NEW RESULTS ABOUT CORNERS IN DIFFERENTIAL GAMES, INCLUDING STATE CONSTRAINTS.

P. BERNHARD

Centre d'Automatique de l'Ecole Nationale
Supérieure des Mines de Paris
35, rue Saint Honoré
77305 - Fontainebleau, France

ABSTRACT

The way corners occur in even simple two-person, zero-sum differential games is where the main difference lies between these problems and more classical optimal control problems. An attempt at a general theory has been reported earlier. However, new situations arise in the study of specific games, which require a broadening of the theory. Some of these will be presented here, together with a general mean of computing the hypersurfaces of discontinuity. Two simple goal keeping problems will be used as examples to motivate the analysis.

INTRODUCTION

Differential Games clearly have much in common with the more classical theory of Optimal Control. However, many differences appear, and almost all of them can be traced back to one fundamental fact. Namely the fact that for each player, the trajectory of the system is not a function of his control only. And this even for deterministic games.

As a consequence of this, the state x carries some important information. Thus closed loop or open loop control become actually different, while for deterministic optimal control problems with known initial state they cannot be distinguished. This, of course, is a feature differential games share with stochastic control. And it is no surprise that worst case design, which evolves as one if the main field of applications of the former, is an alternate way to the later to deal with uncertainties in control systems. Also, stochastic (2) (11) and game theoretic (12) approaches to large scale system optimization are developing.

Usually, differential games will not have an open loop solution. If one exists, for zero-sum two-person games, it coincides with the closed loop solution in the precise meaning developed in (3) or (7). But even then, the sufficiency proofs based on 2nd variation, and thus open-loop variational arguments, will prove inadequate and misleading. And, more important, first order necessary conditions based on variational arguments or the Pontryagin Minimum Principle, do not carry over to differential games, as was proved by Pontryagin in 1966 (14).

We have already shown (8) how one of the technical reasons is that the "adjoint vector" can be discontinuous even in the absence of a state constraint, generating various types of corners. We shall elaborate on this here showing a more complicated kind of corner. Then, we shall give a general mean of computing the classical corner hypersurfaces, and show that while state constraints exibit features unlike those known in optimal control, they are not more complicated than corners to compute.

2. CORNERS IN THE GAME N°1 (*)

2.1 The game

Two players A and B run at constant speeds, respectively $v>1$ and 1, on a planar playing ground. The game is entirely judged at a predetermined final time T ; known of both players. At that time A must be at a distance larger or equal to one unit from B. The payoff is the distance from A to the origin, at time T, that A seeks to minimize and B to maximize.

The simplest mathematical formulation is as follows. Let X_A and X_B be the two dimensional position vectors of A and B. The dynamics are

$$\dot{X}_A = \alpha v, \ \dot{X}_B = \beta \qquad (2.1)$$

where α and β are unit vectors whose directions are A's and B's control respectively. Let $\rho(X_A, X_B, t)$ be the value of the game starting from the state X_A, X_B, at time t. It is constant along an optimal trajectory. the Hamilton Jacobi Isaacs equation reads :

$$\rho_t + \min_{\alpha} \max_{\beta} [(\rho_A . \alpha)v + (\rho_B . \beta)] = 0, \quad (2.2)$$

(*)A substantial part of the analysis of that game was carried out by P. Colleter.

where $\rho_A=\partial f/\partial X_A$ and $\rho_B=\partial X_B$. The minmax is obtained for α opposite to ρ_A and β parallel to ρ_B. Whence :

$$\rho_t - v|\rho_A| + |\rho_B| = 0. \qquad (2.3)$$

Where they apply, the Euler Lagrange equations give :

$$\dot{\rho}_A = \dot{\rho}_B = 0 \qquad (2.4)$$

implying that, away from singular surfaces, A and B run in a straight line.

It is useful for the sequel to use a lower dimensional representation where we choose the x axis aligned with OA. Let then (x,y) be the coordinates of B relative to these axes, and z the abscissa of A. Let φ and ψ be the angles of A's and B's velocities, respectively, with the x axis. The dynamics become :

$$\begin{aligned} \dot{x} &= \cos\psi + \frac{v\sin\varphi}{z}\,y, \\ \dot{y} &= \sin\psi - \frac{v\sin\varphi}{z}\,x, \qquad (2.5) \\ \dot{z} &= v\cos\varphi. \end{aligned}$$

The H.J.I. equation is as in (2.3), but with

$$|\rho_A| = \sqrt{\frac{\rho_\theta^2}{z^2} + \rho_z^2}, \quad \rho_\theta = x\rho_y - y\rho_x,$$

$$|\rho_B| = \sqrt{\rho_x^2 + \rho_y^2}.$$

Where they hold, the Euler Lagrange equations yield

$$\dot{x} = \frac{\rho_x}{|\rho_B|} + v\frac{\rho_\theta}{|\rho_A|}\,\frac{y}{z^2} \qquad x(T) = \rho + \cos\alpha$$

$$\dot{y} = \frac{\rho_y}{|\rho_B|} - v\frac{\rho_\theta}{|\rho_A|}\,\frac{x}{z^2} \qquad y(T) = \rho + \sin\alpha$$

$$\dot{z} = -v\frac{\rho_z}{|\rho_A|} \qquad z(T) = \rho$$

(ρ and α parametrize the end point)

$$\dot{\rho}_x = v\frac{\rho_y}{z^2}\cdot\frac{\rho_\theta}{|\rho_A|} \qquad \rho_x(T) = 2\nu\cos\alpha$$

$$\dot{\rho}_y = -v\frac{\rho_x}{z^2}\,\frac{\rho_\theta}{|\rho_A|} \qquad \rho_y(T) = 2\nu\sin\alpha$$

$$\dot{\rho}_z = -\frac{v}{z^3}\,\frac{\rho_\theta^2}{|\rho_A|} \qquad \rho_z(T) = 1 - 2\nu\cos\alpha$$

Three simple first integrals are $|\rho_A|$, $|\rho_B|$ and ρ_θ which remain constant. Isaacs' "primaries" correspond to the intuitive solution given below, so that by simple geometric arguments one is able to compute $\rho(x,y,z,t)$ and its derivatives explicitely, where this field is optimal. This is of great help in actually carrying out the computations we shall indicate.

2.2. The intuitive solution

The solution of this game is obvious. Draw the circle of points A can reach in time T. Draw the circle (of radius T+1) of points B can forbid at time T. If the closest point from A's circle to the origin is outside B's circle, then A should run to that point and ignore B who can do nothing. Otherwise A should pick the point of intersection of the two cicles the closest to the origin and run at it, while B runs toward the same end point. This we shall call the primary strategy. It is depicted in figure 1.

However, a counter example is easy to find. It may happen that if both players play the primary strategy, after some time has elapsed, if we repeat the above construction from their present position with time to go instead of T, we find that the second intersection point of the two circles draws nearer to the origin than the one they are running toward. This is depicted in figure 2, where the reachability circles at $t=\frac{1}{2}$ are drawn in interrupted lines.

One recognizes that the problem stems from the fact that B has crossed the line OA, which is a symmetry axis. At this time A exerts two threats, and somehow B should take both into account. A typical case for a focal line. For some time, B should manage to stay on the x axis, until the primary strategy is safe. But how this arc is joined and left is a more complicated matter, and we must turn to the mathematics of the problem.

2.3. The focal manifold

The relevant state space is of dimension 4, since time has to be taken into account. However, schematically, the field of primaries has the shape shown in figure 3. The focal manifold is the hyperplane y = 0 . Σ is the two-D. manifold given by

$$y=0, \quad (1+\tau)x - \tau z = 0, \quad \tau \triangleq T-t \qquad (2.6)$$

The requirement for B to stay on OA translates as $\dot{y}=0$ or

$$\sin\psi = v\frac{x}{z}\sin\varphi \qquad (2.7)$$

(2.7) is considered as defining a function that explicitely depends on φ : $\psi=\tilde{\psi}(x,z,\varphi)$ or $\tilde{\psi}(\varphi)$. See earlier discussions of focal lines [10],[13] about this problem.

Now we must find the trajectories in the focal manifold. This manifold is a locus of high values for ρ as compared to neighbouring points, and B must not let the state drift off it, whatever A does. Knowing that, A will try to make the "ridge" as low as possible. Thus we place (2.7) in the dynamics and study this new three-D game, with

Σ as the terminal surface. Let λ be the gradient of the value in that game. Let $(\Sigma_x, \Sigma_z, \Sigma_t)$ be the normal to Σ and (ρ_x, ρ_z, ρ_t) be as computed on the primaries. At Σ we have

$$\lambda_x=\rho_x+\nu\Sigma_x, \quad \lambda_z=\rho_z+\nu\Sigma_z, \quad \lambda_t=\rho_t+\nu\Sigma_t. \qquad (2.8)$$

ν is determined by the three-D H.J.I. equation, which reads in our case :

$$\min_{\varphi}[\lambda_z \; v \cos\varphi + |\lambda_x| \sqrt{1-\frac{v^2 x^2}{z^2}\sin^2\varphi}]=0 \qquad (2.9)$$

For $\frac{x}{z} < \frac{1}{v}$, we have the classical situation : the solution of (2.9) is $\nu=0$. A's strategy is to run toward any of the two symmetric intersection points. The focal manifold is left smoothly when B's strategy to stay on it under A's optimal strategy coincides with his primary strategy. Tributaries can be built as usual, and one obtains a complete field of trajectories defining in that region a value that satisfies the H.J.I. equation.

For $\frac{x}{z} < \frac{1}{v}$, we find that the hamiltonian is minimized for $\varphi=\pi$. And this holds along the (backward) trajectories. The corresponding ν is non zero. Therefore this is no longer a standard focal manifold, since A's strategy is not the primary one, and it is left with a corner. According to (2.8), the gradient of the cost has a discontinuity at Σ. (However, one checks that the two fields merge smoothly in the hyperlane, with no void or overlap).

Let us first deal with this corner. Σ is an n-2 dimensional manifold. The situation is not accounted for by our previous theory (8). However the main argument holds. It is B who is not able to chose an earlier commutation point : if he tries to play earlier the primary strategy for one side of the focal plane, he will find himself drifting on the other side of that plane. Therefore, a discontinuity in the gradient is possible if switching earlier would have been to B's advantage. That is, if we take $(\Sigma_x, \Sigma_z, \Sigma_t)$ pointing toward the primary field, $\nu<0$. This is what we actually find.

A more serious difficulty appears when trying to construct the tributaries to this second part of the focal hyperplane. Here, the optimal strategies are $\varphi=\pi$, $\psi=0$. But the corresponding trajectories are a singular solution of the Euler Lagrange equations in dimension 4 if one places $\rho_y=0$ in them. (Which is not the value of ρ_y in the primaries at Σ, of course). As a consequence, when trying to recover ρ_y through the 4-D H.J.I equation, we find 0 as the only meaningful solution, and no tributaries.

There remains a void between the focal hyperplane and the acceptable primaries, and it must be filled up with trajectories before we can claim we have a solution, and before we can say that the focal strategies are optimal for initial points aligned.

2.4. The complete solution

The solution of this problem is as follows. Σ is n-2 dimensional. Therefore it has two in independent normals. The fact that the value of the game is known on Σ translates, for the gradient λ in a field of trajectories reaching Σ, into

$$\begin{aligned} \lambda_x &= \rho_x+\nu_1\Sigma_x \\ \lambda_y &= \rho_y+\nu_2 \\ \lambda_z &= \rho_z+\nu_1\Sigma_z \\ \lambda_t &= \rho_t+\nu_1\Sigma_t \end{aligned} \qquad (2.11)$$

And the H.J.I. equation only gives one relation linking ν_1 and ν_2, so that from each point of Σ one can construct a one-parameter family of extremals (trajectories). The primary and the previous focal both belong to this family, and correspond to $\nu_2=0$ or $-\rho_y$ respectively. (As the cost is purely final these family constitute semi-permeable surfaces, singular in the meaning of (4) or (7)).

This new field still does not solve the problem. In fact, the two particular trajectories quoted are tangent to the symmetry hyperplane. All the others cut into the field of primaries. But we can show that at each point of Σ, one of these is normal to the gradient $\nabla\rho$ of the value in the primary field, thus tangent to a hypersurface $\rho=$ constant. Equivalently, calling φ^-, ψ^- the controls on this trajectory, we have at Σ : $H(\nabla\rho, \varphi^-, \psi^-) = 0$. This trajectory is a candidate to be incoming to a switch envelope $\mathcal{S}$, that remains to be built from Σ in the field of primaries as proposed below.

Schematically, the situation is as shown in figure 4. Although rather complicated, it is thought to be typical of what happens where an envelope junction hits a symmetry hyperplane. It is very similar to one conjectured in (4). But here we are able to carry out the computations.

3. COMPUTATION OF CORNER SURFACES

In (8), we gave a rather general theory of corners. Here we propose a general mean of computing them. We shall deal with a game of dynamics

$$\dot{x}=f(x,\varphi,\psi) \quad x \in R^n, \quad \varphi \in \Phi \subset R^m, \quad \psi \in \Psi \subset R^p \qquad (3.1)$$

and purely final payoff with final time unspecified. If the dynamics are not autonomous or the final time specified or if the payoff has an integral part L, we can always put the game in this form by adding state variables of derivative unity and L. The Hamiltonian is (an accent on a symbol means "transpose"))

$$H(x,\lambda,\varphi,\psi)=\lambda' f(x,\varphi,\psi) \qquad (3.2)$$

It is assumed to have a saddle point $\varphi^*(x,\lambda)$, $\psi^*(x,\lambda)$. $f^*(x,\lambda)$ and $H^*(x,\lambda)$ are obtained by placing these in f and H. As in (8), we call λ^- and λ^+ the limits of the gradient $\partial V/\partial x$ of the value V(x) in the field before the corner and after it.

3.1. Switch envelope

On a switch envelope $\mathcal{S}$, we know that
a) the trajectories of $\mathcal{S}$ are obtained using the controls $\varphi^*(x,\lambda^-)$, $\psi^*(x,\lambda^-)$.
b) the components of the gradient of the function value tangent to $\mathcal{S}$ satisfy the Euler Lagrange equations while following $\mathcal{S}$ (the component normal to $\mathcal{S}$ is undefined during that part of the game).
c) the discontinuity in the gradient is normal to $\mathcal{S}$.

This leads to the equations (*)

$$\dot{x}=f^*(x,\lambda^-) \qquad (3.3)$$

$$\dot{\lambda}^-= -[\frac{\partial H^*(x,\lambda^-)}{\partial x}]' + \alpha(\lambda^- - \lambda^+) \qquad (3.4)$$

$$(\lambda^- - \lambda^+)' f^*(x,\lambda^-) = 0 \qquad (3.5)$$

(3.5) is equivalent to $\lambda^{+\prime} f^*(x,\lambda^-)=0$ taking the following into account. α in (3.4) is chosen such that (3.5) is satisfied. Or differentiating it with respect to time :

$$f^{*\prime}(\frac{\partial\lambda^+}{dx})' f^* + \lambda^{+\prime}\frac{\partial f^*}{dx} f^* - \lambda^{+\prime}\frac{\partial f^*}{\partial\lambda^-}(\frac{\partial H^*}{\partial x})'$$
$$- \alpha\lambda^{+\prime}\frac{\partial f^*}{\partial\lambda^-}\lambda^+ = 0$$

(it is a classical fact that $(\partial f^*/\partial\lambda^-)\lambda^-=0$). Usually this equation defines α. Otherwise one must go to higher derivatives. Its bad feature is to involve $(d\lambda^+/dx)=(d^2V/dx^2)^+$ which is usually difficult to get. It is where the explicit knowledge of $\rho(x,y,z,t)$ has been helpful.

It is straight forward to check that (3.3)-(3.5) imply, as they should

$$\frac{dH^*}{dt} = 0, \text{ hence } \lambda^{-\prime} f^*(x,\lambda^-) = 0 \qquad (3.6)$$

We must check that they actually generate a hypersurface $\mathcal{S}$ normal to $\lambda^- - \lambda^+$. Let $x=\xi(s)$ be a parametric representation of the initial manifold (in our case Σ. s is of dimension 2). (3.3)-(3.5) are considered as generating functions $x(t,s)$ and $\lambda^-(t,s)$, with boundary conditions $x(t_1,s)=\xi(s)$, and $\lambda^-(t_1,s)$ satisfying (3.5) and (3.6). Now, (3.5) insures that $\partial x/\partial t$ remains normal to $(\lambda^- - \lambda^+)$. We have to prove that

$$(\lambda^- - \lambda^+)'\frac{\partial x}{\partial s} = 0 \qquad (3.7)$$

By construction, this is satisfied at t_1, on Σ. We compute its time derivative interchanging the two differentiations in the evaluation of $\partial/\partial t(\partial x/\partial s)$

$$\frac{\partial}{\partial t}(\lambda^{-\prime}\frac{\partial x}{\partial s}) = -\frac{\partial H^*}{\partial x}\frac{\partial x}{\partial s} + \alpha(\lambda^- - \lambda^+)'\frac{\partial x}{\partial s} + \lambda^{-\prime}\frac{\partial f^*}{\partial x}\frac{\partial x}{\partial s}$$
$$= \alpha(\lambda^- - \lambda^+)'\frac{\partial x}{\partial s}$$

$$\frac{\partial}{\partial t}(\lambda^{+\prime}\frac{\partial x}{\partial s}) = f^{*\prime}(\frac{d\lambda^+}{dx})'\frac{\partial x}{\partial s} + \lambda^{+\prime}\frac{\partial}{\partial s}(f^*(x,\lambda^-))$$

On the other hand, we know that $\lambda^{+\prime} f^*(x,\lambda^-) = 0$ every where on $\mathcal{S}$. Differentiating it in s, it comes

$$\frac{\partial}{\partial s}(\lambda^{+\prime} f^*(x,\lambda^-)) =$$
$$(\frac{\partial x}{\partial s})'(\frac{d\lambda^+}{dx})' f^* + \lambda^{+\prime}\frac{\partial}{\partial s}(f^*(x,\lambda^-)) = 0$$

For each coordinate s_i, the above expressions are scalar. The first term can thus be replaced by its transpose. Using the fact that $d\lambda^+/dx=d^2V/dx^2$ is symmetric, we get

$$\frac{\partial}{\partial t}(\lambda^{+\prime}\frac{\partial x}{\partial s}) = 0$$

and combining the two results

$$\frac{\partial}{\partial t}(\lambda^- - \lambda^+)'\frac{\partial x}{\partial s} = \alpha(\lambda^- - \lambda^+)'\frac{\partial x}{\partial s}$$

Since (3.7) is satisfied on Σ, it is satisfied everywhere, and the proof is complete.

3.2. Equivocal manifolds

At this point, it is not difficult to show also how to compute an equivocal manifold. Assume, for simplicity, that φ is scalar (what is important is that only one component of φ appears linearly) and :

$$f(x,\varphi,\psi) = g(x)\varphi + h(x,\psi)$$

An equivocal junction is to be computed, with (see (5))

$$\lambda^{-\prime} g(x) = 0 \qquad (3.8)$$

We know that
a) the trajectories of the junction are generated by a control $\tilde{\varphi}$ that insures (3.8)
b), c) as in § 3.1.

Let $\tilde{\varphi}(x,n,\psi)$ be, by definition, such that

$$n' f(x,\varphi,\psi) = 0 \qquad (3.9)$$

and introduce

$$\tilde{f}(x,\lambda^+,\lambda^-)=f(x,\tilde{\varphi}(x,\lambda^+-\lambda^-,\psi^*(x,\lambda^-)),\psi^*(x,\lambda^-)) \qquad (3.10)$$

(*) equation (3.4) appears in (14) in a slightly different context.

(We could equivalently define it using $\varphi(x,\lambda^+, \psi^*(x,\lambda^-))$, but the proof is simpler the way we do). Also, $\hat{H}(x,\lambda^+,\lambda^-)=\lambda^{-\prime}\hat{f}(x,\lambda^+,\lambda^-)$. The equivocal surface is generated by

$$\dot{x} = \hat{f}(x,\lambda^+,\lambda^-)$$
$$\dot{\lambda}^- = -\left(\frac{\partial \hat{H}(x,\lambda^+,\lambda^-)}{\partial x}\right)' + \alpha(\lambda^+-\lambda^-)$$

α chosen such that (3.8) be satisfied. Again, α is explicitely determined by differentiating (3.8) with respect to time.

The rest of the proof goes as in the previous §. The only difficulty being in checking that (3.8) implies $\lambda^{-\prime}\partial\hat{f}/\partial\lambda^+=0$. Using this and $\lambda^{-\prime}\partial f/\partial\lambda^-=0$ as usual, one can prove that the hamiltonian remains null, and that the hypersurface generated is normal to $(\lambda^- - \lambda^+)$.

4. STATE CONSTRAINTS (*)

4.1 The game n°2

We consider a game with the same dynamics as previously, but now the constraint that A must stay at a distance larger or equal to one from B holds throughout the pursuit. In addition, we make the game time independent by transforming its payoff to minimax time to reach a finite target : a circle of radius 1 centered at the origin.

B's role in the game is through a state constraint. In the three-D representation it appears as an oblique circular cylinder S :

$$(x-z)^2 + y^2 = 1$$

and A must keep the state of the game in the outside, or on the surface of this cylinder.

The primary trajectories can be computed backward from the terminal surface (which appears as the hyperplane z=1 in the 3-D representation). They correspond to a straight line race toward the origin for A, B's strategy being undefined. In the backward construction, some hit the cylinder. The situation is schematically depicted by figure 5. The field is tangent to S along a curve D which, in our case, is the generatrix of the cylinder defined by

$$x=z+\cos\alpha_0 \quad , \quad y=\sin\alpha_0,$$

with $v\cos\alpha_0=1$. (This defines two symmetric generatrices). At these points, A can just escape in a straight line toward the origin without being captured by B. For points on the cylinder with α slightly larger than α_0, B can forbid this strategy by running essentially toward A. See figure 6. We assume that the optimal play for some points not accounted for by the primaries will include "constrained arcs". Namely arcs of trajectory lying on S in a "safe contact" chase. The problem is to find how to compute these arcs and their tributaries, and in particular how the trajectories behave at the junction.

(*) The analysis of game n°2 and smooth state constraints are mostly due to J.F. Abramatic

The second point will be dealt with shortly. A classical mean to solve the problem that succeded in imprevious examples (6) (13) (4), is the following, borrowed from optimal control theory. When the state is on S, one may find the law $\varphi=\tilde{\varphi}(x,\psi)$ that maintains it on S. It is defined by

$$n'f(x,\tilde{\varphi}(x,\psi),\psi)=0 \qquad (4.1)$$

where n is the outward normal to S.

One places this in the dynamics, and, parametrizing S, obtains a reduced game of dimension n-1 on the surface of S, and solves it with D as final manifold, where the value of the game is known.

However, in the present game, this method gives, on D, $\psi=\alpha_0$, meaning that B would run away from A. Incidently, this also gives $\varphi=0$, opposite to its optimum value on the primaries at this point.

In terms of game, the explanation of this fact is as follows. By placing (4.1) in the dynamics, we implied that A would keep the state on S whatever B does. Then, B takes advantage of this to drag A away from his target, an absurdity.

In (8), we proposed a way to deal with this problem within the framework of the reduced (constrained) game. Here we shall present a simpler method, also generalized from standard practice in optimal control. We first need a mathematical formulation and a general theory, that plays, for state constraints, the role played by our corner conditions for corners.

4.2 Mathematical formulation

Again we must assmume A knows B's control, ending in a max min rather than a saddle point. Let

$$\Phi_{ad}(x,\psi) = \begin{cases} \Phi \text{ if } x \notin S \\ \{\varphi | n'f(x,\varphi,\psi)\geq 0\} \text{ if } x\in S \end{cases} \qquad (4.2)$$

The problem to solve is

$$\max_{\psi\in\Psi} \; \min_{\varphi\in\Phi_{ad}(\psi)} \; J(\varphi,\psi)$$

and the corresponding H.J.I. equation is

$$\max_{\psi\in\Psi} \; \min_{\varphi\in\Phi_{ad}(x,\psi)} \; H(x,\partial V/\partial x,\varphi,\psi)=0 \qquad (4.3)$$

Now, let Σ be the manifold of the (φ,ψ) space on which (4.1) holds, and Σ_ψ be a section of it at constant ψ. The method proposed above amounts to replacing (4.3) by

$$\max_{\psi} \min_{\varphi\in\Sigma_\psi} H(x,\partial V/\partial x,\varphi,\psi)= 0$$

However, this is not equivalent to (4.3), although we know the maxmin occurs on Σ. In optimal control, we look for a minimum, and then, if we know it occurs on a given subset, we can look for it by minimizing over this subset.

We now prove a general theorem about the junction of optimal, unconstrained trajectories with the constrained arcs.

THEOREM.

If a unique solution exists, then at a junction point, on an optimal trajectory the control ψ is continuous. Further, if at this point $H(x, \partial V/\partial x, \varphi, \psi^*)$ has a strict unique minimum in φ, then the control φ also is continuous on the optimal trajectory.

COROLLARY.

If for the given problem, $H(x,\lambda,\varphi,\psi^*(x,\lambda))$ has a unique minimum for all λ on S, then optimal trajectories are smooth (have continuous first derivatives) at the junction with the constrained arcs.

PROOF of the theorem.

Let us write, here, λ for $\partial V/\partial x$. We have by assumption

$$0=\max_{\psi} \min_{\varphi\in\Phi_{ad}(\psi)} H(x,\lambda,\varphi,\psi)\geq$$

$$\min_{\varphi\in\Phi_{ad}(\psi^*)} H(x,\lambda,\varphi,\psi^*)\geq\min_{\varphi\in\Phi} H(x,\lambda,\varphi,\psi^*) = 0$$

Therefore,

$$\min_{\varphi\in\Phi_{ad}(\psi^*)} H(x,\lambda,\varphi,\psi^*)=0,$$

and this is the maximum of this minimum as a function of ψ. Thus this is the solution sought and the first assertion is proved.

Then assume the minimum in φ is strict. Then $\varphi^*\in\Phi_{ad}(x,\psi^*)$.

Otherwise, the hamiltonian in (4.3) could not be zero, since the optimal ψ is ψ^*. The theorem is proved. The corollary follows at once.

4.3. Computation of constrained arcs.

i) the smooth case.

We proceed as we did for corners. We know that.

a) trajectories on S are generated by the optimal controls φ^*, ψ^* on the incoming trajectories.
b) the components of gradient of the function value tangent to S satisfy the Euler Lagrange equations, projected on S.

This leads to (n is the normal to S)

$$\dot{x}=f^*(x,\lambda)$$
$$\dot{\lambda}= - \frac{\partial H^*}{\partial x} + \alpha n$$

and α is chosen such that

$$n'f^*(x,\lambda)=0$$

which, differentiated with respect to time along a trajectory gives α explicitely. It is easy to check that the field thus generated satisfies all the above conditions, and also $H^*=0$.

ii) The singular case

Assume the dynamics are as in § (3.2). Then, if (3.8)holds we are in the case where the theorem allows for a discontinuous junction. Here, the manifold from which the computation is done backward is typically the intersection of a switching surface (on which (3.8) holds) with S. See, for instance, (9). Then we look for a solution where (3.8) stands instead of a) above. This leads to ($\tilde{\varphi}$ is defines by (4.1)

$$\dot{x}=f(x,\tilde{\varphi}(x\lambda,\psi^*), \psi^*(x,\lambda))$$
$$\dot{\lambda} = - \frac{\partial H^*}{\partial x} + \alpha n$$

and α is chosen such that (3.8) holds. Again α is explicitely given by equating to zero the time derivative of this expression. It is interesting to notice that in this case, the method of the reduced game is indeed always valid. As a matter of fact, we see that the trajectories are generated, of course, by $\varphi=\tilde{\varphi}(x,\lambda,\psi)$ but further, at each instant, H is independant of φ. Therefore looking for the maximin reduces to maximizing in ψ. Then, as we observed earlier, the technique of the reduced problem is justified.

5. CONCLUSION

The points where two-person zero-sum differential games are now known to differ from optimal control theory include focal manifolds, corners and state constraints, with or without corners.

We did not discuss in detail focal manifolds. However, we contributed to a general theory of them, still to be done, by exhibiting a new type of such surfaces, imbeded in a new type of singularity, leading to a complicated corner. In short, we may say that this singularity has to do with envelope junction meeting a symmetry hyperplane, while focal manifolds always imply some sort of "local symmetry".

The other points : corner hypersurfaces and state constraints have been dealt with in a very similar set up, so that practicle means of computing them are available. These methods were successfully applied by J.F. Abramatic in the computation of the state constraint of the second game, and by P. Colleter in the computation of the switch envelope of the first one.

In view of the similitary we brought out, one might want to simplify and unify the terminology, which, as it is, is the result of a historical development. We might propose "regular corner", "singular corner", and "regular state constraint" and "singular constraint" for the four types of problems we investigated.

BIBLIOGRAPHY

(1) Abramatic, J.F. "Un jeu différentiel avec une contrainte d'état". Thesis of Ingénieur Docteur, Paris, 1975.

(2) Athans, M, "Survey of Decentralized Control Methods". Report E S L-P-555 of the MIT Electronic Systems Laboratory, 1974.

(3) Berkovitz. L, "Lectures on Differential Games" in Differential Games and Related Topics, Kuhn, H.W, and G.P. SZEGO, ed., North Holland, 1971.

(4) Bernhard P. "Linear Differential Games and the Isotropic Rocket" PhD dissertation, Stanford University Sudaar 413 1971.

(5) Bernhard P. "Corner conditions for Differential Games" Ifac 5th World Congress, Paris, France, 1972.

(6) Bernhard P. "On the evaluation of worst case design with an application to quadratic synthesis" Third Ifac symposium on Sensivity, Adaptivity and Optimality, Ischia, Italy, 1973.

(7) Bernhard P. "Commande Optimale, Decentralisation et Jeux dynamiques", chapter 6, Dunod, Paris, 1975.

(8) Bernhard P., Abramatic J.F., : "Contraintes d'état dans les jeux différentiels", Colloque international sur la théorie du contrôle, les méthodes numériques et la modélisation des systèmes informatiques. Iria, France 1974.

(9) Breakwell, J.V., The Dolichobratistochrone problem. Unpublished lecture notes, 1968, and report C 18 of Centre d'Automatique de l'ENSMP, Fontainebleau, France, 1971.

(10) Breakwell.J.V., "Some differential games with intersting discontinuities" Stanford University, 1973.

(11) Chu, K.C., Y-C Ho "On the generalized linear-quadratic gaussian problem", in Differential Games and Related Topics, (cf (3)).

(12) Cohen G. "A game theoretic approach to decentralized optimal control" submitted to International Journal of applied mathematics and optimization.

(13) Merz, A.W., "The homicidal Chauffeur A differential game". PhD dissertation, Stanford University, 1971.

(14) Marchal C. "Generalization of the Optimality theory of Pontryagin to deterministic two-player zero-sum differential Games. "Fifth Ifip Conference on Optimization Techniques, Roma, Italy, 1973.

(15) Pontryagin L.S, "The theory of differential Games" Usp. Matem. Nauk. 21, n°4, 1966.

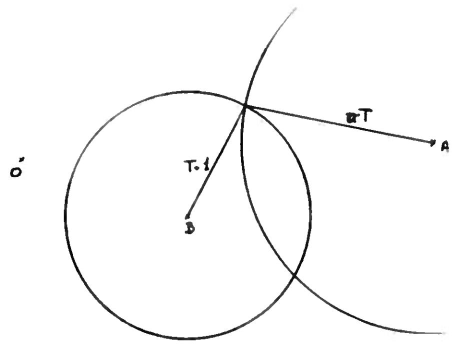

figure 1. The primary strategies.

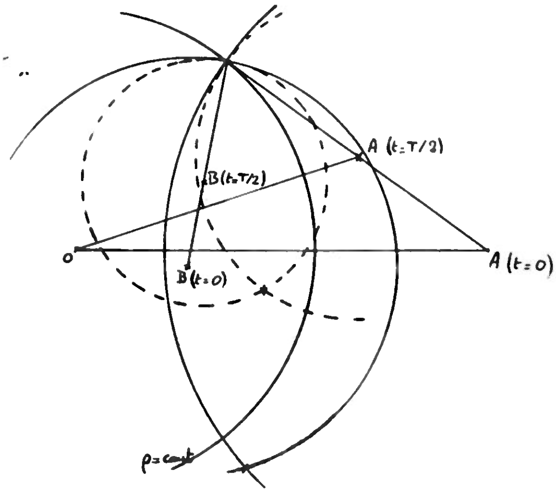

figure 2. The conterexample.

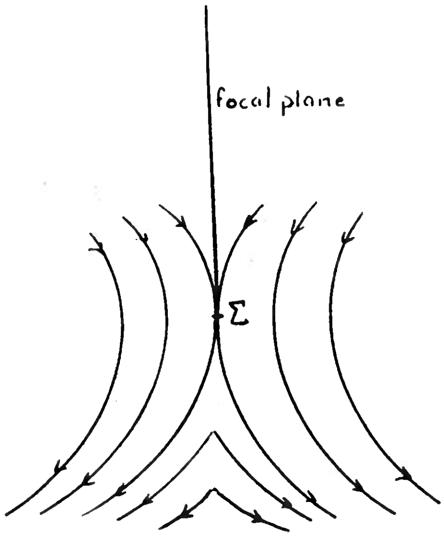

figure 3. The field of primaries.

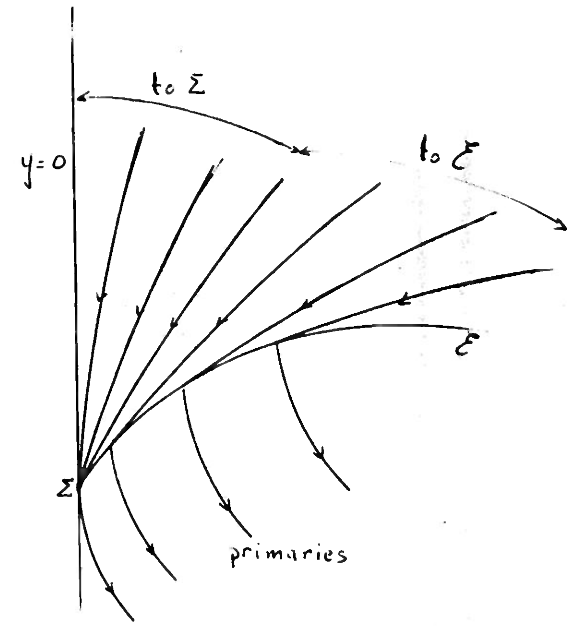

figure 4. Singular field and switch envelope.

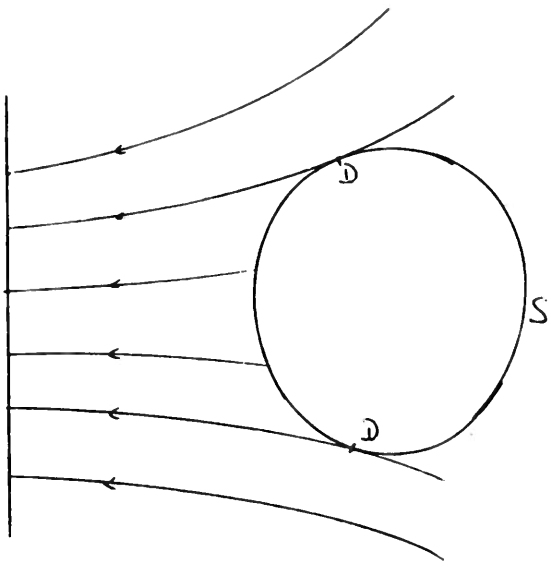

figure 5. Primaries and state constraint.