A critique of

Luca Lambertini

Differential Games in Industrial Economics

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Sections 1.2 “Optimal control theory” and 3.1: “Sticky Prices”

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Abstract

This book, while interesting by its many applications, suffers from serious technical weaknesses, making it a verty misleading introduction to the topic of optimal control and differential games applications in (industrial) economics. I only quote here two very serious mistakes, which make one suspicious about the whole technical development

Chapter 1, section 1.2: Optimal control theory

To start with, I must point once more to a very harmful mistake that pervades the whole book, as well as many other articles on the application of optimal control theory in economics. In page 4, the author is providing the solution of the simplest optimal control problem of maximizing with respect to a control function $u(\cdot)$ a pay-off Π given by

$$\dot{x} = f(x, u, t)\,, \quad x(0) = x_0\,, \qquad \Pi(x_0, u(\cdot)) = \int_0^T \pi(x(t), u(t), t)\, dt\,.$$

He rightly introduces an adjoint variable $\mu(t)$ obeying the classical adjoint equation. The first line of p. 4 specifies that, I quote, “the terminal condition $x(T)$ is left free.” Then comes the formal statement of the Pontryagin maximum principle where,

- on the one hand, the fact that the optimal $u^\star$ maximizes the hamiltonian is *not* formally quoted as part of the theorem (however an ambiguous sentence stated before the formal theorem is “the constrained optimization problem [. . .] is formally equivalent to maximizing the Hamiltonian [. . .]”)
- on the other hand, the transversality condition given is **plain wrong**. I quote: $\mu(T)x^\star(T) = 0$.

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According to this statement, trajectories ending at $x^{\star}(T) = 0$, but not *constrained* to do so, may have an arbitrary $\mu(T)$. Obviously wrong. (Incidentally, varying $\mu(T)$ at $x = 0$ would generate a new field of exremals intersecting with the primary field, requesting a further choice among candidate optimal trajectories from each initial condition.)

I have seen this mistake in several other articles in economics. This does not make it correct. Moreover, here it is repeated in p. 5 with the wrong transversality condition

$$\mu(T) = \frac{\partial S(x)}{\partial x} x(T)\,.$$

Differential games and Maximum principle

I should also mention that the footnote saying that Isaacs' Tenet of transition is "equivalent" to Pontryagin's Maximum principle is also wrong. The Tenet of transition is the game equivalent to Bellman's Optimality principle, leading to a sufficient condition —Isaacs' Verification Theorem—, while the Maximum principle is a necessary condition, embodying the Erdman-Weierstrass corner condition, i.e. the fact that the adjoint variables are *continuous*. This very strong property *does not apply* to the solution of differential games in state feedback strategies. This is the root of complicated singularities such as Isaacs' equivocal surface or Breakwell's switch envelope, two forms of the correct corner condition. See, e.g. [Bernhard, 1972] or [Bernhard, 2009] for nonzero-sum games. This mistake pervades all attempts to find closed-loop saddle points or Nash equilibrium strategies for differential games appealing to Pontryagin's Maximum principle.

Chapter 3, section 3.1: Sticky prices

In a previous note[2] I detailed the accumulation of gross mistakes in the article [Cilleti and Lambertini, 2007] on the topic of this section 3.1. That article is *not* quoted by the present book. However, the author is one of the many that have been misled by wrong statements in [Tsutsui and Mino, 1990], as further explained in [Bernhard, 2024].

References

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[Cilleti and Lambertini, 2007] Cilleti, R. and Lambertini, L. (2007). A differential oligopoly game with differentiated goods and sticky prices. *European Journal of Operational Research*, 176:1131–1144.

[Tsutsui and Mino, 1990] Tsutsui, S. and Mino, K. (1990). Nonlinear strategies in duopolistic competition with sticky prices. *Journal of Economic Theory*, 52:136–161.