Singular Surfaces in Differential Games

An introduction

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Abstract.

We give a general set up and a version of Isaacs' Verification Theorem that allows us to deal with the various singularities we want to investigate. In particular, we are obliged to allow upper or lower strategies, leading to upper or lower saddle points, that may exist even if the Hamiltonian does not have a saddle point. It is shown that this is needed even for separated games. Then we give a general study of junctions of optimal fields with singular surfaces, which requires a special investigation of the situation where this junction is tangential extending Carathéodory's General Envelope Theorem. We then proceed to study special singular surfaces, and we end up with an example which shows how a state constraint may appear in the interior of the game space of a separated problem posed with no such constraint to start with.

Introduction

It can well be said that Isaacs founding work on two person, zero sum, Differential Games, [1] is mainly a study of singular surfaces (together with the fundamentals of Hamilton Jacobi theory). While this topic was investigated further by J.V. Breakwell and his students in particular among other works, see [2], most of the following work has been in the area of existence theory, by refining the concept of strategy (See, e.g. [3] to [11].) While this later work is highly significant, and of relevance to the present one, we wish here to turn back to the topic of better understanding singular surfaces. For the sake of brevity, we shall omit topics related to existence theory, in particular the justification of our choice of upper and lower strategies, and the question of their approximation by simple strategies.

The main emphasis of this short course will be on giving a more rigorous treatment of long used practices, particularily in the case where optimal trajectories have an envelope, a situation that has been recognized early by J.V Breakwell in the study of particular games [12], [13]. In the process, we hope to contribute to a unification, thus a simplification, of the whole topic.

In part I, we shall give the general set up we use, and the relevant Hamilton Jacobi Isaacs equation for this set up. In part 2 we give the fundamental lemmas that allow us to deal with the envelope situation, and general results on junctions of an optimal field with a singular surface. In part 3, we investigate various kinds of singularities. Part 4 gives a simple and interesting example : the One Dimensional Second Order Servomechanism Problem.

1. General set up

1.1. The Game

We shall consider a two player dynamical system governed by the differential equation

(1) $\dot{x} = f(x, u, v)$

where $\dot{x}$ means dx/dt, $t \in R^+$ is the time,

$x(t)$ is the state, $x(t) \in X \subset R^n$
$u(t)$ is the first player's control, $u(t) \in \mathcal{U} \subset R^l$, $\mathcal{U}$ closed,
$v(t)$ is the second player's control, $v(t) \in \mathcal{V} \subset R^m$, $\mathcal{V}$ closed.

Admissible control functions are

$u(.) \in \Omega_u$ = piecewise continous functions fromm R^+ into $\mathcal{U}$
$v(.) \in \Omega_v$ = piecewise continous functions from R^+ into $\mathcal{V}$

$f(.,.,.)$ is a C^2 function fromm $R^n \times R^l \times R^m$ into R^n. We shall assume that f is such that for every pair of admissible controls $u(.)$, $v(.)$, (1) has a bounded solution on every finite interval of the positive real line from any intial state x_o in X. Some further assumptions on f, U and V will be made in a moment.

Notice that we have taken an autonomous system. It is a well known fact that it is always possible to do so, if necessary by having the last component of x have a constant unit time derivative, and thus be equal to time. This also allows us to take, thereafter, an autonomous definition of payoff and strategies, and to assume that the game always begins at time zero.

We could have generalized slightly by allowing measurable control functions, or, more significantly, by allowing $\mathcal{U}$ and $\mathcal{V}$ to vary with time alone, or with all of x in an upper semi-continuous fashion. We avoid it here for the sake of simplicity. However, since x may contain time as one component, taking X fixed as we shall do does not imply that the actual capture set (or capture zone boundaries such as barriers) is (are) fixed.

The <u>playing space</u>. X will be assumed to be a closed subset of R^n, with non empty interior, locally on one side of its C^2 boundary ∂X.

<u>Final time</u> t_1 is the last instant of time before $x(t)$ leaves X:

$$t_1 = \sup \{ t \in R \mid \forall \tau \in [0,t],\ x(\tau) \in X \}.$$

t_1 is a function of initial state and of the chosen control functions $u(.)$ and $v(.)$. It may be infinite.

A <u>pay off</u> is associated to each initial state and pair of control functions :

$$J(x_o\ ;\ u(.),\ v(.)) = K(x(t_1)) + \int_0^{t_1} L(x,\ u,\ v)\,dt \quad \text{if } t_1 < \infty$$

$$J(x_o\ ;\ u(.),\ v(.)) = +\infty \qquad\qquad \text{if } t_1 = \infty$$

(Therefore we arbitrarily decided that the minimizing player wants the game to terminate).

We shall assume that, for all x in X, and v in $\mathcal{V}$, the set $(L(x, \mathcal{u}, v), f(x, \mathcal{u}, v))$ is convex and bounded in R^{n+1}, and similarily in $\mathcal{V}$. In fact, the only property we need is that for relevant vectors λ or R^n, $L + \lambda' f$ have a unique minimum in u or maximum in v.

$K(x)$ is a function from R^n into $R \cup \{-\infty, +\infty\}$. That is, a part ∂X_u of ∂X may exist such that

$$J = K(x(t_1)) = +\infty \qquad \text{if} \quad x(t_1) \in \partial X_u$$

and similary a part ∂X_v where J is equal to $-\infty$ if the game terminates there. We say that ∂X_u defines a state constraint under the first players responsability and similarily for ∂X_v. $K(.)$ is assumed to be of class C^2 in the interior of the region where it is finite.

The first player, or Pursuer P wants to minimize J while the second player, or Evader E wants to maximize it. However this statement must now be made more precise by specifying the information available to the players in making their choice (strategy concept) and the solution sought.

1.2. Strategies and saddle points.

Using an idea of Varaiya and Roxin [5], [6], [7], and a special form of Isaacs tenet of transition (which he stated as early as 1952 in Rand seminars), one can justify the following definitions, that we shall take here as part of the statement of the game.

A u - discriminating strategy or u - D - strategy, (the need for this type of strategy was probably first seen by J.V. Breakwell [14]) is an application $\varphi \in \Phi^-$ from $X \times \mathcal{V}$ into $\mathcal{u}$:

$$u = \varphi(x, v) \tag{3a}$$

such that for every admissible control function $v(.) \in \Omega_v$, the differential equation

$$\dot{x} = f(x, \varphi(x, v), v) \qquad x(o) = x_o , \tag{4a}$$

has a unique solution for every initial state x_o in X, in the precise following meaning : there exists an absolutely continuous function $x(t)$ satisfying (4a) for each t for which $x(.)$

is differentiable (i.e. allmost all t), that together with $v(.)$ generates via (3a) an admissible control function $u(.) \in \Omega_u$. This defines the set Φ^- of admissible u-D-strategies.

A <u>v-ordinary strategy</u> (or v-strategy) is an application $\psi \in \Psi$ from X into $\mathcal{U}$:

(3b) $$v = \psi(x)$$

such that for every admissible control function $u(.) \in \Omega_u$, the differential equation

$$\dot{x} = f(x, u, \psi(x)) , \qquad x(0) = x_o ,$$

has a unique solution for every initial state x_o in X, with the same definition as above, that, together with $u(.)$, generates via (3b) an admissible control function $v(.) \in \Omega_v$. This defines the set Ψ of admissible v-strategies.

A <u>lower saddle point</u> is a pair (φ^-, ψ^-) of admissible u-D-strategy and v-strategy such that

i) the differential equation

(4c) $$\dot{x} = f(x, \varphi^-(x, \psi^-(x)), \psi^-(x)), \qquad x(0) = x_o,$$

has a solution generating admissible control functions $u(.)$ and $v(.)$. (It suffices to assume that one is admissible, because then the other is such).

ii) for every initial state $x_o \in X$, there exists a number $V^-(x_o)$ such that, for every admissible control functions $u(.) \in \Omega_u$ and $v(.) \in \Omega_v$,

(5a) $$J(x_o ; \varphi^-, v(.)) \leq V^-(x_o) \leq J (x_o ; u(.), \psi^-) .$$

The notations $\hat{J}(x_o ; \varphi^-, v(.))$ and $J(x_o ; u(.), \psi^-)$ having an obvious non ambiguous meaning.

Let $u^-(.)$ and $v^-(.)$ be the control functions generated by a solution of (4c) then necessarily

(5b) $$J(x_o ; u^-(.), v^-(.)) = V^-(x_o)$$

because

$$J(x_o ; u^-(.), v^-(.)) = J(x_o ; \varphi^-, v^-) = J(x_o ; u^-, \psi^-),$$

which together with (5a) implies (5b).

V^- is called the lower value, or lower value function, of the game. Our definition (5a) of a lower saddle point seems restrictive in that it requires comparison controls to be open loop. However, as was pointed out by Berkovitz [15], this is not so since for any closed loop v-strategy ψ, if the pair (φ^-, ψ) generates a solution $x(t)$ admissible in the sense that $\psi(x(.))$ is admissible, then letting $v(.) = \psi(x(.))$ gives the same payoff and allows us to use (5a) to evaluate this payoff. On the other hand, our definition avoids some difficult problems of play ability which may end up in the fact that different saddle points exist for the same game, with different values. [16].
Notice also that since $x(t)$ may have the time as one component our strategies include open loop controls.
We similary define a u-ordinary strategy as an application $\varphi \in \Phi$ of X into $\mathcal{U}$ and a $\mathcal{V}$- D- strategy as an application $\psi \in \Psi^+$ from $X \times \mathcal{U}$ into $\mathcal{V}$. An upper saddle point is a pair $(\varphi^+, \psi^+) \in \Phi \times \Psi^+$, again such that the corresponding differential equation has a solution generating admissible control functions, and such that for any admissible control functions $u(.)$ and $v(.)$, the inequalities (5a) are satisfied with φ^-, ψ^- and V^- replaced by φ^+, ψ^+ and V^+. The upper value V^+ satisfies the equivalent of equality (5b).

We define an ordinary saddle point, or daddle point, as a pair $(\varphi^o, \psi^o) \in \Phi \times \Psi$ of admissible ordinary strategies, with the same properties as before, and such that (5a) hold with φ^o, ψ^o, and V in place of φ^-, ψ^- and V^-.
In most examples, the game shall be "separated", i.e., we shall have

$$f(x, u, v) = g(x, u) + h(x, v)$$

$$L(x, u, v) = M(x, u) + N(x, v)$$

In these cases, the hamiltonian (that we introduce below) has a saddle point in (u, v), and it turns out that, except on the singular surfaces we want to investigate, the optimal D-strategies wont use the extra information allowed on the opponents current control, but will only use the current state. Moreover, on the singular surfaces, only one of the players will need this extra information in order for us to be able to exhibit a solution of the game.

For this reason, it shall then make sense to introduce, without further precautions pertaining to existence, the concept of a D-saddle point where both players are allowed to use D-strategies. A D-saddle point shall be made of the concatenation (in time) of upper and lower strategy pairs, admissible in the same sense as previously, leading to inequalities of the type (5a).

Of course, we do not imply that all these saddle points, or any of them, exist. However, we shall investigate the case where one exists, since we use a theory of sufficiency conditions, the theory of necessary conditions being extremely involved and closely linked to existence theory.

1.3 Isaacs Main equation.

We shall now adapt Isaacs Verification Theorem [1] to the case of a lower saddle point with discontinuities of the Value function. We introduce a function $V^-(x)$ which is allowed to have discontinuities of a simple kind. (We shall relax this later on). We assume that there exists a partition of X by C^2 n-1-dimensional manifolds, such that

i) $V^-(x)$ is of class C^2 in the interior of each region

ii) its restrictions to these manifolds is C^2

iii) $V(x)$ is continuous and continuously differentiable on at least one side of each manifold, which means that in at least one of the regions, V^- coïncides with a continuously differentiable function defined on an open set containing the manifold.

iv) Upon leaving one of these manifolds on a discontinuous side, V^- has a simple jump. We will refer to discontinuity manifolds as being of positive or negative jump according to the sign of this jump (Upon reaching such a manifold on the discontinuous side, the jump is of opposite sign).

Remark that for the following theorem, it suffices to assume for $V^-(x)$ the regularity C^1 where we have assumed C^2. However, since we shall construct solutions making use of the characteristics of Isaacs equation (his retrograd path equations or Euler

Lagrange equations) we shall not make use of any more generality. This is not so for the point iii) which is precisely the hypothesis we even want to relax further later on.

Notice also that at the intersection of discontinuity manifolds there exists manifolds of lower dimension where both positive and negative jumps may occur. In our local investigation of discontinuity manifolds, we shall not consider these higher order singularities. For the global theorem we state in this paragraph, this only translates into intersections of the set we now proceed to define, and causes no special problem.

For each $x \in X$ and $v \in \mathcal{V}$, let $\tilde{\mathcal{U}}_v(x)$ be the set of controls u that prevent a positve jump, that is

$\tilde{\mathcal{U}}_v(x) = \{u \in \mathcal{U} | \nu'(x)f(x, u, v) \leq 0\}$ if $x \in$ positive jump manifold, with normal $\nu(x)$ pointing toward the discontinuity.

$\tilde{\mathcal{U}}_v(x) = \mathcal{U}$ otherwise.

Similarily, let $\tilde{\mathcal{V}}(x)$ be the set of controls $v \in \mathcal{V}$ that prevent a negativd jump <u>for all</u> $u \in \mathcal{U}$.

We shall hereafter assume that the property $u(t) \in \tilde{\mathcal{U}}_{v(t)}(x(t))$ for all t implies that the trajectory does not leave a positive jump discontinuity manifold on the discontinuous side. This is not rigorously true, some more care is required, but we shall not go into this question in any more detail. Similarily for $v \in \tilde{\mathcal{V}}(x)$ and negative jump manifolds.

In order to state our theorem, we introduce the hamiltonian of the game

$$H(x, \lambda, u, v) = L(x, u, v) + \lambda' f(x, u, v)$$

which is a function from $X \times R^n \times \mathcal{U} \times \mathcal{V}$ into R.

THEOREM 1. Assume there exists a function $V^-(x)$ defined over X, with the regularity described above, and a lower strategy pair $(\varphi^-, \psi^-) \in \Phi^- \times \Psi$ such that

i) for every admissible control function $v(.) \in \Omega_v$, trajectories generated by $(\varphi^-, v(.))$ never reach a negative jump manifold from the discontinuous side, and

(6a) $$\forall x \in X, \quad \forall v \in \mathcal{V}, \quad \varphi^-(x, v) \in \tilde{\mathcal{U}}_v(x)$$

ii) for every initial state x_0 in X and every admissible control function $v(.) \in \Omega_v$, the game terminates at a finite time t_1 (1).

iii) for every admissible control function $u(.) \in \Omega_u$, trajectories generated by $(u(.), \psi^-)$ never reach a positive jump manifold from the discontinuous side, and

(6b) $$\forall x \in X, \quad \psi^-(x) \in \tilde{\mathcal{V}}(x)$$

iv) ∂X is treated as a (possible) discontinuity manifold, with $V^-(x) = K(x)$ in the exterior of X (Thus ∂X is a positive jump manifold where $V^-(x) < K(x)$, and conversely). There necessarily exists a region of ∂X, called the usable part, where $V^-(x) = K(x)$, otherwise ii) could not hold for the trajectory φ^-, ψ^-.

v) The following relations hold everywhere in X :

(7a) $$H(x, \frac{\partial V^-}{\partial x}(x), \varphi^-(x, \psi^-(x)), \psi^-(x)) = 0,$$

(7b) $$\forall v \in \tilde{\mathcal{V}}(x), \qquad H(x, \frac{\partial V^-}{\partial x}, \varphi^-(x, v), v) \leq 0,$$

(7c) $$\forall u \in \tilde{\mathcal{U}}_{\psi^-(x)}, \quad H(x, \frac{\partial V^-}{\partial x}, u, \psi^-(x)) \geq 0.$$

Then, (φ^-, ψ^-) is a lower saddle point and $V^-(x)$ the associated lower value.

PROOF

Assumption i) and iii) imply that a trajectory generated by (φ^-, ψ^-) has no jump in $V(x(t))$. Therefore, noticing that

$$H(x, \frac{\partial V^-}{\partial x}, u, v) = \frac{dV^-}{dt}(x, u, v) + L(x, u, v)$$

where the total time derivative is taken along the trajectory, (7a)

(1) We assume that the game of kind has been dealt with before and that X is the capture region from where P is able to force the game to terminate. ∂X may contain barriers, that form part of ∂X_u.

yields for a trajectory generated by (φ^-, ψ^-)

$$V^-(x(t_1)) + \int_0^{t_1} L(x, u, v)\, dt = V^-(x_0)$$

(Because of ii), t_1 exists). Because of iv) and relations (6);

$$V^-(x(t_1)) = K(x(t_1))$$

so that the previous relation yields

$$J(x_0 \,;\, \varphi^-, \psi^-) = V^-(x_0)$$

which is relation (5b).

Now, consider an arbitrary admissible control function $v(.)$, and the trajectory generated, from a prescribed initial phase x_0, by the pair $(\varphi^-, v(.))$. By assumption it terminates at a finite time that we again note t_1. Because of i), any jump in $V^-(x(t))$ will be negative. However, as V^- remains finite, there shall be at most countably many such jumps. At these instants, $v \notin \tilde{V}(x)$. However, (7b) will still hold for almost all t, or :

$$\frac{dV}{dt}(x, \varphi^-, v, t) + L(x, \varphi^-, v, t) \leq 0 \quad \text{almost all } t.$$

We can therefore integrate, and we get (from the sign of the jumps) :

$$V^-(x(t_1)) - V^-(x_0) \leq \int_0^{t_1} \frac{dV^-}{dt} \leq - \int_0^{t_1} L(x, u, v)\ dt$$

Now, from iv) and (6 a) results that

$$V^-(x(t_1)) \leq K\,(x(t_1))$$

so that we get

$$J(x_0 \,;\, \varphi^-, v(.)) \leq V^-(x_0)$$

Finally, consider an arbitrary admissible control function $u(.)$, and the trajectory generated by $(u(.), \psi^-)$. Either it does not terminate, then $J = +\infty$, either it does terminate and we have a similar argument using (7c). In both cases we conclude that

$$J(x_0 \,;\, u(.), \psi^-) \geq V^-(x_0)$$

and the proof is complete, since the last two in equalities are identical to (5a).

REMARK. While (7a) and (7b) imply that $\psi^-(x)$ yields the maximum over $\tilde{\mathcal{V}}(x)$ of $H(x, V_x^-, \varphi^-(x, v), v)$, (7c) does not imply that $\varphi^-(x, v)$ yields the minimum over $\tilde{u}_v(x)$ of $H(x, V_x^-, u, v)$ for all v, but only for $v = \psi^-(x)$. However, since

$$\min_{u\in\tilde{u}_v} H(x, \frac{\partial V^-}{\partial x}, u, v) \leq H(x, \frac{\partial V^-}{\partial x}, \varphi^-(x, v), v) \leq 0$$

The argument of the minimum then satisfies all the hypothesis in the theorem, except perhaps ii). In any event, condition v) implies

$$\max_{v\in\tilde{\mathcal{V}}(x)} \min_{u\in\tilde{u}_v(x)} H(x, \frac{\partial V^-}{\partial x}, u, v) = H(x, \frac{\partial V^-}{\partial x}, \varphi^-, \psi^-) = 0$$

The rather complicated set up used here is devised to allow an optimal trajectory to reach the boundary of the capture set, for instance, or another particular manifold such as a barrier, and possibly stay on it for a while. It also takes care, through the requirement (6a) of the case where the game starts from a barrier limiting the capture zone and where by playing the barrier strategy, E may prevent P from using his ordinary saddle point strategy (Breakwell's lunge maneuver. See [17],)

We just noticed that we dot need that H have a saddle point. Our examples shall be <u>separated games</u>, i.e. games where f and L are such that we have

$$H(x, \lambda, u, v) = H_P(x, \lambda, u) + H_E(x, \lambda, v)$$

For such games as we said , H as a saddle point and optimal D-strategies turn out to be ordinary strategies, except, and this is very important, on discontinuity manifolds because the requirement (6a) couples u and v. It is why separated games may not have ordinary saddle points. The example of part 5 is an instance of this fact. (See [18], [19] for special uses of upper saddle point).

2. Junction with a singular surface.

We shall show later on that in general, when an optimal trajectory reaches a singular manifold, the situation we wish to allow, it does so tangentially. For reasons we shall make clear in a moment most of the classical litterature rules out this situation. The aim of this part is to show that we can deal with this case as with the non tangential case.

2.1. Non differentiability of a continuous V

The following developpement holds for any kind of saddle point. We note (φ^*, ψ^*) the optimal strategies, and $V(x)$ the corresponding value, and to investigate these junctions."

$$f^*(x) = f(x, \varphi^*, \psi^*), \qquad L^*(x) = L(x, \varphi^*, \psi^*)$$

Let S be a n-1-dimensional manifold locally parametrised by the C^2 map

$$x = \xi(s) , \qquad s \in \theta \subset R^{n-1}$$

By assumption, the restriction of $V(x)$ to S is a C^2 function :

$$V(\xi(s)) = U(s)$$

A field of optimal trajectories reaches S. Let $t = \tau(s)$ be the time at which the trajectory through $\xi(s)$ reaches S.
We have, along that trajectory, and for $t < \tau(s)$.

(8) $$x(t) = \xi(s) + \int_{\tau(s)}^{t} f^*(x(\alpha))d\alpha = y(s,t)$$

By assumption, this field is regular, in the precise meaning that $(\partial y/\partial s, \partial y/\partial t)$ exists is bounded in the closed half space considered, and is invertible in the open half space (to allow a tangent contact). Notice that :

$$\frac{\partial y}{\partial t} = f^*(y)$$

Finally, the optimal trajectories define a value function by

(9) $$V(y(s,t)) = U(s) + \int_{t}^{\tau(s)} L^*(y(s,t))\, dt = W(s,t)$$

This last relation gives, in matrix notations

(10) $$\frac{\partial V}{\partial x} \left(\frac{\partial y}{\partial s} \frac{\partial y}{\partial t}\right) = \left(\frac{\partial W}{\partial s} \frac{\partial W}{\partial t}\right).$$

Therefore, at every point where the inverse exists

$$\frac{\partial V}{\partial x} = \left(\frac{\partial W}{\partial s}\ \frac{\partial W}{\partial t}\right)\left(\frac{\partial y}{\partial s}\ \frac{\partial y}{\partial t}\right)^{-1}$$

However, if the optimal trajectories reach S tangentially, this implies that $\partial y/\partial t$, which is equal to f^*, is linearly dependent with $\partial \xi/\partial s$ which is, by definition, the set of (column) vectors generating the tangent plane to S. In this case, $\partial V/\partial x$ need not exist, and usually does not.

It is interesting to see what (10) gives at $t = \tau(s)$ when $\partial V/\partial x$ exists there. Assuming f^* defines a regular field on S (i.e., $\partial f^*/\partial s$ exists), we differentiate (8) and (9) partially at $t = \tau(s)$ it comes, usign $y(s, \tau(s)) = \xi(s)$:

$$\frac{\partial y}{\partial s}(s, \tau(s)) = \frac{\partial \xi}{\partial s} - f^*(\xi(s))$$

$$\frac{\partial W}{\partial s}(s, \tau(s)) = \frac{\partial U}{\partial s} + L^*(\xi(s))$$

Now, the last column of (10) gives (for all t)

$$\frac{\partial V}{\partial x} f^*(x) + L^*(x) = 0$$

(which is (7a)), and this together with the first block column of (10) gives

$$\text{(11a)} \qquad \frac{\partial V}{\partial x}(\xi(s))\frac{\partial \xi}{\partial s} = \frac{\partial U}{\partial s}$$

which is the classical fact that the gradient of V has its projection on the tangent plane to S equal to the gradient of the restriction U of V to S.

2.2. The envelope lemma.

We now consider the case of the tangent field. S is then the enveloppe of this field. This means that along a trajectory, φ^* and ϕ^* are continuous functions of time with, $f^*(y(s, \tau(s))) \in$ tangent plane to S. We make an assumption of regularity on the field near the contact :

ASSUMPTION. The direction f^* of the optimal trajectories has, as a function of s and t, a continuous partial derivative $\partial f^*/\partial s$ in the closed half space.

REMARK We specifically avoid to assume that $\partial f^*/\partial x$ exists and is continuous, since φ^* and ψ^* usually depend on the gradient V_x. Our assumption is that the field of optimal directions in S is regular, and varies smoothly in the neighborhood of S.

LEMMA. Under the above conditions, the gradient of V has a limit λ_o ax $x \to S$ on an optimal trajectory, and this limit satisfies the relation

$$\lambda_o' \frac{\partial \xi}{\partial s} = \frac{\partial U}{\partial s} \tag{11b}$$

PROOF. In the open half space, $\tilde{u}_\psi = u$ and $\tilde{v} = v$. Thus, using standard techniques of control theory, we have

$$\frac{\partial \overline{H}}{\partial x} = \frac{\partial H}{\partial x}\left(x, \frac{\partial v}{\partial x}, \varphi^*(x), \psi^*(x)\right),$$

where $\overline{H}(x, \lambda) = H(x, \lambda, \varphi^*(x), \psi^*(x))$.

Therefore, if λ has a limit as $x \to S$, so does $\partial \overline{H}/\partial x$.

In the open half space, let $\lambda = (\partial V/\partial x)'$ be the gradient of V. V being assumed to be of class C^2, λ satisfies the Euler Lagrange equation :

$$\dot{\lambda}' = -\frac{\partial \overline{H}}{\partial x} = -\frac{\partial H}{\partial x} = -\lambda' \frac{\partial f}{\partial x}(x, \varphi^*, \psi^*) - \frac{\partial L}{\partial x}(x, \varphi^*, \psi^*).$$

This is a linear differential equation in λ with bounded coefficients. Therefore λ remains finite as $t \to \tau(s)$, and has a limit λ'_o given as a function of λ at a previous time t : by

$$\lambda'_o(s) = \lambda'(t) - \int_t^{\tau(s)} \frac{\partial H}{\partial x}(y(s, \alpha))\, d\alpha.$$

(with a transparent abuse of notation).

Consider a particular s and $t < \tau(s)$. Then by assumption $\partial f^*/\partial s$

exists from t to τ, and thus also $\partial L^*/\partial s$. Differentiate partially (8) and (9) and place in (10). The first block column yields :

$$\frac{\partial V}{\partial x}\left(\frac{\partial \xi}{\partial s} + \int_{\tau(s)}^{t} \frac{\partial f^*}{\partial s}(y(s, \alpha))d\alpha - f^*(\xi(s))\frac{d\tau}{ds}\right) - \frac{\partial U}{\partial s}$$
$$+ \int_{\tau(s)}^{t} \frac{\partial L^*}{\partial s}(y(s,\alpha))d\alpha - L^*(\xi(s)) \frac{d\tau}{ds} = 0.$$

As previously, we use tha last column of (10) to cancel the terms multiplying $d\tau/ds$, and using (12) again, we get

$$\frac{\partial V}{\partial x} \frac{\partial \xi}{\partial s} - \frac{\partial U}{\partial s} + \int_{\tau(s)}^{t} \frac{\partial H}{\partial x} \frac{\partial y}{\partial s} d\alpha = 0.$$

We take the limit as $t \uparrow \tau(s)$. By assumption the integrand remains bounded since

$$\frac{\partial y}{\partial s}(\alpha) = \frac{\partial y}{\partial s}(t) + \int_{t}^{\alpha} \frac{\partial f^*}{\partial s}(\beta)\, d\beta,$$

and we obtain the result sought.

COROLLARY. The theorem 1 still holds if V has the type of singularity we have described above.

PROOF. We just have to check that for all trajectories avoiding a jump,

$$V(x(t_1)) - V(x(t_o)) = \int_{t_o}^{t_1} \lambda'(t)\ dt,$$

where $\lambda'(t)$ is either $\partial V/\partial x$ where it exists, or λ'_o on an envelope. If an arc of trajectory lies on S, then this is a consequence of 11b). If an arc has a point $x(t_2)$ not belonging to S, then, by continuity, this is true in an open interval of time. If at an end point t_3 of this interval, $x(t_3)$ belongs to S, then

$$\forall\, t \in (t_2, t_3)\ , \quad V(x(t)) - V(x(t_2)) = \int_{t_2}^{t} \lambda'(t)\ dt$$

and as V is continuous at S, we take the limit of both sides and have the result.

Thus we have extended theorem 1 to a very common type of discontinuity, broadening the applicability of Hamilton Jacobi Isaacs theory, as compared to previous papers attempting to apply it to state constraints for instance.

2.3. Generalized Euler Lagrange equations.

We now need a differential equation for λ_0 along an otpimal trajectory lying on S. We have assumed that U(s) is of class C^2. We shall further assume that λ_0 is of class C^1 in s.

Let H_0 (resp $\overline{H}_0$) be H (resp $\overline{H}$) with λ replaced by λ_0. Since $\overline{H}(x) = 0$ for $t < \tau$, we have in the limit $\overline{H}_0 = 0$. Also, (12) gives, in the limit, differentiating $\overline{H}_0$ partially with respect to s_i :

$$\frac{\partial H_0}{\partial x}(x, \lambda_0, \varphi^*, \psi^*) \frac{\partial \xi}{\partial s_i} + \frac{\partial \lambda_0'}{\partial s_i} f^* = 0.$$

Now, along such a trajectory, we have

$$f^* = \frac{\partial \xi}{\partial s}\frac{ds}{dt} = \sum_{j=1}^{n-1} \frac{\partial \xi}{\partial s_j}\frac{ds_j}{dt} .$$

Further, from (11,b) we derive

$$\frac{\partial \lambda_0'}{\partial s_i}\frac{\partial \xi}{\partial s_j} = \frac{\partial^2 U}{\partial s_i \partial s_j} = \frac{\partial \lambda_0'}{\partial s_j}\frac{\partial \xi}{\partial s_i} .$$

Placing the last two equations in the previous one, and using the fact that the scalar ds_j/dt commutes with vectors, we obtain

(12a) $$\left(\frac{d\lambda_0'}{dt} + \frac{\partial H_0}{\partial x}\right)\frac{\partial \xi}{\partial s_i} = 0, \qquad \forall i.$$

This is a first form of the relation sought. Noticing that $\partial\xi/\partial s_i$ generate the tangent plane to S, we can rewrite it in the following form, where $\nu(\xi)$ is a normal to S and $\alpha(t)$ an unknown scalar function :

(12b) $$\dot{\lambda}_0' = -\frac{\partial H_0}{\partial x} + \alpha\nu$$

If S is given by an equation $S(x) = 0$, then one possible choice of $\nu(x)$ is $\partial S/\partial x$, and (12b) has a familiar form.

Remark that equations (12) have been established only as a consequence of (7a) and (11b). If a field of optimal trajectories reaches S transversally and then follows S, (with a discontinuity of f^* upon reaching S), then $\partial V/\partial x$ is defined in the closed half space, (11a) replaces (11b), and our generalized Euler Lagrange equations hold.

2.4. Conditions at the junction.

Still in the context of a field of optimal trajectories reaching and then traversing a surface S, and generating a function V satisfying the enlarged theorem 1, we investigate the behaviour of the optimal strategies at the junction with S.

Let $\varphi^*(x, v)$ and $\psi^*(x)$ be the limits of the lower saddle point strategies as x reaches S, and $\hat{\varphi}(x, v)$, $\hat{\psi}(x)$ the "traversing" optimal lower strategies on S. Applying (7a) and (7b) with $\hat{\varphi}$, $\hat{\psi}$, and using the fact that $\tilde{\mathcal{V}}(x) = \mathcal{V}$, it comes (we omitt unnecessary arguments on the functions)

(13a) $$0 = H(\hat{\varphi}(\hat{\psi}), \hat{\psi}) \geq H(\hat{\varphi}(\psi^*), \psi^*).$$

Now, in the open halfspace, $\tilde{u}_v(x) = \mathcal{U}$, therefore applying (7c) and (7a)

(13b) $$H(\hat{\varphi}(\psi^*), \psi^*) \geq H(\varphi^*(\psi^*), \psi^*) = 0.$$

Therefore (13a) gives

$$H(\hat{\varphi}(\psi^*), \psi^*) = 0 = \max_{v \in \mathcal{V}} H(\hat{\varphi}(v),v) = H(\hat{\varphi}(\hat{\psi}), \hat{\psi}).$$

We therefore have,

THEOREM 2. If the maximum of $H(\hat{\varphi}(v),v)$ is unique, the optimal strategy ψ is continuous at the junction.

The geometry of the set $(f(\hat{\varphi}(\mathcal{V}), \mathcal{V}), L(\hat{\varphi}(\mathcal{V}),\mathcal{V}))$ is somewhat difficult to investigate. We shall not attempt here to understand better the unicity assumption of Theorem 2.

Relation (13b) gives similarily :

$$H(\hat{\varphi}(\psi^*), \psi^*) = 0 = \min_{u \in \mathcal{U}} H(u, \psi^*) = H(\varphi^*(\psi^*), \psi^*)$$

We therefore have the interesting result :

THEOREM 3. If the minimum of $H(u, \psi^*)$ is unique, then the optimal trajectories reach S tangentially. If furthermore the optimal strategy ψ is continuous (see theorem 2) then so is the optimal D-strategy φ.

PROOF. Under the assumption of theorem 3, we have

$$\hat{\varphi}(\psi^*) = \varphi^*(\psi^*)$$

and thus $\varphi^*(\psi^*) \in \tilde{u}_{\psi^*}$. Therefore the field generated by (φ^*, ψ^*) cannot be transverse to S.

Under our assumption of convexity of $(f(u,v), L(u, v))$, the minimum is guaranteed to be unique if this set is strictly convex. Otherwise, part of its boundary is linear (a hyperplane, or an intersection of), and we have the following situation :

COROLLARY. Under the convexity assumption, if an optimal field of a lower saddle point reaches a singular surface transversally and then follows it, the hamiltonian is linear with respect to at least one component of u, and is singular at the junction.

Theorem 3 explains why we were interested in allowing envelopes in the field of optimal trajectories.

3. Particular singular surfaces.

Here, we shall study in more details the various situations that may arise according to the shape of the field on both sides of the singular surface. We shall not consider surfaces that are left on both sides by the optimal field (dispersal lines), as they pose no particular problem in the present set up. (This may not be so if we do not allow D-strategies). Neither shall we consider surfaces that are the limit of a field, but not reached by its trajectories. Those are barriers and we purposedly avoid them here.

In the three types we consider, there is a regular case and a singular case according to whether conditions of theorem 3 or of the corollary prevail.

In order to give formulas to actually compute singular surfaces of various kinds, we need the following notations :

$\bar{\varphi}(x, \lambda, v)$, $\bar{\psi}(x, \lambda)$: arguments of $\max_{v \in V} \min_{u \in U} H(x, \lambda, u, v)$

$\overline{f}(x, \lambda)$, $\overline{H}(x, \lambda)$, $\overline{H}(x, \lambda)$ are obtained by placing $\overline{\varphi}$ for u and $\overline{\psi}$ for γ in f and H.

$\tilde{\varphi}(x, \lambda, v)$ argument of the constrained min : $\min_{u \in \mathcal{U}_v} H(x, \lambda, u, v)$

$\tilde{f}$ and $\tilde{H}$ are as $\overline{f}$ and $\overline{H}$ but with $\tilde{\varphi}$ in place of $\overline{\varphi}$.

With $\nu(x)$ the normal to the singular surface, we always have

(14) $$\nu'(x)\tilde{f}(x, \lambda) = 0$$

and we have know that $\tilde{\varphi}(\overline{\psi})$ differs from $\overline{\varphi}(\psi)$ only in the singular case. In that case, also, we have

$$H(x, \lambda, u, v) = a(x, \lambda, u_2, v)\, u_1 + b(x, \lambda, u_2, v)$$

where u_1 is one component of u, and u_2 all the others. The singularity condition reads:

(15) $$\tilde{a}(x, \lambda) = a(x, \lambda, \tilde{\varphi}_2(x, \lambda, \overline{\psi}), \overline{\psi}(x, \lambda)) = 0.$$

3.1. State constraints.

This is the case where optimal trajectories exist only on one side of S. Actually, a completely similar situation arises if, on the other side, an optimal field leaves S, but that in addition $\overline{V}(x)$ has a jump on S. Then S has to be a barrier, of course, but it further plays the role of a local state constraint. We shall give below an example of this very interesting phenomenon.

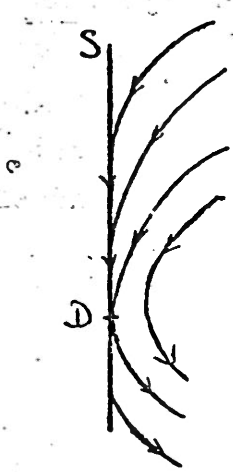


Fig. 1

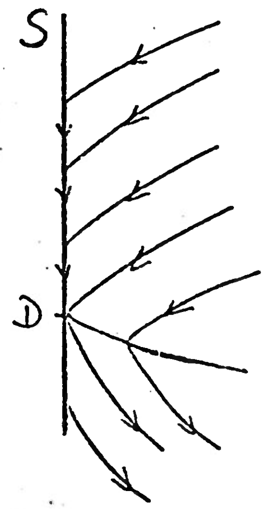
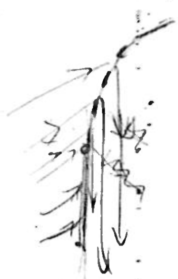


Fig.2



Typically, the field of optimal trajectories constructed backwards hits S along a n-2-dimensional manifold D. Notice that theorem 3 and its corollary apply as well to the point, on D, where an optimal trajectory leaves S. The two typical situations are as depicted in figures (1a) (regular) and (1b) (singular). In the second case, D is the intersection of S with a switching surface for φ^*.

In both cases, the value of λ_0 on the incoming trajectory is assumed known, as this trajectory belongs to the previously computed unconstrained field. Then, the field of optimal trajectories can be computed with the following equations.

Regular case :

$$\dot{x} = \bar{f}(x, \lambda),$$

$$\dot{\lambda} = -\frac{\partial \bar{H}}{\partial x} + \alpha(t)\left(\frac{\partial S}{\partial x}\right)',$$

where α is given by the following equation, obtained by differentiation of

$$(16) \qquad \frac{\partial S}{\partial x}\bar{f}(x, \lambda) = 0$$

with respect to time :

$$\alpha \frac{\partial S}{\partial x}\frac{\partial \bar{f}}{\partial \lambda}\left(\frac{\partial S}{\partial x}\right)' + \bar{f}'\frac{\partial S^2}{\partial x^2}\bar{f} + \frac{\partial S}{\partial x}\frac{\partial \bar{f}}{\partial x}\bar{f} - \frac{\partial S}{\partial x}\frac{\partial \bar{f}}{\partial \lambda}\frac{\partial \bar{H}}{\partial x} = 0.$$

Singular case :

$$\dot{x} = \tilde{f}(x, \lambda),$$

$$\dot{\lambda} = -\frac{\partial \tilde{H}}{\partial x} + \alpha(t)\left(\frac{\partial S}{\partial x}\right)'$$

Now, (14) is automatically satisfied, but α is chosen in such a way as to insure (15), which, differentiating with respect to time, yields :

$$\alpha \frac{\partial \tilde{a}}{\partial \lambda}\frac{\partial S}{\partial x} - \frac{\partial \tilde{a}}{\partial \lambda}\frac{\partial \tilde{H}}{\partial x} - \frac{\partial \tilde{a}}{\partial x}\tilde{f} = 0.$$

See [20] for a more detailed investigation of state constraints.

3.2. Universal surfaces.

Isaacs called universal surfaces singular surfaces that are reached on both sides by the optimal trajectories.

Let indices 1 and 2 refer to various quantities in the two half spaces, and ν be a normal to S pointing toward region 2. Our lemma shows that :

$$\lambda_1 = \lambda_2 + \alpha\nu \tag{17}$$

By assumption, we have, on S

$$\nu' f(\varphi_1^*, \psi_1^*) \geq 0, \qquad \nu' f(\varphi_2^*, \psi_2^*) \leq 0.$$

Assume, for instance, that

$$\nu' f(\varphi_1^*, \psi_2^*) \geq 0.$$

By assumption also, both fields satisfy (7), hence,

$$H(\lambda_1, \varphi_1^*, \psi_2^*) = L(\varphi_1^*, \psi_2^*) + (\lambda'_2 + \alpha\nu') f(\varphi_1^*, \psi_2^*) \leq 0$$

$$H(\lambda_2, \varphi_1^*, \psi_2^*) = L(\varphi_1^*, \psi_2^*) + \lambda'_2 f(\varphi_1^*, \psi_2^*) \geq 0$$

Hence

$$\alpha\, \nu' f(\varphi_1^*, \psi_2^*) \leq 0, \qquad \alpha \leq 0.$$

Now, looking at $H(\lambda_i, \varphi_2^*, \psi_1^*)$, with $i = 1, 2$, it comes similarily :

$$\alpha\, \nu' f(\varphi_2^*, \psi_1^*) \geq 0$$

and therefore finally

$$\nu' f(\varphi_2^*, \psi_1^*) \leq 0.$$

So, we see that, when the two players chose different strategies, the state always drifts in the half space where E's strategy is optimal. Therefore he can stick to his choice, ψ_1^* in region 1, ψ_2^* in region 2, either of the two on S. P will be unable to keep φ_1^* or φ_2^* since both lead the state in a region where they are not optimal. This situation ends up in a "chatter" for P, or, in our formalism, a strategy φ^* which is not admissible. The only solution for P is to chose, on S, a strategy $\tilde{\varphi}(x, v)$, that will insure that the state remains on S, for all v's. Notice that the situation is different from the state constraint in that, now, P will not let the state drifton either side of S. For that reason, the pairs $(\tilde{\varphi}(x, v), \psi_1^*)$ can be regarded as giving the optimal field of the "reduced game" which is the game where P is obliged to keep the state on x, that is where $u \in \hat{u}_v(x)$ of all controls that satisfy $\nu' f = 0$. Now, except perhaps on exceptional dispersal lines, this game has under our assumptions of convexity, a unique field of extremals. Therefore

$$f(x, \tilde{\varphi}(x, \psi_1^*(x)), \psi_1^*(x)) = f(x, \tilde{\varphi}(x, \psi_2^*(x)), \psi_2^*(x)).$$

(But this does not imply that $\psi_1^* = \psi_2^*$).

Again, two situations arise regular and singular. In the first case, we have A. Merz's focal line [14], [17]. The fields come in tangentially. In the second case there is a corner at the junction. We have the equivalent of singular arcs of optimal control theory. The typical situations are described in figures (2a) and (2b) respectively.

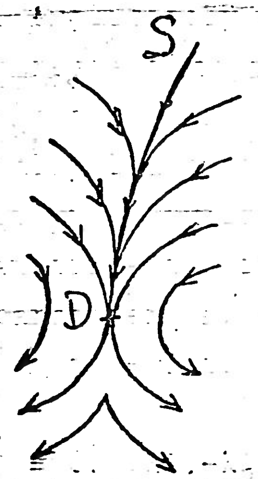


Fig.2a

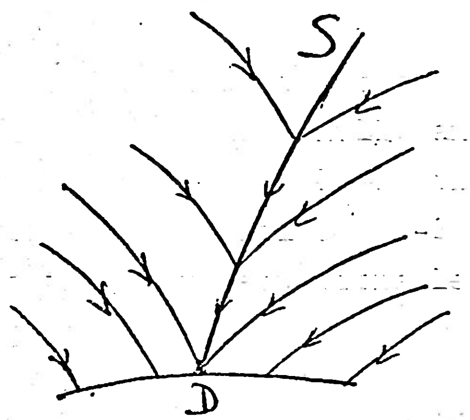


Fig.2b

Here, D has to be found as a locus of points where the trajectories of two fields are tangent, or as a switching point on a surface otherwise joined by the trajectories. Again, λ on the incoming trajectory at D is thus known. (See, however, a more complicated situation in [20]).

In the present case we do not know S a priori, but we know that $\lambda_2 - \lambda_1$ must be normal to it. We can therefore proceed as follows, from D :

Regular case :

$$\dot{x} = \overline{f}(x, \lambda_1) ,$$

$$\dot{\lambda}_1 = -\frac{\partial \overline{H}_1}{\partial x} + \alpha_1(\lambda'_1-\lambda'_2).$$

$$\dot{\lambda}_2 = -\frac{\partial \overline{H}_2}{\partial x} + \alpha_2(\lambda'_1-\lambda'_2).$$

$\alpha_1(t)$ and $\alpha_2(t)$ being chosen in such a way that

$$(\lambda'_1-\lambda'_2)\overline{f}(x, \lambda_1) = 0$$

$$(\lambda'_1 - \lambda'_2)\overline{f}(x, \lambda_2) = 0$$

We leave to the reader to carry out the time differentiations to get a system for α_1 and α_2.

Singular case : same equations as above, except that the bars must be replaced by tildas and $\tilde{u}_v$ is defined using $\lambda_2-\lambda_1$ for ν. Now, α_1 and α_2 are chosen to insure (15) with both λ_1 and λ_2, and are still explicitely given by differentiation of these two formulas with respect to time.

What remains to be checked at this point is that the above construction actually leads to a hamiltonian that remains null, and, more difficult, to a surface S normal to $(\lambda_1-\lambda_2)$. To do this, one must parametrize the final conditions on D with a (n-2) dimensional parameter s, and show that the above equations lead to a surface $x(t) = \xi(t,s)$ satisfying :

$$(\lambda'_1-\lambda'_2)\,\frac{\partial\xi}{\partial t} = 0, \quad (\lambda'_1-\lambda'_2)\,\frac{\partial\xi}{\partial s} = 0$$

The first of these relations will be satisfied by contruction. Checking the second is more difficult. It can be done using the same argument as for the corner surfaces of the next paragraph. The details can be found in [20] and [21].

3.3. Corner surfaces.

Here we are interested in surfaces that are reached by the optimal trajectories on side 1 and left on side 2. Hence, V(x) will again be continuous on both sides, but, as previcusly will be joined tangentialy in the regular case. ν is again pointing toward region 2.(17) still holds.

By assumption, we have

$$\nu' f(\varphi_1^*, \psi_1^*) \geq 0, \quad \nu' f(\varphi_2^*, \psi_2^*) \geq 0.$$

We distinguish two cases, depending on whether the following <u>permeability condition</u> is met or not :

CONDITION

$$\nu' f(\varphi_1^*, \psi_2^*) > 0, \qquad \nu' f(\varphi_2^*, \psi_1^*) > 0.$$

This condition says that neither of the two players can, by refusing to switch to the strategy 2 upon reaching S, prevent the other from doing so and be right to.

We then have the equivalent of Weierstrass' corner condition :

THEOREM 4. If the permeability condition is met, the gradient must be continuous across S.

PROOF. As previously, we use (7b) and (7c) in both fields using (17) :

$$H(\lambda_1, \varphi_1^*, \psi_2^*) = L(\varphi_1^*, \psi_2^*) + (\lambda_2' + \alpha\nu') f(\varphi_1^*, \psi_2^*) \leq 0$$

$$H(\lambda_2, \varphi_1^*, \psi_2^*) = L(\varphi_1^*, \psi_2^*) + \lambda_2' f(\varphi_1^*, \psi_2^*) \geq 0$$

hence

$$\alpha\nu' f(\varphi_1^*, \psi_2^*) \leq 0$$

Similarily, using (φ_2^*, ψ_1^*), we get

$$\alpha\nu' f(\varphi_2^*, \psi_1^*) \geq 0$$

This together with the permeability condition implies

$$\alpha = 0$$

and the theorem is proved.

Therefore, a corner can occur only if a continuous gradient causes a switch in the optimal controls, which happens with a singular hamiltonian, a classical situation, met in the Dolichobrachistochrone problem for instance [1], and in the example below.

Now, let us assume that

$$\nu' f(\varphi_2^*, \psi_1^*) \leq 0.$$

From the proof of the above theorem, we infer that

$$\alpha \leq 0,$$

$$\nu' f(\varphi_1^*, \psi_2^*) \geq 0.$$

As in the case of the universal surfaces, if E decides to keep his strategy ψ^*, on S, the strategy φ^* is not admissible for P, nor any other that does not keep x on S.
Therefore, against ψ_1^*, P must play a strategy $\tilde{\varphi}$ that makes the state transverse S, and we are in the situation of our general theory of junction of singular surfaces. Notice that here, at any time while traversing S, E may chose to switch to ψ_2^*, and then the state will leave S on side 2. P must then switch to φ_2^*.
The regular case corresponds to J.V. Breakwell's switch envelope [13], [17], and the singular case to R. Isaac's equivocal surface [1]. See [22] or [23] for a more detailed discussion.

Now, as far as constructing S is concerned, the situation is somewhat different. We assume that the field 2 has been constructed previously, and a singularity D is known. It can be a corner of the game of kind, or a more complex situation. (See [20]). The equations used are the same as the first two we wrote above for universal surfaces. A difficulty appears in differentiating the relation

$$(\lambda_1' - \lambda_2')\bar{f}(x, \lambda) = 0,$$

or relation (15) in the singular case, with respect to time. It occurs because $\partial\lambda_2/\partial t$ is not directly known along the trajectories we construct. We must then assume that, in the field 2, we know $\lambda_2 = \partial V/\partial x$ as a function of x, and are able to compute

$$\frac{\partial \lambda_2}{\partial t} = \frac{d\lambda_2}{dx} \bar{f}(x, \lambda_2)$$

As previously, to have a satisfactory theory, we must still check that S_{so} constructed is actually normal to $(\lambda_1 - \lambda_2)$. See [20] for a complete proof.

4. The second order servomechanism problem.

4.1. Statement of the problem

A simple second order plant, with state y, is governed by

$$\ddot{y} = v \qquad v \in \mathcal{V}, \text{ a bounded set.}$$

It is to match a set point z that may drift in an impredictable fashion, but with bounded speed :

$$\dot{z} = u \qquad u \in \mathcal{U}, \text{ a bounded set.}$$

The specification of the servomechanism is its precision :

$$z - y \in \mathcal{X}, \quad \text{a bounded set.}$$

We consider the simpler case where y and z are scalars, all data sets are symetrical. By normalizing, and setting $x_1 = z - y$, we get the following equations :

$$\dot{x}_1 = u - x_2 \qquad |u| \leq 1,$$

(18) $$\dot{x}_2 = \frac{v}{2p} \qquad |v| \leq 1, \qquad p \text{ a parameter}$$

The playing space is

(19) $$|x_1| \leq 1.$$

The real problem is to know whether we can, with v, insure (19) whatever u does. We formulate this as a game, with P trying to escape (violate (19)), E trying to forbid it. When escape occurs (the situation we shall consider), the payoff will be escape time :

$$J = \int_{t_0}^{t_1} dt.$$

The hamiltonian of this game is

$$H = 1 + \lambda_1(u-x_2) + \lambda_2 \frac{v}{2p}.$$

Away from singular surfaces, the optimal controls shall be

$$u = -\operatorname{sgn}\lambda_1, \quad v = \operatorname{sgn}\lambda_2,$$

so that optimal trajectories will be arcs of parabolas in the x space.

We readily find that the useable part of the capture set is made of two symetrical pieces :

$$x_1 = 1, \qquad x_2 < 1,$$

$$x_1 = -1, \qquad x_2 > -1.$$

From these points two pieces of barriers can be built. They are arcs of parabolas tangent to $|x_1| = 1$. To go further, one must consider the relative position of these parabolas.

We consider the case where p is smaller than 1, but not by much, say $0.8 \le p < 1$. Then the two pieces of barrier are as shown on figure 3. (We use reverse axes for the benefit of space).

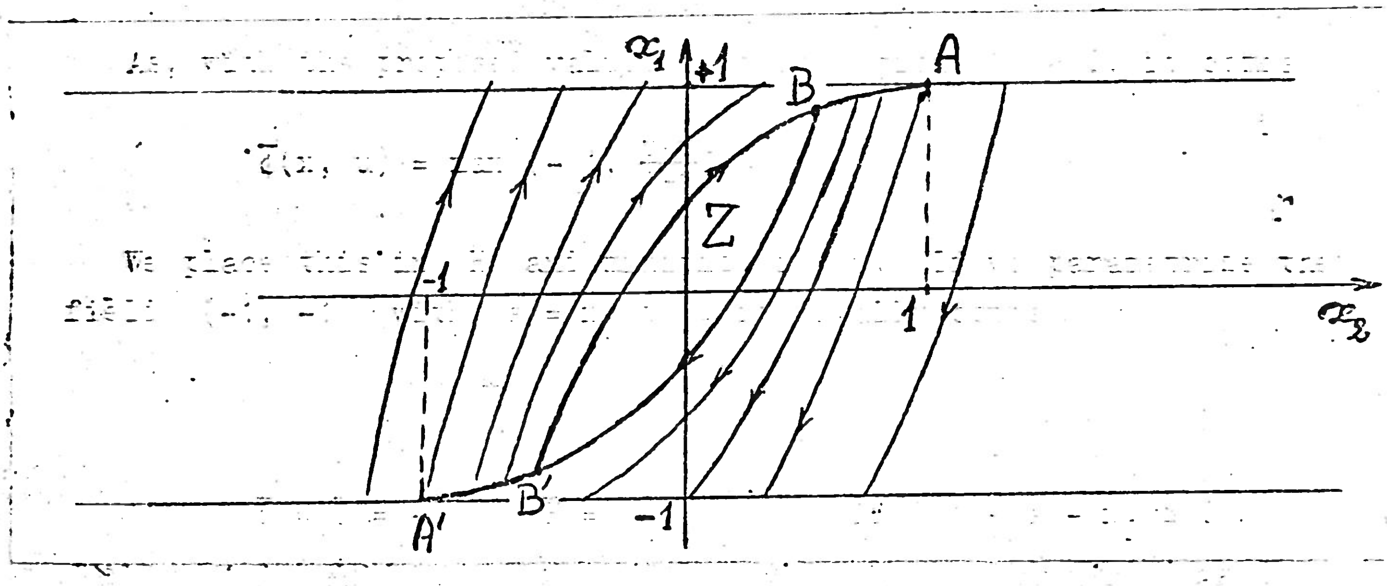


Fig.3. Barrier and primaries.

It seems that they define a closed region Z where the state is trapped, if E wants. However it is not so, because at points B and B' where the two pieces of barrier cut each other, E is not able to prevent the state to cross the two pieces at the same time. (In that respect, see [20] or [23] for a theory of junction of barriers). But before we completely solve the question thus raised, we must investigate more closely what happens outside of Z.

4.2. Lunge maneuver and equivocal line

The two pieces of barrier are the limits of two fields of primaries, one with $u = v = 1$, ends on $x_1 = 1$, one with $u = v = -1$ ends on $x_1 = -1$. Let B the point where the two pieces cut each other, in the region $x_1 > 0$, $x_2 > 0$. (The same things happen, by symmetry, on the other side).

The arc BA is a surface of discontinuity of the value function corresponding to these fields, continuous on the side of the field $(-1, -1)$, and thus with negative jump. We therefore look for an upper strategic pair $\varphi(x)$, $\psi(x, u)$ that avoids the jump.

The direction of the barrier and its normal (continuous side) are

$$\text{barrier}: \begin{pmatrix} 1 - x_2 \\ \frac{1}{2p} \end{pmatrix} \qquad \text{normal} \begin{pmatrix} -\frac{1}{2p} \\ 1-x_2 \end{pmatrix}$$

Thus,

$$\tilde{\gamma}_u(x) = \{v \mid -\tfrac{1}{2p}(u-x_2)+(1-x_2)\tfrac{v}{2p} \geq 0\} = \{v \geq \tfrac{u-x_2}{1-x_2}\}.$$

As, with the proposed value, in this region $\lambda_2 < 0$, it comes

$$\bar{\psi}(x, u) = \max\left(-1, \frac{u-x_2}{1-x_2}\right).$$

We place this in H and minimize in u. If we parametrize the field $(-1, -1)$ with $s = x_2(t_1)$, it finally comes

$$\bar{\varphi} = \bar{\psi} = +1, \quad \bar{H} = 2\,\frac{1+s-x_2}{1+s}, \quad \text{if} \quad 1 + s - x_2 \leq 0,$$

$$\bar{\varphi} = \bar{\psi} = -1, \quad \bar{H} = 0, \quad \text{if} \quad 1 + s - x_2 \geq 0.$$

Therefore this field does not satisfy our sufficient conditions.

In the part of the arc BA where $1+s-x_2$ is negative, P can, by playing his barrier strategy, oblige E to do so, and let time to go decrease at a rate less than one in the process. The limit point C on the arc BA can be computed easily. At thid point, we have

$$0 = H(\bar{\varphi}, \bar{\psi}(\bar{\varphi})) \leq H(\varphi^*, \bar{\psi}(\varphi^*)) \leq H(\varphi^*, \psi^*) = 0 .$$

Hence, it can be the starting point of an equivocal line, with singularity in v. The theory gives us the equations of this line :

$$\dot{x}_1 = 1-x_2, \qquad \dot{\lambda}_1^- = \alpha(\lambda_1^+ - \lambda_1^-),$$

$$\dot{x}_2 = \frac{v}{2p}, \qquad \dot{\lambda}_2^- = \lambda_1^- + \alpha(\lambda_2^+ - \lambda_2^-).$$

(we have used superscipt + and - for outgoing and incoming trajectories) Equations (15) and (14) read

$$\lambda_2^- = 0 ,$$

$$(1-x_2)(\lambda_1^+-\lambda_1^-) + \frac{v}{2p}(\lambda_2^+ - \lambda_2^-) = 0 ,$$

We can also use the first integral $H(x, \lambda^-, \bar{\varphi}, \bar{\psi}) = 0$, which gives

$$\lambda_1^- = \frac{1}{x_2-1}.$$

The condition $\dot{\lambda}_2^- = 0$ would have given α, but we do not need it here .

We therefore have the differential equation for the equivocal line :

$$\dot{x}_2 = -\frac{1}{\lambda_2^+}[\lambda_1^+(1-x_2) +1].$$

It turns out that, in the field (-1, -1), λ_1^+ and λ_2^+ are easy to obtain as functions of the state, so that we can readily compute this line. It is a commutation line, with the incoming field having the same controls as the barrier (+1, +1), the outgoing field being (-1, -1).

As we integrate backward from C, the equivocal line cuts trajectories of the outgoing field with decreasing s. As the state approaches the barrier of that field, the magnitude of the gradient λ^+ tends to infinity. Therefore the equivocal line tends to become normal to it, and thus it reaches the barrier $(-1, -1)$ tangentially at a point D.

The limit outgoing trajectory is the barrier DA'. However, with our definition of an open capture set, the barrier itself is not a capture trajectory. Therefore, from D, P should choose to keep $u = +1$ for awhile before switching.

4.3. The state constraint.

Consider a point on the barrier DA', close to D. If capture should occur by first following the barrier, then the optimal strategy would be to switch to $(+1, +1)$ at C', symmetric point to C, and then follow the new parabola until capture. This takes much more time than the optimal strategies we have proposed from D. Therefore P would rather try to reach D. Now, by playing $u = +1$ he can actually insure that $\dot{x}_1$ be positive.

If the state must actually reach D before leaving the zone Z, along the equivocal line, then the barrier is a surface of discontinuity of the value, discontinuous on the outside of Z as we have seen, with a negative jump. Therefore, E must prevent the state from crossing the barrier, which in that region will act as a state constraint.

We can apply the previous theory, with

$$\tilde{\gamma}_u(x) = \{v| \le \frac{u-x_2}{1+x_2}\}$$

and, as in that region, by continuity with the incoming field of the equivocal line, λ_2 is positive, we find :

$$\bar{\phi}(x, u) = \min\left(1, \frac{u-x_2}{1+x_2}\right)$$

Starting from D, we can easily integrate a trajectory that traverses the barrier in the opposite direction from the "natural" one. The theory shows that we must still have, on the barrier

$$\lambda_2 = 0$$

and again, the first integral $H=0$ gives λ_1 as previously.

The state constraint is joined by the same incoming field $(+1, +1)$ as the equivocal line.

This strategy is optimal, along the barrier, as long as it yields

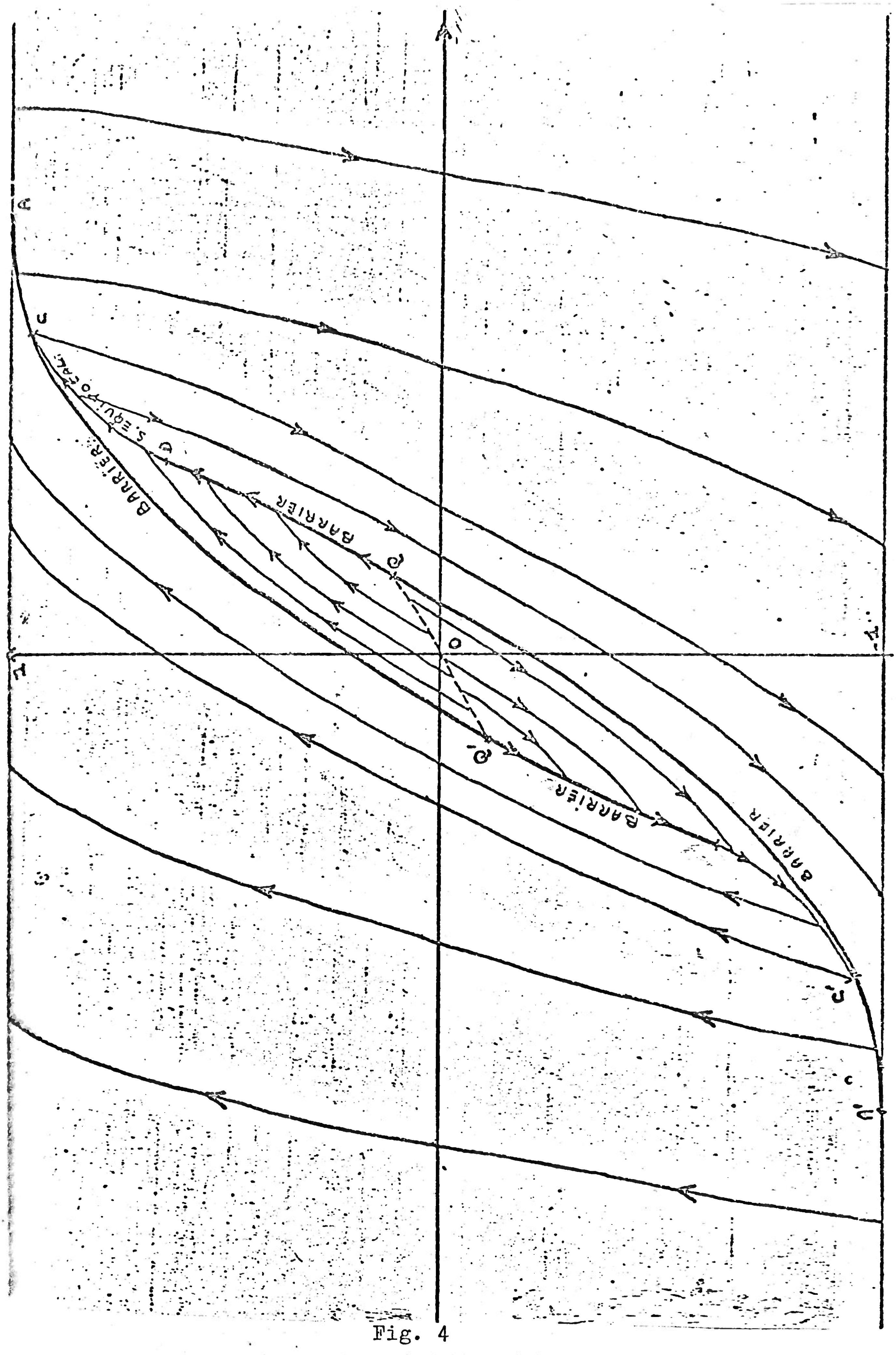


Fig. 4

a shorter capture time than following the barrier in its natural direction, and switching at C'. This defines a point Q where from the two strategies yield the same time to go. For the range of parameter values we have set here, Q is the starting point of a simple dispersal line separating the field (+1, +1) from the field (-1, -1), and readily computed using the requirement that it be normal to $(\lambda^+-\lambda^-)$.

For smaller values of p the situation is more complicated, Q being the starting point of a new equivocal line, itself followed smoothly by a dispersion line. Then, this extremely simple game may have a seven-stage optimal capture trajectory.

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