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CORNER CONDITIONS FOR DIFFERENTIAL GAMES

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In his historical book "Differential Games" (4), Isaacs described a feature which has no counterpart in optimal control theory, what he termed the "equivocal line". Later on, a new original phenomenon was discovered by Breakwell and Merz(2), in the study of the same game (the homicidal chauffeur), again with no counterpart in classical optimization. They named it "switch envelope". It appears that these two phenomenons involve "corners" of the trajectories, that is, a discontinuity of the slope. When the need for some corner condition was again felt, in the game of the Isotropic Rocket ((4), (1)) that we shall use here as an example, the time seemed to be ready for an attempt at a more general treatment of that question.

THE PROBLEM

Differential Game. We are working with a two-person zero-sum differential game. That is, a system is acted upon by two players: the "Pursuer" P, who choses the control u, and the "Evader" E, who choses the control v. The system dynamics are given in terms of the state x.

$$\dot{x} = f(x, u, v) \quad x \in X \subset R^n$$
$$u \in U \subset R^p \quad v \in V \subset R^m$$

Both players have perfect knowledge of the model and of the state. Moreover, a criterion J is given, that the pursuer seeks to minimize, while the evader wants to maximize it:

$$\min_{u(.)} \max_{v(.)} \left[J = \varphi(x_f) + \int_{t_o}^{t_f} L(x, u, v)dt\right]$$

where $x_f = x(t_f) \in C$ is the first point where the trajectory penetrates a given "capture set" $C \subset X$. (It is a well known fact of control theory that such a problem can always be formulated as time independant, by adding a state variable if necessary).
We say that there exists a solution if there exists a pair of strategies (closed loop control laws) $u^*(x)$, $v^*(x)$ such that for every starting point x, (u^*, v^*) leads to a saddle point:

$$J(u^*, v) \le J(u^*, v^*) \le J(u, v^*)$$

Sufficient conditions are known for a solution to exist (see (1), (3)). They involve the existence, for each state and adjoint λ, of a saddle point in (u, v) for the hamiltonian H, with in addition:

$$\min_u \max_v \{H = L + \langle\lambda, f\rangle\} = 0$$

In the solution, one must distinguish between two types of trajectories: "Normal trajectories" are inbeded in a regular field of extremals. On such trajectories, the adjoint λ is the gradient of the return function $V(x)$. "Abnormal trajectories" are part of a "barrier", locus of discontinuity of V (or of points of infinite gradient). There, the adjoint has a different interpretation ("semi permeability" property) and is normal to the barrier.

Assumption: We shall make the additional assumption that the maneuverability domains are convex. More precisely,

$$f(x, U, v) \text{ convex } \forall v \in V$$
$$f(x, u, V) \text{ convex } \forall u \in U$$

or, if the dynamics are "separated" in $f(x, u, v) = h(x, v) - g(x, u)$

$$h(x, V) \text{ convex, } g(x, U) \text{ convex.}$$

Problem and notations. We want to find a condition that must be satisfied by a manifold S on which optimal controls have a discontinuity. Let the superscripts - and + denote values of the variables at S before and after switching.
In the case of normal trajectories, S is an hypersurface locally separating the state space into two half-spaces. Let them be called region - and region +. We shall assume that S has everywhere a normal n, chosen pointing into region +.
In the case of abnormal trajectories, we are studying the intersection of two barriers, which are hypersurfaces themselves. S is an $(n-2)$ dimensional manifold. Since the barriers have a normal, S has a normal plane. (Unless the two constituing semi-permeable surfaces are tangent along S, case that we shall rule out).

We have the following very simple result:

Lemma. The difference $\lambda^- - \lambda^+$ is normal to S.
Proof. i) Normal trajectories: Since V(x) must be uniquely defined on S, its directional derivatives tangent to S are uniquely defined, and they are the projection of the gradient on the tangent plane to S. Therefore

$$\lambda^- = \lambda^+ + \alpha n \qquad \alpha \text{ a scalar.}$$

ii) Abnormal trajectories: Both λ^- and λ^+ are normal to S, since they are normal to their respective semi-permeable surfaces, the intersection of which is S. Thus, the lemma is proved. Frome here on, we are obliged to do two separate treatments of the two cases: normal or abnormal trajectories. We begin with the first case and proceed to establish the "indifference condition".

PERMEABILITY CONDITION

Definition. We shall say that this property is satisfied if, in a neighborhood of S,

$$\langle n,\ f(u^-,\ v^+)\rangle \geq 0$$

$$\langle n,\ f(u^+,\ v^-)\rangle \geq 0$$

namely, none of the players can cause the state to return in region −, after reaching S, by keeping his strategy of region −.

Proposition: if the dynamics are separated, one at most of the above two inequalities can be violated at a time.
Proof: assume, for instance, that

$$\langle n,\ f(u^+,\ v^-)\rangle = \langle n,\ h(v^-)\rangle - \langle n,\ g(u^+)\rangle < 0$$

by definition of n, we have

$$\langle n,\ f(u^+,\ v^+)\rangle = \langle n,\ h(v^+)\rangle - \langle n,\ g(u^+)\rangle \geq 0$$

$$\langle n,\ f(u^-,\ v^-)\rangle = \langle n,\ h(v^-)\rangle - \langle n,\ g(u^-)\rangle \geq 0$$

Then by simple addition:

$$[\langle n,\ h(v^+)\rangle - \langle n,\ g(u^-)\rangle] + [\langle n,\ h(v^-)\rangle - \langle n,\ g(u^+)\rangle] \geq 0$$

and the second bracket being negative, the first is strictly positive.

Theorem 1. (Erdman-Weierstrass Corner Condition). When the permeability condition is satisfied, a necessary condition is

$$\lambda^+ = \lambda^- \quad (\text{or} \quad \alpha = 0).$$

Proof. Assume, for definiteness, $\alpha > 0$. Let the evader decide to switch to the strategy v^+ on a surface translated of S by $\delta 1$ into region − (he switches before reaching S). Because of the permeability property, the state will penetrate the new region +, and the pursuer will have to switch to u^+ to play optimally against v^+. To first order, the variation in payoff will be:

$$\delta V = \langle (\lambda^+ - \lambda^-),\ \delta 1\rangle = -\alpha\langle n,\ \delta 1\rangle > 0$$

and thus, this new strategy is better than the one defined by the original S which, therefore, is not optimal.
For $\alpha < 0$, it is the pursuer who should switch earlier. The theorem is proved. This situation is that of classical optimal control theory.
Notice the importance of the permeability condition in the proof. If, say,

$$\langle n,\ f(u^-,\ v^+)\rangle < 0$$

then when E switches to v^+, the state goes back in region − and the proof does not hold any more. We must investigate that case.

THE INDIFFERENCE CONDITION

The traversing strategy. Let us, for definiteness, study the case just envisioned. We must expect that a switching surface with $\alpha > 0$ should be possible. (And conversely, if the other permeability condition is violated, $\alpha < 0$ should be possible). Because of the difference $\lambda^- - \lambda^+$, the Evader should switch as soon as possible. Assume that the Pursuer refuses to switch and keeps his strategy u^-. Then, as we said, the state will go back into region −, where the optimal strategy for E is, by assumption v^-. Therefore E must switch back to v^-, causing the state to reach S again, and so on. We end up with a chattering strategy for E, between v^- and v^+, the state following S. However, because of our convexity assumption, it is known that a chattering strategy cannot be better than the simple strategy giving the same direction of displacement. Therefore, E should chose the control $\tilde{v}$ such that the state follows S:

$$\langle n,\ f(u^-,\ \tilde{v})\rangle = 0$$

This pair $(u^-,\ \tilde{v})$ is called the "traversing" strategy. (Notice that without the convexity assumption, $\tilde{v}$ might be a "relaxed" control, or limit of very high frequency chattering controls). Since we have seen that the pursuer can force the evader to chose this strategy, a necessary condition is that it be as "good" as the optimal one, that is, optimal itself. This is the indifference condition.
In that case, P will have the choice between an infinity of equivalent trajectories, being allowed to follow the corner surface, or to leave it at any time (as long as the end of that surface is not met).

Theorem 2. In the case studied: $\langle n,\ f(u^-, v^+)\rangle < 0$, a necessary condition is $\alpha \geq 0$ and

$$\max_{v} H(\lambda^-,\ u^-,\ v) = H(\lambda^-,\ u^-,\ \tilde{v})$$

Proof. We notice first that because of the definition of $\tilde{v}$, and with the lemma, we have

$$\langle \lambda^-,\ f(u^-,\ \tilde{v})\rangle = \langle \lambda^+,\ f(u^-,\ \tilde{v})\rangle$$

Thus

$$H(\lambda^-, u^-, \tilde{v}) = H(\lambda^+, u^-, \tilde{v})$$

and the indifference condition can be checked with either definition of H. This condition requires that H be zero. But we notice that $H(\lambda^-,u^-,v^-)$ is zero, and moreover, v^- is the argument of the maximum of $H(\lambda^-,u^-,v)$. Therefore, the theorem is proved.
The obvious parallel is, for the other case:

$$\langle n, f(u^+, v^-)\rangle < 0, \quad \alpha \le 0,$$

and

$$\min_u H(\lambda^-, u, v^-) = H(\lambda^-, \tilde{u}, v^-).$$

Discussion. The indifference condition can be satisfied in two ways:

- either $\tilde{v} = v^-$

- or $\operatorname{Arg}\max_v H(\lambda^-, u^-, v)$ non unique.

In the first case, the optimal trajectories of region – reach the switching locus tangentially, hence the name given by Breakwell: "switch envelope". The second case usually corresponds to a singular arc in v. If the Lagrangian L is independant of v, (the minimax time problem, for instance), and with our assumption on the convexity of the maneuvarbility domain, $f(u^-,v)$ (or h(v) if the dynamics are separated) must have an affine set as part of its boundary, and λ^- must be normal to it.
For a two dimensional game, the classical situation is that where one of the domains of maneuverability is a line segment, and λ^- being normal to it gives a "switching function" that remains zero. It is Isaacs' equivocal line. That is why we shall call this case the (generalized) equivocal case.

Construction of a solution. Sufficient conditions always rest on the construction of a complete field of trajectories, and using Isaacs' "verification theorem". The previous necessary condition can be used in an attempt to construct this field. Once S is known, the condition $H^- = 0$ is sufficient to determine α, and hence λ^-, knowing λ^+. The only problem is to find S. The corner condition usually gives one condition on n. It is equivalent to a partial differential equation on S, that degenerates in an ordinary differential equation for a two-dimensional game. However, as soon as the state space is of higher dimension, this equation may be extremely difficult to integrate, even numerically, as we shall see on a very simple example.
A last question is that of the choice of the right corner condition, and finding the "initial conditions" to integrate the equations for S. Although ingenuity in each particular case is the only answer, a general rule is that these questions, mainly the second one, are generally answered by the solution of the "game of kind", which must be investigated first, and gives the barriers, families of abnormal trajectories.

JUNCTION OF SEMI-PERMEABLE SURFACES

Problem, notations. We recall the definition of a semi-permeable surface, as a surface (hyper-surface) B such that, its normal being λ, we have

$$\min_u \max_v \langle\lambda, f\rangle = \max_v \min_u \langle\lambda, f\rangle = 0$$

with the interpretation that each player can prevent crossing of it in a direction. We want to investigate under which condition two such surfaces can join, at an angle, and the composite surface still be a barrier. Let B^- and B^+ be the two surfaces, J their line of intersection, the trajectories of B^- incoming at J, those of B^+ leaving J. B^- locally divides the space into two regions R_1^- and R_2^-, similarily for B^+. Region 1 and 2 are determined by the direction of λ (we purposely avoid to specify whether λ points into region 1 or 2). The composite surface B locally separates the space into two regions R_1 and R_2 defined by

$$R_1 = R_1^- \cap R_1^+$$

$$R_2 = R_2^- \cup R_2^+$$

and B is obtained by deleting the portions of B^- and B^+ lying in R_2. Let u_1 be the control of the player who wants to bring the state into region 1, and conversly for u_2. (We avoid specifying which is u and which is v).
Assumption. On the trajectories of B^-, at J, the semi-permability condition uniquely defines u_1.
Under this assumption we have the followint result:

Theorem 3. For B to be a barrier, it is necessary that incoming trajectories do not cross the junction.
Proof. Let us assume that the paths of B^- cross J. This can happen only if $u_1^- \neq u_1^+$. Then, when the state reaches J, player 2 will keep his strategy u_2^-. If player 1 keeps his strategy u_1^-, by our current assumption the state will cross J and fall in $R_2^+ \subset R_2$. If player 1 switches to u_1^+, by the unicity assumption the state will fall in $R_2^- \subset R_2$. In both cases B has failed to be semi-permeable.
Remark. We do not claim that $u_1^- = u_1^+$ is necessary. There can exist a pair (u_1^-, u_2^-) that is different from (u_1^+, u_2^+) but generates paths tangent to B^+.

Discussion. In the case we just described, J is an envelope of the trajectories of B^-, and we have an "envelope junction", the counterpart of the switch envelope. If player 2 decides not to switch to u_2^+, the two players will use the pair – corresponding to the local incoming trajectory and the state will follow the junction, as in the case of normal trajectories.
The equivalent of the equivocal surface is less simple. It corresponds to the case where our unicity assumption is not satisfied. Then assuming

that λ points into region 2, say ($u_1 = u$, $u_2 = v$), there should exist a control $\tilde{u}_1$ such that

$$<\lambda^-, f(\tilde{u}_1, \bar{u}_2)> = 0$$

$$<\lambda^+, f(\tilde{u}_1, \bar{u}_2)> \leq 0$$

but as the last expression is not necessarily zero, we can have several situations:

i) $<\lambda^+, f(\tilde{u}_1, \bar{u}_2)> = 0$ $\tilde{u}_1$ is a possible traversing strategy,

ii) $<\lambda^+, f(\tilde{u}_1, \bar{u}_2)> < 0$ The evader must switch to u_2^+.

One also sees that there might exist an infinity of acceptable $\tilde{u}_1$'s.

THE ISOTROPIC ROCKET GAME

The game. To exhibit both a junction of barriers, and a case where the computation of a switch envelope involves a partial differential equation, we are obliged to chose a three dimensional game. One of the simplest such games that have been investigated in detail is Isaacs' Isotropic Rocket Game. It was presented in a Rand report in 1955, then in (4). (1) contains further, but still incomplete, developments. We shall, here, describe results by Isaacs without deriving them. The reader is refered to (4) or (1) for derivations.
The game can be described in a three dimensional space by

$$\dot{x} = -\frac{Fy}{v}\sin\varphi + w\sin\psi$$

$$\dot{y} = \frac{Fx}{v}\sin\varphi + w\cos\psi - v$$

$$\dot{v} = F\cos\varphi$$

x, y, v are state variables, φ and ψ the controls of the pursuer and the evader. Payoff is capture time, where capture is defined by $x^2 + y^2 < 1^2$ and F, w, and 1 are given constants.
These equations model, in a moving coordinate system, attached to the pursuer and the y axis pointing in the direction of its velocity, the plane chase of a cat pursuing a mouse on a sliding floor. The whole chase occurs at a speed well under the cat's maximum speed, so that the pursuer is limited only by its sliding, that allows it a finite acceleration F, the same in every direction. The mouse on the other hand, is runing at its maximum speed, w, and its sliding is negligible. x and y are the mouse's relative position, v is the cat's speed, φ and ψ are the angles of the cat's control (its acceleration) and the mouse's control (its velocity) with the y axis.

Smooth trajectories. It is found that a barrier attaches to the capture cylinder C, $x^2 + y^2 = 1^2$, along the B.U.P. given by $v\cos\theta = w$, where $x = r\sin\theta$, $y = r\cos\theta$. For the limiting case

$$w^2 = 2F1$$

that we shall be concerned with from now on, this barrier just tuches the (y, v) symmetry plane along the straight line $y = vw/F$, thus apparently defining a closed capture region.
It was found by Isaacs, however, that for final conditions at $v^2 < w^2 + F1$, the trajectories of the barrier penetrate C before coming tangent to it along the B.U.P. He then introduced another curve, $\mathcal{D}$, on C, tangent to the B.U.P.; at the critical point, and constructed a barrier: the "envelope barrier" (because its trajectories come tangent to $\mathcal{D}$) that provides a smooth extension to the previous, or "natural", barrier.
However, we showed in (1) that the curve $\mathcal{D}$ does not reach the symmetry plane, and that the envelope barrier does not close on this plane (fig. 1). Similarily, it was found by Breakwell that, because the "primary" trajectories penetrate C before planned capture, part of the field of trajectories must be constructed allowing for a state constraint, or "safe contact" on the boundary of C. However, this construction does not fill the whole region of the state space not accounted for by the primaries.

Envelope junction. (See (1)). Physically, the two barriers known so far correspond to the evader's side-stepping in an attempt to out-maneuver the less agile pursuer. It ceases to exist in the low v's because there, the pursuer becomes very agile. But in the region of interest, the evader is faster than the pursuer, we should therefore expect that it will take advantage of this superiority to flee from the pursuer.
In fact if we consider the case of a straight chase in real space, it gives a parabola in the (y, v) plane of the state space. Parabolas far enough from the capture cylinder hit the natural barrier that closes on the (y, v) plane. Such parabolas clearly are semi-permeable trajectories. Consider the parabola that hits the last trajectory of the natural barrier. Being in the symmetry plane, it is tangent to the barrier. From this point, it is possible to construct an envelope junction J extending on the envelope barrier, and from this locus a semi-permeable surface, made of trajectories fleeing from C, which together with the envelope barrier creates a composite barrier, with a corner (fig. 2).
Notice that the semi-permeability condition defines at each point of the state space a cone of semi-permeable directions. The requirement that it be tangent to the envelope barrier allows to isolate one such direction (two in fact) at each point of the barrier. Thus constructing the envelope junction amounts to integrating a differential equation on the surface considered, suitably parametrized. The main difficulty comes from the fact that the curve $\mathcal{D}$ could only be integrated numerically itself. It is interesting to notice that there is a range of marameters,

$$1.056 < \frac{w^2}{2F1} < 1.092$$

where this construction, completed toward the symmetry plane by an additional barrier of a different type, actually defines a closed capture region. The previous barriers only achieve this for

$$\frac{w^2}{2Fl} > 1.062$$

and suffice to define the smallest such capture region for

$$\frac{w^2}{2Fl} > 1.092.$$

Switch envelope. We expect that the field of optimal trajectories should be completed with trajectories qualitatively similar to the last piece of barrier constructed. Actually, the envelope junction provides the initial conditions to compute a switch envelope S.
We established the partial differential equation satisfied by S. It is actually given implicitly by conceptually eliminating α between the two equations

$$H^- = 0$$

$$\langle n, f(\varphi^-, \psi^-)\rangle = 0$$

where

$$\lambda^- = \lambda^+ + \alpha n$$

and

$$n = \begin{pmatrix} -\frac{\partial v}{\partial x} \\ -\frac{\partial v}{\partial y} \\ 1 \end{pmatrix} .$$

The equations are highly nonlinear, and elimination of α is unpracticle. This is not the main difficulty, it is possible, for instance, to find the equations of the characteristics directly from the pair of equations at hand. However, two forbiding difficulties arise in trying to numerically integrate these equations. The first is that λ^+ is known, numerically, not as a function of the state, but parametrized, together with the state, by a set of three parameters (time to go on the three successive legs of the trajectory until capture).
The second difficulty is that the initial conditions are given on a line where the λ^+ of the normal field is infinite. It can be checked that the information lost on the modulus of λ^+ is compensated for by the knowledge of the direction of λ^-. However using this information appears hardly feasible in practice.
We were able to derive analytically some results on the shape of the solution, and they indicate that it would probably give the desired field of incoming trajectories.

CONCLUSIONS

We have seen that the Erdman Weierstrass necessary conditions do not always hold for differential games. In fact, corners appear to be much more frequent in games than in classical optimal control problems. This fact might be traced back to the less stringent necessary conditions that prevail in the former case as compared to the latter.
It must be pointed out, also, that this is one of the reasons why the Pontryagin Maximum Principle cannot be directly generalized to differential games: the adjoints need not be continuous along an optimal trajectory.
Finally we must emphasize how the interaction between two players is the critical fact, which produces novel features with no possible counterpart in one-player problems.

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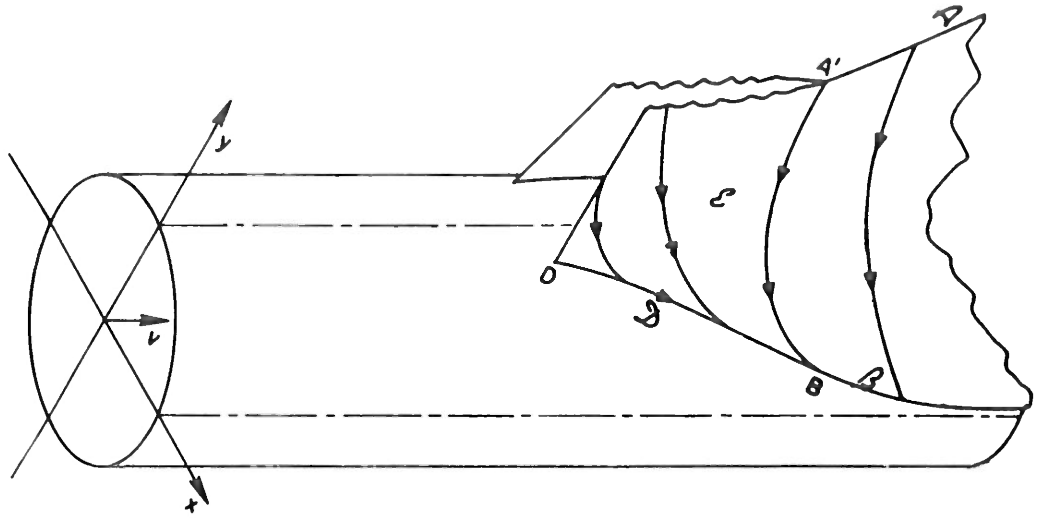

Figure 1. -

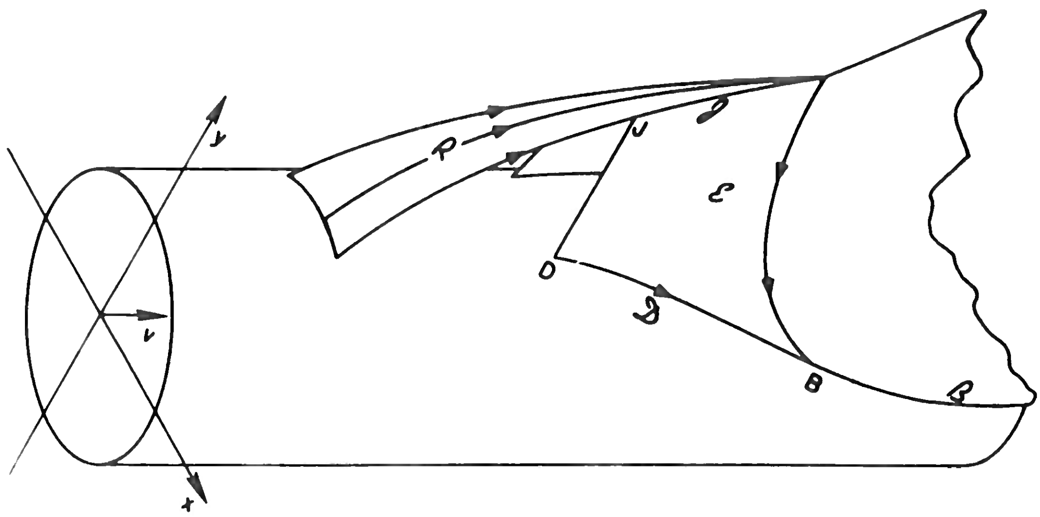

Figure 2. -