, \tau_2]$ over which the equation (15), initialized at time τ with (14) has a solution.

Let $\mathcal{W}(t)$ be the subspace of states that can be controlled by u alone to $\mathcal{M}$ on $[t, t_1]$:

$$\mathcal{W}(t) = \Phi(t, t_1)\left[\mathcal{M} + \mathcal{R}\left(\int_t^{t_1} \Phi(t_1, s)G(s)G'(s)\Phi'(t_1, s)\, ds\right)\right]. \quad (16)$$

$\Phi(t, s)$ is the transition matrix associated with F.

Proposition 3.1. The subspace $\mathcal{W}(t)$ is of piecewise constant decreasing dimension.

Proof. The range subspace that appears in the definition (16) is a decreasing set, and therefore piecewise constant and of decreasing dimension. □

Definition 3.1. The system (1) is said to be *G-reducible* if

$$\forall t, \qquad \mathcal{R}(E(t)) \subset \mathcal{W}(t). \tag{17}$$

As a matter of fact, in that case v cannot by himself drive the state out of $\mathcal{W}$, so that, if the initial state belongs to it, one can, by the classical technique, restrict the state space to $\mathcal{W}$, losing no information, and making the system completely u-controllable modulo $\mathcal{M}$ at t_1. Hence, the terminology.

We have the following obvious result.

Lemma 3.2. A necessary condition for the existence of a nondegenerate (i.e., with finite value) saddle point is that the system be G-reducible and that

$$x_0 \in \mathcal{W}(t_0). \tag{18}$$

Proof. It is elementary. □

Now, we shall study the problem on a time interval $[t_2, t_1]$ on which $\mathcal{W}(t)$ is of constant dimension, so that performing the G-reduction, we shall have a system differentially completely u-controllable modulo $\mathcal{M}$ at t_1 (d.c.u-c mod $\mathcal{M}$). We shall show that, if a saddle point exists at all, the set of initial states for which it does exist is a linear space $\mathcal{M}_2$. For initial states $x(t_2) \notin \mathcal{M}_2$, v can make J arbitrarily large. Therefore, if on $[t_0, t_1]$ $\mathcal{W}$ changes dimension at some instants t_i, $i = 2, \ldots$, we can first consider the problem on $[t_2, t_1]$, then on $[t_3, t_2]$ with capture set $\mathcal{M}_2$ at t_2, a final cost $V(x, t_2)$ (which will be quadratic in x), and so on up to t_0. Furthermore, the equations (12)–(13) for the problem on $[t_{i+1}, t_i]$ may be initialized with $X(t_i)$, $\Lambda(t_i)$ as given by the system over $[t_i, t_{i-1}]$ (see Ref. 3 or Ref. 7). Therefore, there is no loss of generality in making the following assumption:

$$\forall t \in [t_0, t_1], \qquad \mathcal{W}(t) = R^n. \tag{19}$$

We are obliged to state a further definition.

Definition 3.2. The problem (1)–(6) is said to be *normalizable* if, for all t on $[t_0, t_1]$, except possibly at isolated points, called *focal points*,

$$\mathcal{R}(X(t)) = \mathcal{W}(t). \tag{20}$$

If (20) holds, by performing the G-reduction, one can make the problem *normal* according to the classical definition (see Refs. 4 and 7), that we now recall.

Definition 3.3. The problem is *normal* if, for all t on $[t_0, t_1]$, except possibly at isolated focal points,

$$\det X(t) \neq 0. \tag{21}$$

It is a central fact of second-variation theory that, in optimal control theory, all nonsingular problems are normalizable according to the above definition. Therefore, the concept is not needed. The new fact is that this is not true for differential games. As a counterexample, choose

$$F=0, \qquad G=E, \qquad Q=0, \qquad S=0, \qquad T=0, \qquad R=B.$$

Definition 3.4. A focal point different from t_1 is called a *conjugate point.*

We can now state the main theorem.

Theorem 3.1. A necessary and sufficient condition for the existence of a nondegenerate saddle point to the problem (1)–(6) with the assumptions (7) and (8), is that:

(i) the system (1) be G-reducible (Definition 3.1);
(ii) the problem be normalizable (Definition 3.2);
(iii) $x_0 \in \mathcal{R}(X(t_0))$, $X(t)$ is defined by (12)–(13);
(iv) $\forall t \in [t_0, t_1]$, $P(t) \geq 0$, $P(t)$ defined by (14).

Then, the optimal strategies are

$$u(t) = \varphi^*(x(t), t), \qquad \varphi^*(x, t) = -R^{-1}(t)(G'(t)P(t) + S'(t))x, \tag{22}$$

$$v(t) = \psi^*(x(t), t), \qquad \psi^*(x, t) = B^{-1}(t)(E'(t)P(t) + T'(t))x, \tag{23}$$

and the value of the game is

$$V(x_0, t_0) = x_0'P(t_0)x_0. \tag{24}$$

If the above conditions do not hold, v can make the payoff arbitrarily large.

Remark 3.1. In the neighborhood of a focal point, φ^* and ψ^* are unbounded (for some x's). We must check that these strategies are consistent with the requirements set to define admissible strategies. This is done as follows.

(i) The strategies φ^* and ψ^* generate trajectories from any initial state satisfying condition (iii) of the theorem, against any opponent's open-loop control. This is not trivial only if t_0 is a conjugate point. The fact that it is true for ψ^* will be a consequence of the sufficient condition in Section 4, and the study of the *reverse game* in Section 5. For φ^* this has to be proved independently and is asserted by Lemma 3.3.

(ii) The controls generated by $(\varphi^*, v(\cdot))$ or $(\psi^*, u(\cdot))$ are square integrable if the opponent's one is. This again is a consequence of the proof of the theorem. For ψ^*, it holds only if $u(\cdot)$ ensures (5). We may admit that, if it does not, v chooses to bound his control once he has made a sufficient profit, chosen arbitrarily large. He can then play $v=0$. Notice, however, that this is no longer a pure feedback strategy.

Lemma 3.3. For any $x_0 \in \mathcal{R}(X(t_0))$, and for any admissible control function $v(\cdot)$, there exists at least one trajectory generated from x_0 by $(\varphi^*, v(\cdot))$.

Proof. It is omitted (see Ref. 3). □

4. Proof of the Main Theorem: No Conjugate Point

4.1. Simple Game. We first study a particular case of the game (1)–(6), that we call the simple game. It is defined by

$$F=0, \qquad \mathcal{M}=\{0\}, \qquad \text{i.e., } M=0, \qquad \Pi = I_n,$$
$$A=0, \qquad \forall t\in[t_0, t_1], \qquad Q(t)=0, \qquad S(t)=0, \qquad T(t)=0.$$

In this case, assumption (8) is void.

We further *assume that the system is* d.c.u-c mod $\mathcal{M}$, i.e., (19) holds. Equations (10)–(14) now reduce to

$$X(t)=\int_t^{t_1}[G(s)R^{-1}(s)G'(s)-E(s)B^{-1}(s)E'(s)\,ds, \qquad \Lambda(t)=I,$$
$$P(t)=X^{\dagger}(t), \qquad P(t)=X^{-1}(t), \tag{25}$$

if the system is normalizable.

Necessary Condition. The necessity of (i) follows from Lemma 3.2. The necessity of (iii), (iv) follows from Heymann, Pachter, and Stern (Ref. 9, Corollary 3.10): they state that, if the conditions

$$X(t_0)\geq 0, \tag{26}$$
$$x(t_0)\in \mathcal{R}(X(t_0)), \tag{27}$$

are not both satisfied, then

$$\sup_{u(\cdot)\in\Omega_u} \inf_{v(\cdot)\in\Omega_v} J(x_0, t_0; u(\cdot), v(\cdot)) = +\infty.$$

This means that, even knowing the whole future control function $v(\cdot)$, u cannot prevent his opponent from making J arbitrarily large. This is a fortiori true if u is constrained to using causal strategies. And, in that case, (26) must hold for all $t\in[t_0, t_1]$, and not only at t_0. Otherwise, if it were violated at a time t_2, v could *wait* (say, play $v = 0$) until that time, and use the above result from t_2 on.

For the same reason, u must also ensure (27) at all $t\in[t_0, t_1]$, from any admissible initial state. At a conjugate point, this is possible only if φ is allowed to be unbounded in a left neighborhood, and this, together with Definition 2.2, forbids accumulation points of conjugate points. The necessity of (ii) is then a consequence of the following result.

Lemma 4.1. If $X(t)$ is positive semidefinite and singular on an interval $[t_2, t_3]$, no strictly causal strategy φ can hold the state in $\mathcal{R}(X)$ for every control $v(\cdot)\in\Omega_v$.

Proof. It is omitted (see Ref. 3). □

Sufficient Condition. Using (22)–(23) and (25), one sees that, for all u and v, assuming $X(t)>0$,

$$u'Ru - v'Bv = -(d/dt)(x'X^{-1}x) + (u-\varphi^*)'R(u-\varphi^*) - (v-\psi^*)'B(v-\psi^*).$$

Integrating by parts, this yields (using the notation $u'Ru = \|u\|_R^2$):

$$\int_{t_0}^{t} (\|u\|_R^2 - \|v\|_B^2)\, ds = x_0'X(t_0)^{-1}x_0 - x'(t)X(t)^{-1}x(t) + \int_{t_0}^{t} \|u-\varphi^*\|_R^2\, ds - \int_{t_0}^{t} \|v-\psi^*\|_B^2\, ds. \tag{28}$$

Assume that the pursuer uses the strategy φ^*. The above relation gives

$$\int_{t_0}^{t} \|u\|_R^2\, ds = \int_{t_0}^{t} \|v\|_B^2\, ds + x_0'X(t_0)^{-1}x_0 - x'(t)X(t)^{-1}x(t) - \int_{t_0}^{t} \|v-\psi^*\|_B^2\, ds$$

In the right-hand side of this equation, the first two terms are positive and bounded as long as $v(\cdot)$ is square integrable on $[t_0, t_1]$. The last two terms are negative. However, the left-hand side being positive, these last two terms are both bounded. From this, we can conclude: firstly, the function $u(\cdot)$ thus generated is square integrable on $[t_0, t_1]$; secondly, there exists a positive real number a such that

$$x'(t)X(t)^{-1}x(t)\leq a^2, \qquad \forall t<t_1,$$

so that

$$\|x(t)\|\leq\|X^{1/2}(t)\|\,\|X^{-1/2}(t)x(t)\|\leq\|X^{1/2}(t)\|a;$$

and, since $X(t)$ goes to zero as t goes to t_1, in the limit

$$x(t_1)=0; \tag{29}$$

thirdly, taking the above fact into account,

$$J(x_0, t_0; \varphi^*, v(\cdot))\leq x'(t_0)X(t_0)^{-1}x(t_0). \tag{30}$$

Now, assume that against φ^*, v plays ψ^*. With these strategies, we get

$$(d/dt)(X^{-1}(t)x(t))=0.$$

Therefore $X^{-1}x$ is constant along a trajectory; thus, since $x(t)$ goes to zero,

$$x'(t)X^{-1}(t)x(t)\to 0 \qquad \text{as } t\to t_1. \tag{31}$$

Equations (23) therefore give

$$J(x_0, t_0; \varphi^*, \psi^*) = x_0'X(t_0)^{-1}x_0 \tag{32}$$

We must now establish the second inequality of the saddle point, i.e., that

$$J(x_0, t_0; u(\cdot), \psi^*)\geq x_0'X(t_0)^{-1}x_0. \tag{33}$$

This will be done using the following result.

Lemma 4.2. Against the strategy $v=\phi^*$, all L^2 controls $u(\cdot)$ that ensure capture (29) result in property (31).

Proof. We now consider the one-player system

$$\dot{x} = EB^{-1}E'X^{-1}x+Gu$$

that we want to control to the origin. A difficulty comes from the fact that its matrix is unbounded in the neighborhood of t_1 and is undefined at t_1. Thus, we cannot use its transition matrix $\Psi(t_1, t)$. However, using the proof of Faurre (Ref. 6), which carries over unchanged to the linearly constrained final state case, we know that there exists a matrix $W(t)$ such that, for all capturing controls,

$$\min_{u(\cdot)}\int_t^{t_1}\|u\|_R^2\,ds = x'(t)W(t)x(t), \tag{34}$$

and that W satisfies the Riccati equation

$$\dot{W} = -WEB^{-1}E'X^{-1} - X^{-1}EB^{-1}E'W + WGR^{-1}G'W.$$

W is positive definite [since $u=0$ cannot cause (29) from $x_0 \neq 0$], and thus

$$(W^{-1})^{\cdot} = EB^{-1}E'X^{-1}W^{-1} + W^{-1}X^{-1}EB^{-1}E' - GR^{-1}G'. \tag{35}$$

From classical control theory, and using the fact that, as a product of positive semidefinite matrices, $EB^{-1}E'X^{-1}$ has all its eigenvalues unstable, we see easily that the coercivity constant $\nu(W)$ goes to infinity as t goes to t_1; therefore, $W^{-1}(t)$ can be extended to t_1, by posing

$$W^{-1}(t_1) = 0, \tag{36}$$

which together with (35) uniquely defines $W(t)$.

We now notice that (25) and (35)–(36) give

$$(X - W^{-1})^{\cdot} = -EB^{-1}E' + EB^{-1}E'X^{-1}(X - W^{-1}) + (X - W^{-1})X^{-1}EB^{-1}E'.$$

Therefore, for $t < s < t_1$,

$$X(t) - W^{-1}(t) - \Psi(t, s)[X(s) - W^{-1}(s)]\Psi'(t, s)$$
$$= \int_t^s \Psi(t, \sigma)E(\sigma)B^{-1}(\sigma)E'(\sigma)\Psi'(t, \sigma)\, d\sigma \geq 0. \tag{37}$$

From the nonnegativity of the eigenvalues of $EB^{-1}E'X^{-1}$ and Gronwall's inequality, it follows that $\Psi(t, s)$ remains bounded as s goes to t_1. And since

$$X(t_1) = W^{-1}(t_1) = 0,$$

(37) yields

$$X(t) - W^{-1}(t) \geq 0.$$

X and W being both positive definite, this implies that

$$X^{-1}(t) \leq W(t), \qquad \forall t < t_1;$$

and, using (29), we have, for all $t < t_1$,

$$0 \leq x'(t)X^{-1}(t)x(t) \leq x'(t)W(t)x(t) \leq \int_t^{t_1} \|u(s)\|_R^2\, ds.$$

Since $u(\cdot)$ is by hypothesis square integrable, the rightmost term goes to zero as t goes to t_1; therefore, (31) holds, which proves the lemma. □

Now, use $v = \psi^*$ and (31) in Eq. (28), and (33) follows.

Finally, if the problem is not normal, it must still be normalizable, with $x(t)$ in $\mathcal{R}(X(t))$ (see necessary condition). Then, X^+x coincides with $\tilde{X}^{-1}x$, where $\tilde{X}$ is the restriction of X to $\mathcal{R}(X)$. Therefore the theorem is proved for the simple game in the absence of a conjugate point (but with the final constraint).

4.2. Reduction of the General Game to the Simple Game. We give here a shortened treatment, that hides some underlying facts. See Ref. 3 for a more detailed account.

We consider anew the problem (1)–(6), with assumption (7). But up to and including Lemma 4.3, we are careful not to use hypothesis (8). With this problem, we associate the same problem, but without the final constraint. Let X^1, Y^1, P^1 be the matrices X, Y, P for this last problem. By continuity, X^1 is invertible in a left neighborhood of t_1; therefore, P^1 satisfies the Riccati equation (15) in that neighborhood, with

$$P^1(t_1) = A. \tag{38}$$

We now take that Riccati equation as the *definition* of P^1.

We make the classical change of control variables, possible as long as the free end-state problem has no conjugate point (P^1 defined):

$$u = \hat{u} - R^{-1}(G'P^1 + S')x, \qquad v = \hat{v} + B^{-1}(E'P^1 + T')x; \tag{39}$$

and, via the same type of calculation as we did to obtain (22), we have, as long as P^1 exists on $[t_0, t_1]$,

$$J(x_0, t_0; u(\cdot), v(\cdot)) = x_0'P^1(t_0)x_0 + \int_{t_0}^{t_1} (\|\hat{u}\|_R^2 - \|\hat{v}\|_B^2)\, dt, \tag{40}$$

the state being now governed by

$$\dot{x} = [(F - (GR^{-1}G' - EB^{-1}E')P^1 - GR^{-1}S' + EB^{-1}T']x + G\hat{u} + E\hat{v}. \tag{41}$$

The remarkable fact is that now (41) serves only to define the constraint (5) on $\hat{u}$ and that the criterion does not involve x any more (since we can ignore the first term, which depends only on the initial condition). We can therefore replace (41), (5) by any equivalent constraint. This is done using the transition matrix $\hat{\Phi}$ of (41), and the projection Π on $\mathcal{M}^\perp$, and using the new state

$$\tilde{x}(t) = \Pi\hat{\Phi}(t_1, t)x(t), \tag{42}$$

which is governed by the equation

$$\dot{\tilde{x}}(t) = \tilde{G}(t)\hat{u}(t) + \tilde{E}(t)\hat{v}(t), \tag{43}$$

$$\tilde{G}(t) = \Pi\hat{\Phi}(t_1, t)G(t), \qquad \tilde{E}(t) = \Pi\hat{\Phi}(t_1, t)E(t), \tag{44}$$

and the final constraint reads

$$\tilde{x}(t_1) = 0. \tag{45}$$

Now, (40), (43), (45) define a simple game; the only things that remain to be checked, in the case where P^1 and $\tilde{X}^{-1}$ (corresponding to this game) exist on $[t_0, t_1]$, are these: the known saddle point of this game translates back in φ^*, ψ^* as given by (22)–(23); the value is V as given by (24); and the necessary conditions on the simple game translate into the same on the general game.

We first establish the relations that link the various matrices involved. Introduce

$$X^0(t) = \int_t^{t_1} \hat{\Phi}(t_1, s)[G(s)R^{-1}G'(s) - E(s)B^{-1}E'(s)]\hat{\Phi}'(t_1, s)\,ds. \quad (46)$$

It is a simple matter of tracing back into the proper equations to check that

$$\tilde{X}(t) = \Pi X^0(t)\Pi', \quad (47)$$

$$X(t) = \hat{\Phi}(t, t_1)[X^0(t)(I - MM^\dagger) + MM^\dagger], \quad (48)$$

$$\Lambda(t) = \hat{\Phi}'(t_1, t)[I - MM^\dagger + P^1(t)X(t)]. \quad (49)$$

We now are in a position to prove the following important results.

Lemma 4.3. On an interval $[t_0, t_1]$ on which P and P^1 are bounded, the following results hold.

(i) A necessary and sufficient condition for the problem to have a nondegenerate saddle point is that it be normalizable, $x_0 \in \mathcal{R}(X(t_0))$, and $\tilde{X}(t) > 0$. In that case, the solution is given by (22)–(24). In the absence of terminal constraint, P and P^1 coincide, and their existence suffices as (40) shows, and it also ensures invertibility of X by Lemma 3.1.

(ii) For the normalized problem, if P is positive semidefinite, $\tilde{X}$ is positive definite; therefore, the saddle point exists for every initial point.

Proof. We first show that normalizability of the two problems is equivalent. On the one hand, (x, t) is controllable to $(\mathcal{M}, t_1)$ iff $(\tilde{x}, t)$ is controllable to $(0, t_1)$. On the other hand, in view of (47),

$$\tilde{x} \in \mathcal{R}(\tilde{X}) \Leftrightarrow \Pi\hat{\Phi}(t_1, t)x \in \mathcal{R}(\Pi X^0\Pi') \Leftrightarrow \hat{\Phi}(t_1, t)x \in \mathcal{R}(X^0\Pi'\Pi) + \mathcal{M}. \quad (50)$$

We also have the simple fact that

$$\mathcal{R}(X^0\Pi'\Pi) + \mathcal{M} = \mathcal{R}(X^0(I - MM^\dagger)) + \mathcal{R}(MM^\dagger) = \mathcal{R}(X^0(I - MM^\dagger) + MM^\dagger),$$

which in view of (43) and (45) gives

$$\tilde{x} \in \mathcal{R}(\tilde{X}) \Leftrightarrow x \in \mathcal{R}(X).$$

Since normalizability and normality only involve comparison of the subspace of controllable states with $\mathscr{R}(X)$, the first result is proved.

It turns out to be convenient to introduce for the normal game the matrix:

$$Z(t)=\hat{\Phi}'(t_1, t)\Pi'\tilde{X}^{-1}\Pi\hat{\Phi}(t_1, t). \tag{51}$$

By differentiating with respect to time and checking the initial conditions, one can see that

$$(P^1(t)+Z(t))X(t)=\Lambda(t),$$

so that, when these matrices exist and X is invertible,

$$P(t)=P^1(t)+Z(t). \tag{52}$$

Placing back the solution of the simple game in $\tilde{x}$ in (34), and using (46), (47), (51), (52), one recovers (21)–(24). Again, if the system is not G-reduced, we can check that we can replace X^{-1} by $X^{\dagger}$. Therefore, using the equivalence between the two games and the results of Section 4.1, we have the first assertion of the lemma.

Assume now that $\tilde{X}$ is not positive definite in the neighborhood of t_1, but invertible. Then, $\tilde{X}^{-1}$ has eigenvalues that diverge to $-\infty$ as t goes to t_1. Therefore, for t large enough, there exist vectors ξ of unit norm such that $\xi'\tilde{X}^{-1}\xi$ is arbitrarily large negative. Since $\Pi\hat{\Phi}(t_1, t)$ is surjective, there exist vectors η of unit norm such that $\eta'Z\eta$ is arbitrarily large negative; thus, according to (52), $\eta'P\eta$ can be made negative. Therefore, if P is positive semidefinite in a left neighborhood of t_1, $\tilde{X}$ is positive definite in that neighborhood. But $\tilde{X}$ cannot become singular without P of P^1 diverging. Therefore, the second assertion of the lemma is proved. □

We now complete the proof of the main theorem, in the absence of a conjugate point, with the following lemma, which is the first place where we use assumption (8).

Lemma 4.4. Under assumption (8) for a normal problem, the following results hold.

(i) On any interval on which P is positive semidefinite and bounded, P^1 exists.

(ii) If, in a neighborhood of t_1, where P^1 exists, $\tilde{X}$ is positive definite, P is positive semidefinite on any interval $[t_0, t_1]$ over which it is bounded.

Proof. (i) We introduce the solution $P_*(t)$ of the Riccati equation of the free end-state *control* problem, i.e., Eq. (15) with the term $(PE+T)B^{-1}(E'P+T')$ deleted, and initialized as P^1, i.e.,

$$P_*(t_1)=A.$$

We know that P_* exists for all $t < t_1$ and (see Ref. 11)

$$0 \le P_*(t) \le P^1(t) \tag{53}$$

Now, if P is positive semidefinite, $\tilde{X}$ is positive definite in a neighborhood of t_1; therefore, in this neighborhood, according to (51)–(53);

$$0 \le P^1(t) \le P(t).$$

We have seen that, if P remains bounded, $\tilde{X}$ remains positive definite; therefore, P^1 cannot diverge without P doing so.

(ii) If $\tilde{X}$ is positive definite, according to (51)–(53) again, P is positive definite, and we have just seen that then, as long as it bounded, this situation prevails. □

Therefore, the condition $P \ge 0$, instead of P^1 exists, and $\tilde{X}$ positive definite is sufficient because of Lemmas 4.3 and 4.4, and necessary because of Lemmas 4.3 and 4.4, and the fact that there always exists a neighborhood of t_1 where P^1 exists.

5. Proof of the Main Theorem: Conjugate Point

We shall make use of the following fact (see Ref. 7).

Proposition 5.1. At any time $t_2 < t_1$, the matrices $X(t_2)$, $\Lambda(t_2)$ which are solutions of (12)–(13) can be used to initialize the canonical equations of the game with the same dynamics and integral payoff, final time t_2, target set $\mathcal{R}(X(t_2))$, and final cost $x'(t_2)P(t_2)x(t_2)$.

Notice that this proposition allows us to extend our solution beyond an instant t_2 at which $\mathcal{W}$ changes dimension, by concatenating the two games, after t_2 and before t_2.

5.1. Extension of the Solution to t_1^*. We again assume that the system has been G-reduced, and consider the case where $t_0 = t_1^*$ is the first rear conjugate point of t_1. Let

$$\mathcal{R}(X(t_1^*)) = \mathcal{M}^*.$$

By definition,

$$\mathcal{M}^* \ne R^n.$$

We assume that the necessary conditions of the previous paragraph hold over $[t, t_1]$, for any $t > t_1^*$.

Assume first that $x_0 \in \mathcal{M}^*$. We consider a new game (referred to as J^-), which is in fact the present one with time reversed, in the neighborhood of t_1^*. Let τ be its time:

$$dx/d\tau = -Fx - Gu - Ev,$$

$$J^- = -x'(\tau_1)P(t_1^*)x(\tau_1) + \int_{\tau_0}^{\tau_1} L(x, u, v)\, d\tau \qquad \text{if } x(\tau_1) \in \mathcal{M}^*,$$

$$J^- = -\infty \qquad \text{if } x(\tau_1) \neq \mathcal{M}^*,$$

where the matrices F, G, E, and the integrand $L(x, u, v)$ are the same as in (1), (4), evaluated at $t = t_2 - \tau$. t_2 is a time larger than t_1^*, that we shall choose later. Let

$$\tau_0 = 0, \qquad x(\tau_0) = x_2, \qquad \tau_1 = t_2 - t_1^*.$$

This new game is similar to the first one, except for the fact that the roles of $\mathcal{U}$ and $\mathcal{V}$ have been reversed. The assumption corresponding to (8) is not satisfied either. We chose t_2 in such a way that the matrix P^1 corresponding to this new game exists on (τ_0, τ_1). It is straightforward to write the canonical equations of this game, that can be initialized with $X(t_1^*)$ and $-\Lambda(t_1^*)$. One sees immediately that their solution is $X^-(\tau)$, $\Lambda^-(\tau)$:

$$X^-(\tau) = X(t_2 - \tau), \qquad \Lambda^-(\tau) = -\Lambda(t_2 - \tau), \qquad P^-(\tau) = -P(t_2 - \tau).$$

We can apply Lemma 4.3 to this game (with signs suitably reversed), and we see that

$$\varphi^-(x, \tau) = -R^{-1}(-G'P^- + S')x = \varphi^*(x, t_2 - \tau),$$
$$\psi(x, t) = B^{-1}(-E'P^- + T')x = \psi^*(x, t_2 - \tau)$$

are a saddle point. In particular, ψ^- induces the capture against any $u(\cdot)$, and produces trajectories through any point of $\mathcal{M}^*$, showing that ψ^* is an admissible strategy from $x_0 \in \mathcal{M}^*$.

Consider a control $v(\cdot)$ applied to the game J^-. According to Lemma 3.2, there exists a trajectory of (1) generated by (φ^*, v) through x_0. Let $x_2 = x(t_2)$ on this trajectory. Applying the above construction with that x_2, we obtain:

$$J^-(x_0, \tau_0; \varphi^*, v(\cdot)) \leq -x_2'P(t_2)x_2.$$

Thus,

$$\int_{t_1^*}^{t_2} L(x, \varphi^*, v)\, dt \leq x_0'P(t_1^*)x_0 - x_2'P(t_2)x_2.$$

Still playing $u=\varphi^*$ from t_2 to t_1, using the results of Section 4, and adding the saddle-point inequality to the previous one, we get

$$J(x_0, t_1^*; \varphi^*, v(\cdot)) \leq x_0' P(t_1^*) x_0.$$

We proceed in the same way to obtain the other inequality of the saddle point.

Consider now the case $x_0 \notin \mathcal{M}^*$. Assume that there exists an optimal strategy φ^0 that ensures a finite cost against any admissible control $v(\cdot)$. Consider for $v(\cdot)$, the strategy

$$v=0 \quad \text{on } (t_1^*, t_1^*+\epsilon), \qquad v=\psi^* \quad \text{on } [t_1^*+\epsilon, t_1].$$

The switch occurs at a point x_ϵ that goes to x_0 as ϵ goes to zero. However, we have the following lemma.

Lemma 5.1. On any sequence (x_ϵ, t_ϵ) converging to (x_0, t_1^*), with $t_\epsilon > t_1^*$, $x_0 \notin \mathcal{M}^*$, one has

$$x_\epsilon' P(t_\epsilon) x_\epsilon \to \infty$$

Since, with the strategy that we have proposed, the total cost is larger or equal to $x_\epsilon' P(t_\epsilon) x_\epsilon$, by choosing ϵ sufficiently small, $\mathcal{V}$ can make the cost arbitrarily large.

Therefore, the main theorem is proved for the case $t_0 = t_1^*$.

5.2. Extension beyond t_1^*. Assume now that $t_0 < t_1^*$. It is still possible that $P(t) \geq 0$ on $[t_0, t_1]$. When this is the case, t_1^* will be called an *even* conjugate point. In that case, the game with same dynamics, same integral part of the payoff, final time t_1^*, target set $\mathcal{M}^*$, and final cost $x'(t_1^*)P(t_1^*)x(t_1^*)$ [and $+\infty$ if $x(t_1^*) \notin \mathcal{M}^*$, i.e., still $V(x(t_1^*), t_1^*)$] has a nondegenerate saddle point given by the same formulas (thanks to the proposition above). Therefore, the global game has a saddle point. But if P fails to be positive semidefinite for $t \leq t_1^*$, we know that the game ending at t_1^* has an infinite value. The theorem is now completely proved.

Example 5.1. The following game has a saddle point that *survives* a conjugate point. All variables are scalar, and there is no terminal constraint.

$$\dot{x} = (2-t)u + tv, \qquad \mathcal{M} = R,$$

$$J = \tfrac{1}{2}x(2)^2 + \int_0^2 (u^2 - v^2)\, dt.$$

We leave to the reader to check that $t=1$ is an even conjugate point:

$$P(t) = 1/2(1-t)^2.$$

Remark 5.1. It is possible that, in a given game, our sufficient condition will be satisfied and that, still, there exists a strategy $\hat{\psi}$ such that, against any L^2 control $u(\cdot)$, we are ensured that $x(t_1) \notin \mathcal{M}$, or $x(t_1^*) \notin \mathcal{M}^*$, an apparent paradox.

On the one hand, the pair $(\varphi^*, \hat{\psi})$ would generate controls $(u(\cdot), v(\cdot))$, none of which is L^2; therefore, from the condition (iii) on admissible strategy sets and the requirement that $\varphi^* \in \Phi$, it follows that $\hat{\psi} \notin \Psi$. But this is a rather arbitrary dictum, that lets one player play his capturing strategy, and not his opponent play his anticapturing one.

On the other hand, this set-up allowed us a clean theory, and is justified by the remark that Eq. (28) shows that a strategy pair $(\varphi^*, \hat{\psi})$ would cause the criterion to diverge to $-\infty$ as $t \uparrow t_1$ (or $t \uparrow t_1^*$), so that $\mathcal{V}$ would be driven out of the market before the game ends. We could decide a priori that, in such a case, $J = -\infty$, and relax the constraints on Ψ to include $\hat{\psi}$.

6. Conclusions

We have a complete theory with the positivity hypothesis (8), which corresponds to the worst-case design of a classical positive control problem. The two remarkable facts are that, if they exist at all, the optimal strategies are pure feedbacks, and that the saddle point can exist even in the presence of a conjugate point, provided it is even. It is easy to generalize everything to a nonhomogeneous problem or to a problem with intermediary costs $x'(t_i)A_i x(t_i) + 2a_i' x(t_i)$, or both (see Ref. 3).

Lemma 3.3 and the notion of even conjugate point give the basis for a theory without the positivity assumption. It would also be interesting to investigate what happens at an accumulation point of conjugate points, a situation that we have ruled out with the help of Definition 2.2 and then classified as nonnormalizable.

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TECHNICAL COMMENT

Linear-Quadratic Two-Person Zero-Sum Differential Games, Necessary and Sufficient Conditions: Comment

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Abstract. We give a simpler, easier-to-check, version of the theorem of the paper referred to, i.e., a necessary and sufficient condition for the existence of a saddle point to the linear-quadratic two-person zero-sum perfect information differential game.

Key Words. Linear differential games, conjugate points, saddle points.

According to the analysis of Ref. 1, it is clear that the subspace $W(t)$ to be taken into account for the definitions of both G-reducibility and normalizability is that of states u-controllable to $\mathcal{M}^* = \mathcal{R}(X(t^*))$ at the next conjugate point t^*. However, the conditions of the theorem are thus complicated to state and difficult to check, since this requires that conjugate points be recognized and analyzed and that the corresponding controllability matrix be computed.

It is possible to give a simpler form of these conditions.

Theorem. A necessary and sufficient condition for the existence of a nondegenerate saddle point to the given differential game is that:

(i) $x(t_0) \in \mathcal{R}(X(t_0))$;
(ii) rank $X(t)$ is piecewise constant;
(iii) $\mathcal{R}(E(t)) \subset \mathcal{R}(X(t))$, for all $t \in (t_0, t_1)$, except, at most, at isolated points in time;
(iv) $P(t) \geqslant 0$, for all $t \in (t_0, t_1)$.

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Proof. Necessity of (iii) follows, as in Lemma 4.1 of Ref. 1, from the fact that, otherwise, no strictly causal strategy can ensure (i) for $t > t_0$, a necessary condition. Strictly causal strategies have been defined in such a way that, if φ holds $C(x, t, v) = 0$, for all $v(\cdot)$, then necessarily $\partial C/\partial v = 0$ along all trajectories.

Sufficiency will first be proved on the simple game. Again along the lines of Lemma 4.1, it follows from (iv) and (iii) that $\mathcal{R}(G(t)) \subset \mathcal{R}(X(t))$, and thus that, on any interval where rank X is constant, $\mathcal{R}(X(t))$ is constant and contains the u-controllable and v-controllable state subspaces. But, as $\mathcal{R}(X(t))$ is trivially contained in the sum of the last two, it is equal to this sum. Moreover, $X(t) \geq 0$ immediately implies G-reducibility for the simple game, so that

$$\mathcal{R}(X(t)) = W(t).$$

We can then apply the earlier theorem.

Finally, translating (iii) for the simple game $\tilde{J}$ into the same condition for the general game J goes as we did for normalizability at the beginning of Lemma 4.3 of Ref. 1.

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