# There is no known non-linear Markov Nash equilibrium strategies for the infinite horizon LQ differential game

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**Abstract**

This note is a belated critical analysis of [Tsutsui and Mino, 1990], referred to in the sequel as T&M. We note that while "for each price between a competitive stationary price and a near collusive stationary price, there exists some stationary Markov feedback equilibrium which supports it as its stationary price" (T&M, p. 138), *this is not true for the infinite horizon linear quadratic differential game*, but only for a family of related, free end-time games which are not state constrained versions of the infinite horizon game, and whose economic meaning is unclear. (And not true for any single one of them.) The confusion maintained by the article has spread in the economic literature.

## 1 Introduction

The article [Tsutsui and Mino, 1990], hereafter referred to as T&M, has been largely quoted (e.g [Dockner and Long, 1993, Kossioris et al., 2007, Lambertini, 2018]), most often ignoring the sufficiency part of the original article. This is a serious lapse on two counts.

On the one hand, these articles often proceed, as does T&M, by differentiating the main equation (often Isaacs "main equation"), thus obtaining a derived equation that they integrate producing a spurious integration constant. Typically, assume that the main equation is $A(x) = 0$, differentiating yields a derived equation equivalent to $\mathrm{d}A/\mathrm{d}x = 0$, a necessary condition for the main equation to be satisfied. Integrating back gives an equivalent of $A(x) = C$, $C$ an integration constant. This constant is then treated as being *arbitrary*, thus yielding an infinite number of solutions. Yet, clearly, $C$ is arbitrary as far as solving the derived equation is concerned. But with respect to the main equation, it is *unknown*, not *arbirary*.

On the other hand, T&M avoids that criticism by actually providing sufficient conditions associated with a range of values of the integration constant. And it is in the derivation of these sufficient conditions that the exact nature of the problem actually solved reveals itself most clearly. A problem rigorously stated at the begining of the article, but obscured by the ensuing commentaries that wrongly identify it with a state constrained version of the original infinite horizon problem.

Our objective is, while providing a somewhat simplified derivation of the results of T&M, to emphasize their difference (and links) with the classical infinite horizon linear quadratic (LQ) theory.

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## 2 The setup

### 2.1 Dynamics and profits

The article discusses the following nonzero-sum two-player differential game, where $p$ is a price, $s$ the inverse of a time constant, $u_1$ and $u_2$ in quantity by unit of time, production rates chosen by two duopolists, $r$ a discount rate. We introduce self explanatory constant positive parameters $a$, $b$, $c$, and $d$.[2]

The dynamics of the price are sticky:

$$\dot{p} = s(a - bu_1 - bu_2 - p)\,, \tag{1}$$

and player $i$'s profit is first stated as (equation (2.4) in T&M):

$$J^i = \int_0^\infty \mathrm{e}^{-rt} \left( pu_i - cu_i - \frac{d}{2} u_i^2 \right) \mathrm{d}t\,. \tag{2}$$

But we will see that this is not the pay-off of the game for which nonlinear equilibrium strategies are derived. Hence some confusion in the ensuing literature.

T&M looks for a Nash equilibrium in state feedback strategies $u_i = u_i(p)$, requiring that they be Lipshitz continuous, and defines a *Stationary Markov Strategy Space over a domain* $L$ of the state space. We must define more precisely the meaning of this phrase.

### 2.2 The games considered

T&M introduces a "domain $L$ of the state space". To be more precise, we will ask that $L$ be a compact interval of the real line, $\mathring{L}$ its interior, $\partial L$ its boundary. To each domain $L$, it associates a differential game defined as follows:

For any $p(0) = p_0 \in \mathring{L}$ and any pair of admissible controls $(u_1(\cdot), u_2(\cdot))$, let

$$T_L(p_0, u_1, u_2) = \sup_{t \geq 0} \{t \mid \forall \tau < t\,,\ p(\tau) \in \mathring{L}\}. \tag{3}$$

Thus, $T_L(p_0, u_1, u_2)$ is the first instant when $p(t)$ reaches $\partial L$ under the action of $(u_1, u_2)$, $+\infty$ if it never does[3].

The game considered has its pay-off defined as

$$J_L^i(p_0, u_1, u_2) = \int_0^{T_L(p_0,u_1,u_2)} \mathrm{e}^{-rt} \left( p(t)u_i(t) - cu_i(t) - \frac{d}{2} u_i^2(t) \right) \mathrm{d}t\,. \tag{4}$$

This is a *very different* game from the one resulting from the definition (2). The fact that it admits, inter alia, linear state-feedback equilibrium strategies is not obvious.

[2] T&M set $b = d = 1$, which we have rather avoid, because these parameters are not dimensionless.

[3] T&M does not define $T$ very precisely. We interpreted as (3) the sentence "$p(t)$ reaches a boundary of $L$ at a finite time", and then the phrasee "a terminal time $T$ of $p(t)$." An alternate definition would be to replace $\mathring{L}$ by $L$ in (3), but this requires that the strategies $u_i(p)$ be defined in an open neighborhood of $L$. And it is less consistent with the statement in T&M.

## 2.3 Remarks

Several remarks are in order.

### Main remarks

1. While the game (1)(4) is mathematically well defined, it is *not* the infinite horizon game (1)(2) advertised, nor a state constrained version of it. It is fundamentally a (related) "free end-time" problem. Superficially, it resembles the original problem, and the confusion with it is nurtured by the article. But, to be clear: *the nonlinear strategies exhibited are not a Nash equilibrium of any infinite horizon game problem.*

2. The economical, meaning of the related free end-time games solved is unclear. What is the meaning of the assertion that "the game stops" at $T_L$ ? In that formulation, no one of the duopolists is held responsible for the violation of the constraint. The discussion in T&M refers to a possible legal constraint limiting the admissible price, but fails to explain what happens when that price is reached, as a legal constraint ought to do. "The game stops…".

### Technical remarks

1. The requirement that the strategies be Lipshitz continuous in the state is there to ensure existence (and uniqueness, not stated by T&M, but for existence, simple continuity suffices) of a solution to the state dynamics. However, as is well known, this defines a set of admissible strategies which is not closed for any reasonable form of concatenation operation, thus preventing the derivation of Isaacs' Tenet of Transition and therefore preventing even sub-game perfectness. Isaacs' pioneering book [Isaacs, 1965] has been criticized for its lack of rigor in that respect. (Yet, it is genial.) Many authors have proposed ways to overcome that difficulty, including [Fleming, 1961, Roxin, 1969, Friedman, 1971, Elliot and Kalton, 1972, Krasovskii and Subbotin, 1977], or our less explicit but simpler setup [Bernhard, 1992], while [Blaquière et al., 1969] used less regular state feedback strategies and the concept of "playability".

2. This requirement rules out bang-bang control, or, for that matter, allmost all known examples of nonlinear differential game solutions.

3. The real, even stronger, requirement used by T&M, quoted in passing but not in the formal setup, is that the Value function be twice continuously differentiable. The classical quadratic Value function of the LQ differential game is one of the scarce (the only ?) instances that we know of such a regularity. A rich literature is devoted to avoiding even the requirement that the Value function be differentiable via the notion of viscosity solution, invented by [Fleming, 1964], made into a powerful theory by [Crandal and Lions, 1983] and applied to differential games by [Lions and Souganidis, 1985].

4. T&M essentially avoids the technical difficulties raised above by providing a sufficient condition, that does satisfy these regularity assumptions for the Value function and the strategies. One should just check carefully that the equilibrium strategies exhibited are indeed optimal against more general, namely measurable (or piecewise continuous) open-loop controls. T&M fails to check that, but it is true for its construction.

# 3 The analysis

## 3.1 Isaacs' equation

Very quickly, T&M specialises its search for a solution of Isaacs' equation to the symmetric case where both players get the same Value, using the same equilibrium strategies. We restrict our attention to this case. Moreover, the problem is clearly stationary, therefore we may look for $V$ as a function of $p$ alone (as opposed to a function of $t$ and $p$). And we choose here (contrary to T&M) to ignore during the derivation the constraints $u_i \geq 0$ and $p \geq 0$ which we will consider later on.

Let $V(p)$ be Isaacs' Value function of the game, and, following T&M, use the shorthand notation

$$\frac{\partial V}{\partial p} = y\,, \qquad \frac{\partial^2 V}{\partial p^2} = y'\,,$$

both functions of $p$.

Isaacs' "Main Equation" associated with this problem is

$$rV = \max_u \left\{ \left( p - c - \frac{d}{2} u \right) u + ys(a - bu - bv - p) \right\}.$$

At this stage, we miss a boundary condition, both on $\partial L$, and at $T = \infty$. We refrain from giving one, as we will see that T&M uses a non-classical one.

The maximum in Isaacs' equation is reached for

$$u = u^\star := \frac{1}{d}(p - c - sby)\,, \tag{5}$$

and yields

$$rdV = \frac{3}{2}s^2b^2y^2 + [ad + 2bc - (2b + d)p]sy + \frac{1}{2}(p - c)^2\,. \tag{6}$$

We remind the reader that Isaacs'equation (6) is classically used as a *sufficient condition* via Isaacs' Verification Theorem.

## 3.2 Necessary condition

From here, we give a slightly more direct form of the theory developped in T&M, and avoiding to divide by possibly null factors.

We first find a necessary condition for a twice differentiable function to be a solution of (6), differentiating both sides with respect to $p$. Reordering we get

$$[3sb^2y + ad + 2bc - (2b+d)p]sy' - [dr + (2b+d)s]y + p - c = 0\,. \tag{7}$$

We choose to write this in an homogeneous way as

$$[3sb^2(y-\beta) - (2b+d)(p-\alpha)]sy' - [dr + (2b+d)s](y-\beta) + (p-\alpha) = 0\,. \tag{8}$$

which imposes

$$\alpha = \frac{ad[(dr + (d+2b)s] + cb(2dr + (2d+b)s)}{(2b+d)dr + (b^2+4bd+d^2)s}\,,$$
$$\beta = \frac{(a-c)d}{(2b+d)dr + (b^2+4bd+d^2)s}\,.$$

We rewrite once more (8) as

$$[(y-\beta) - \frac{2b+d}{3b^2s}(p-\alpha)]y' - \frac{dr + (2b+d)s}{3b^2s^2}(y-\beta) + \frac{1}{3b^2s^2}(p-\alpha) = 0\,. \tag{9}$$

This allows one to look for a solution of the form[4]

1. either $\qquad y - \beta = K(p-\alpha)$
2. or $\qquad [y - \beta - K_1(p-\alpha)]^\gamma[y - \beta - K_2(p-\alpha)]^{1-\gamma} = C \neq 0, \qquad (10)$
   with $C$ an integration constant.

We check each of these possibilities.

**Case 1** That case corresponds to $y = \partial V/\partial p$ affine in $p$, thus the classical quadratic Value function. Therefore, placing this into (9), we necessarily get the classical Algebraic Riccati equation (ARE) associated to the quadratic solution[5]. Indeed, identifying to zero for every $p$, we obtain:

$$K^2 - \frac{2(2b+d)s + dr}{3b^2s^2}K + \frac{1}{3b^2s^2} = 0\,, \tag{11}$$

which has two positive roots. It is well known, and easy to check directly, that the controls given by placing the smallest of these solutions into equation (5) stabilizes the dynamics, while the other solution does not.

[4] T&M uses the variable $Z = (y-\beta)/(p-\alpha)$, which we have rather avoid to dispense with a discussion, absent in T&M, of $p$ in a neighborhood of $\alpha$, and calls $z_a$ and $z_b$ our $K_1$ and $K_2$.

[5] T&M does not mention the fact that its equation $F(z) = 0$, our equation (11), is the classical ARE.

**Case 2** Start from equation (10), differentiate with respect to $p$, multiply by the product

$$[y-\beta-K_1(p-\alpha)][y-\beta-K_2(p-\alpha)]$$

recognize $C$, which is assumed non zero, and divide by $C$ to obtain

$$\gamma(y'-K_1)[y-\beta-K_2(p-\alpha)]+(1-\gamma)(y'-K_2)[y-\beta-K_1(p-\alpha)]=0\,.$$

Reorganize as

$$y'[y-\beta-(\gamma K_2+(1-\gamma)K_1)(p-\alpha)]-[\gamma K_1+(1-\gamma)K_2](y-\beta)+K_1K_2(p-\alpha)=0\,.$$

Identifying with equation (9) yields

$$\begin{aligned}(1-\gamma)K_1+\gamma K_2 &= \frac{(2b+d)s}{3b^2s^2}\,,\\ \gamma K_1+(1-\gamma)K_2 &= \frac{(2b+d)s+dr}{3b^2s^2}\,,\\ K_1K_2 &= \frac{1}{3b^2s^2}\,.\end{aligned}$$

Summing the first two equations yields

$$K_1+K_2=\frac{2(2b+d)s+dr}{3b^2s^2},$$

so that we see that $K_1$ and $K_2$ are necessarily the two roots of the ARE (11). We choose $K_1<K_2$. The power $\gamma$ is then uniquely defined using either of the first two equations above, with the help of the notation $R=(b^2+4bd+d^2)s^2+(2b+d)dsr+(d^2/4)r^2$ as

$$\gamma=\frac{1}{2}+\frac{(2b+d)s}{2\sqrt{R}}\,,\qquad 1-\gamma=\frac{1}{2}-\frac{(2b+d)s}{2\sqrt{R}}\,.$$

The affine case $y=K(p-\alpha)+\beta$ has a well known status as a solution of the infinite horizon problem with no stopping condition at $T_L(p_0,u_1,u_2)$ as we have here. Its relevance to the present problem must still be verified. The other, non-linear, case yields a family of curves, depending on $C$, which satisfy a necessary condition to be solutions of the Hamilton Jacobi Isaacs (HJI) equation, itself written so far for the infinite time problem, or rather for no precise problem yet since we have not stated a boundary condition, neither on the boundary of $L$ nor at infinity.

Before we investigate this question, we need to draw a graph of these curves in the $(p,y)$ plane classically used in the calculus of variations. This is our figure 1, that we attempted to draw slightly more precisely than in T&M.

The two bold straight lines through the point $(\alpha,\beta)$, labeled $K_1$ and $K_2$ are the affine solutions, their slopes are $K_1$ and $K_2$. The line labeled SSL, for Steady State Line, called "blue line" hereafter, represents the line

$$2b^2sy=(2b+d)p-(ad+2bc)\,,$$

it separates the region $\dot{p}>0$ to its left from the region $\dot{p}<0$ to its right under the action of both $u_i=u^\star$, and is therefore the locus of possible dynamic equilibria with both

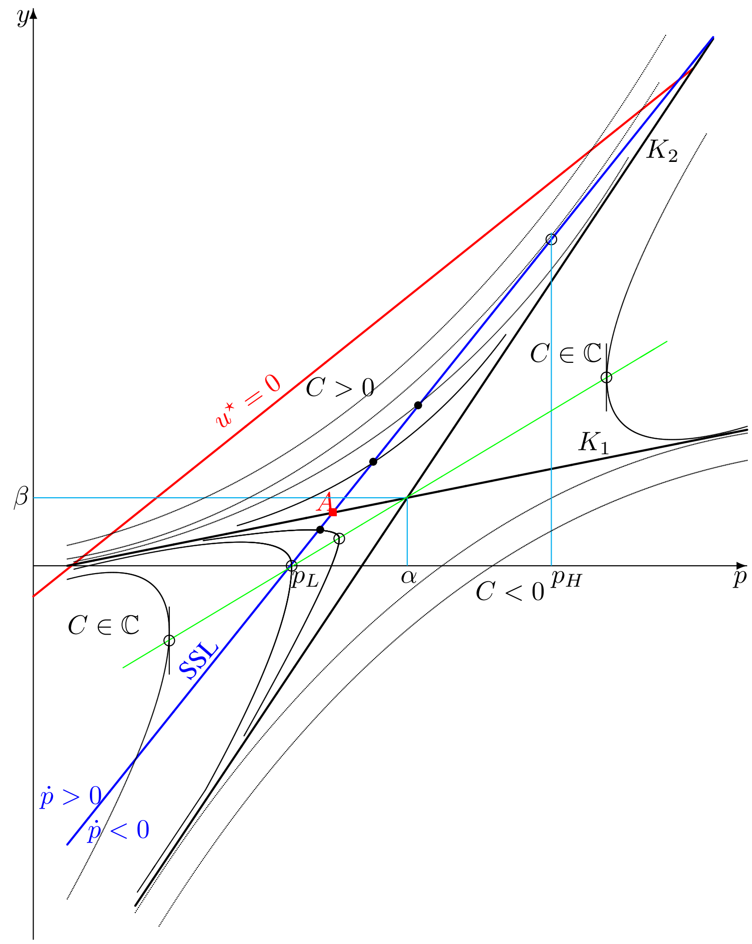


Figure 1: Sketch of the curves solution of equation (7)

controls at $u^\star$. We may notice that the slope of this line lies between the slopes $K_1$ and $K_2$ of the two boldface lines. A simple way to see that is to plug $K = (2b + d)/2b^2 s$ into the ARE (11) and see that the result is negative, namely

$$K^2 - \frac{2(2b+d)s + dr}{3b^2 s^2} K + \frac{1}{3b^2 s^2} = \frac{-1}{3b^4 s^2} \left[ bd + \frac{1}{4} d^2 + \frac{1}{2}(2b+d) d \frac{r}{s} \right].$$

The hyperbolic-like curves are various curves of the type of case 2 above, with various values of $C$. (Notice that in the "south-west" and "north-east" regions defined by the affine solutions, one of the factors is negative. Its non-integer power $\gamma$ or $(1 - \gamma)$ is therefore a complex number.) The light line through the point $(\alpha, \beta)$, called "green line" hereafter, is the line

$$3b^2 sy + ad + 2bc - (2b + d)p = 0$$

which joins the points where the curves have vertical tangents. Finally, the line labeled $u^\star = 0$ limits the region $u^\star \geq 0$ in case we want to take that constraint into account.

## 3.3 Sufficient conditions

### 3.3.1 A particular form of the verification theorem

On each side of its intersection with the green line, each hyperbolic-like curve defines a function $y = h(p)$ in some domain of the state space. To any of these, T&M associates the function

$$W(p) = \frac{1}{rd}\left\{\frac{3}{2}b^2 s^2 h^2(p) + [ad + 2bc - (2b+d)p]sh(p) + \frac{1}{2}(p-c)^2\right\}, \qquad (12)$$

copied after equation (6). Differentiate with respect to $p$ and use (7) to recognize that $\partial W/\partial p = y$, and thus, replacing in the above definition, we see that $W$ satisfies equation (6). And we have one such solution of (6) per segment of curve where a function $h$ is defined.

Select one curve, generating a function $y = h(p)$ in a certain domain of the state space. Consider the dynamics generated by $u_1 = u_2 = (p-c-bsh(p)/d$ from some $p_0$ in that domain. The point $(p, h(p))$ of the $(p, y)$ plane moves along the curve selected, the sign of $\dot{p}$ being given by the location of this point w.r.t. the blue line. We have marked with a solid dot the stable dynamic equilibria of such trajectories.

We are only interested in those dynamic evolutions that converge to a stationary point. We find them either in the "south-west" region, with a domain bounded above by the intersection with the green line, and in the "north-west" region, with the domain bounded above by the second intersection of the curve $h(p)$ with the blue line. (Which has a slope smaller than $K_2$.) All these domains are bounded below if we want to take the constraint $u^\star \geq 0$ into account, or by $p \geq 0$.

**Assertion** *The domain $L$ of the game considered will always be contained into the domain of the function $h(\cdot)$ used to define $W$.*

Here comes the sufficient condition tailored to the particular game problem (1)(4) considered by T&M.

**Theorem 1** *If there exists a continuously differentiable function $W$ from a bounded interval $L \subset \mathbb{R}$ into $\mathbb{R}$ such that*

1. *it satisfies the equation* (6) *in $L$,*
2. *the trajectory generated by $u_1(p) = u_2(p) = (p - c - bs\,\partial W/\partial p)/d$, from any $p(0) \in L$ converges to a stationary point $p_\infty \in L$,*
3. *$\forall p \in \partial L$, $W(p) \geq 0$,*

*then the srategies $u_i$ quoted above form a Nash equilibrium of the differential game problem with dynamics* (1)*, domain $L$ and pay-off* (4)*, the function $W$ is the Isaacs Value function of the game.*

**Proof** We need to prove that, if $u_2(t) = u^\star(p(t))$, then the strategy $u_1(t) = u^\star(p(t))$ maximizes player one's pay-off among all measurable time functions $u_1(\cdot)$. Under these controls, Isaacs equation may be written as a Lagrangian derivative:

$$\frac{\mathrm{d}\,[\mathrm{e}^{-rt}W(p(t))]}{\mathrm{d}t} + \mathrm{e}^{-rt}\left[a - c - \frac{d}{2}u_1(t)\right]u_1(t) \le 0\,,$$

equality being reached for $u_1 = u^\star$. Integrating from 0 to $T$, we get

$$\mathrm{e}^{-rT}W(p(T)) - W(p_0) + \int_0^T \mathrm{e}^{-rt}\left[a - c - \frac{d}{2}u_1(t)\right]u_1(t)\,\mathrm{d}t \le 0$$

equality being obtained for $u_1 = u^\star$.

If $u_1(t) = u^\star(p(t))$, then the inequality above is an equality by hypothesis 1, by hypothesis 2, $p(t) \to p_\infty \in L$, therefore, on the one hand, $T = \infty$, and on the other hand $W(p(t))$ remains bounded as $t \to \infty$, and we get that the infinite horizon integral, then player one's pay-off $J^1(u^\star, u^\star)$, is equal to $W(p_0)$.

If $u_1(t)$ is any control, then the inequality above stands. If for these controls, $p$ remains in $L$ for all $t > 0$, $L$ being by hypothesis bounded, $W(p(t))$ remains bounded, and we conclude that the infinite horizon integral is less or equal to $W(p_0)$, its value with $u_1 = u^\star$. On the other hand, if $p$ reaches $\partial L$ at a time $T$, we have, taking hypothesis 3 into account:

$$\begin{aligned} J^1(u_1, u^\star) = \int_0^T \mathrm{e}^{-rt}\left[a - c - \frac{d}{2}u_1(t)\right]u_1(t)\,\mathrm{d}t &\le W(p_0) - \mathrm{e}^{-rT}W(p(T)) \\ &\le W(p_0) = J^1(u^\star, u^\star)\,. \end{aligned}$$

QED.

It is worthwhile to mention that it is because we have this stopping rule $p(T) \in \partial L$ that the boundary condition on $W$ is so weak, allowing for an infinite number of functions $W$.

### 3.3.2 Domains $L$

There remains to define the corresponding allowable domain $L$ for each choice of function $h$, each defining a different differential game problem. This domain $L$ must be

1. contained in the domain of the corresponding function $h$,
2. contained in the attraction basin of a stable dynamic equilibrium for the equilibrium dynamics
3. contained in the domain $p \ge 0$ and, if we want to take that constraint into account, $u^\star(p) \ge 0$.

Consider figure 1. Under the equilibrium dynamics, states $(p, h(p))$ to the left of the steady-state (blue) line move "to the right" (i.e. $\dot{p} > 0$), and states $(p, h(p))$ to the right of the same line move "to the left", i.e. $\dot{p} < 0$. This allows us to easily spot possible stable stationary points, marked with solid dots on our figure. The (red) square dot labeled $A$ corresponds to the classical linear (or rather affine) state feedback of the infinite horizon unconstrained game. Clearly, the set of solid dots fills the open interval $p_L < p < p_H$ as indicated on the graphic. It is a simple matter to check that the lower bound where the steady-state blue line and the green line both intersect the $p$ axis is

$$p_L = \frac{ad + 2bc}{2b + d}\,,$$

the "purely competitive price" where price equals marginal production cost. (T&M quotes [Fershtman and Kamien, 1987]). It is a somewhat heavier, but still straightforward calculation to see that the upper bound, where a hyperbolic-like curve is tangent to the steady-state blue line, is

$$p_H = \frac{a[(2b + d)s + 2dr] + 2cb(s + 2r)}{(4b + d)s + (4b + 2d)r}\,.$$

The corresponding attraction basins are all limted below by the positivity constraint. The upper bound is obtained

- for the points on a $y = h(p)$ curve of the "north-west" region, by the limit of the attraction basin, i.e. the second intersection of that curve with the blue line (above it, the representative point moves to the left and up, away from the intersection)

- for the points in the "south-west" region, by the intersection of the curve with the green line marking the upper bound of the domain of the function $h$.

There remains to check the boundary condition, i.e. the positivity of $W$ on $\partial L$. As a matter of fact, T&M proves that $W$ defined as (12) is positive in the whole domains as characterized above. This proof is lengthy, and not very enlightning. Instead of summarizing it, we note the following fact:

The state-affine strategy corresponding to the relation $y = K_1(p - \alpha) + \beta$ and the steady state labeled $A$ in the graphic, of abscissa $p_A$, is known to be the Nash equlibrium strategy of the infinite horizon unconstrained game. Hence, for player 1, say, $u^\star$ is optimal against the same strategy of player 2. However, player 1 may always play $u_1 = 0$ and so doing get a zero pay-off. And it can be checked that the unique optimal strategy does not lead to $u_1 = 0$. Thus the optimum pay-off $J^1 = W(p_A)$ is strictly positive. By continuity, there exists a nighborhood of the point $A$ where $W(p) > 0$. Therefore, by choosing a small enough neighborhood of $p_A$ as domain $L$, we define a game, different from the classical infinite horizon LQ game, which admits an infinite number of non-linear state feedback Nash equilibrium strategies. Admittedly an unforeseen conclusion.

Finally, notice that to obtain equilibrium strategies with a stationary price that approaches $p_L$, we need to take domains $L$ with upper bounds that approach $p_L$, but taking that upper bound as being $p_L$ is not admissible, because that point is not stable for the equilibrium strategies generated. A similar remark holds for the upper stationary price. Hence no single game admits the whole open interval $(p_L, p_H)$ as possible stationary equilibrium prices.

# 4 Conclusion

Some confusion has arisen in the literature as the nature of the results of the article T&M. It has been understood as proving that the infinite horizon linear quadratic differential game defined by (1)(2) admits an infinite number of nonlinear, state feedback Nash equilibrium strategies, this variety being a consequence of the choice of admissible strategies, and/or of a constraint on the maximum admissible state (price). Particularily harmful in that respect is the last sentence of its section 3 : "Before we conclude this section, it is useful to stress that the equilibrium concept discussed above involves nothing new, but rather it attempts to clarify the meaning of stationary Markov feedback equilibrium when the domain of the state space is explicitely taken into account". As far as we know, the particular game (1)(4) is "something new" and has never been investigated in the previous literature.

In this note, we emphasize the fact that it is not just a choice of admissible strategies or of a constraint on the admissible states (as claimed by the article), but a choice of *game* that underlies this state of affairs. As far as we know, *the classical infinite horizon, linear quadratic differential game admits a unique pair of pure state feedback Nash equilibrium stategies. And these strategies are affine in the state*. (This might be the claim of Corollary 2 of Theorem 3 in T&M, although its statement is ambiguous. Noticeably, uniqueness is not stated.) The nonlinear strategies of T&M yield a Nash equilibrium of the free end time game game (1)(4) *which is not a state constrained version of an infinite horizon game.*

It has long been known (see [Başar and Olsder, 1982]) that there is, for the infinite horizon game, an infinite number of closed-loop Nash equilibrium strategies if different memory-strategies are allowed, an informational non-uniqueness. But this is not what is at play in T&M which concentrates on pure state feedback or "Markov" strategies.

Many authors using the method of T&M seem unaware of the fact that the nonlinear strategies that they advocate are *not* Nash equilibrium strategies of the infinite horizon game that they mean to investigate, not even with an exogeneously imposed bound on the state, but of another related free end-time game, which may depend on the particular nonlinear strategy chosen.

It is difficult to give a reasonable economic interpretation to these related games. T&M quotes the case where the government imposes a ceiling on allowable prices. But the model deals with that constraint in an unsatisfactory way: upon reaching it the game "stops". But presumably, life does not. Nothing is said of what happens after. In our opinion, in case a government imposed ceiling $p_{\max}$ applies, a more realistic model would be to say that the game keeps running for an infinite time, with the price frozen at $p_{\max}$ as long as its free dynamics (1) would make $\dot{p} \geq 0$. This would indeed be the infinite horizon game with a state constraint, $p$ obeying the variational inequality commonly used to modelize an "obstacle":

$$\max\{p - p_{\max}\,, \dot{p} - s[a - b(u_1 + u_2) - p]\} = 0\,.$$

Yet, the article offers a new approach to the analysis of the Hamilton Jacobi Isaacs equation, and does exhibit games with linear dynamics and quadratic running costs (but not the classical pay-off (2)) that admit an infinite number of nonlinear state feedback Nash equilibrium strategies. A surprising fact.

As a “mathematical” explanation for this indeterminacy, T&M cites an “incomplete transversality condition”, but what it calls “transversality condition” is just the constraint that everything becomes constant at the equilibrium point. What plays the role of transversality conditions of these particular games are that, on the one hand, $W(p(t))$ remains bounded when $t \to \infty$, (guaranteed by the fact that $p(t) \in L$ and $h(p)$ remain bounded) and on the other hand, condition 3 of theorem 1: $\forall p \in \partial L \,, W(p) \geq 0$. Admittedly rather loose constraints.

T&M also proposes an economic interpretation of what it terms a non uniqueness of the Markov Nash equilibrium strategies of its game. It first (rightly) discounts an informational non uniqueness, quoting [Başar and Olsder, 1982], and in a footnote writes “The non uniqueness in this paper is based solely on the game’s infinite time horizon”. It continues with a discussion by comparison with (infinitely) repeated games and the Folk theorem. To warrant that comparison, it states “[...] after all, our dynamic game has an infinite time horizon”. Therefore, this analysis is entirely based upon the false pretense that the game solved is an infinite horizon game, in spite of the precise definition (4). Thus strengthening the ambiguity.

As a last question, one wonders what the method of T&M does if applied to a (one agent) optimal control problem. Indeed, particularizing T&M’s set-up to a simple problem with only one decision maker yields a perfectly well defined optimization problem. Yet a maximization problem with “several supremums” would be paradoxical.

The answer to that puzzle is that with a single decision maker, the “blue”, steady state, line coincides with the “green” line joining the points with “vertical” tangents to the hyperbolic-like curves. Therefore no intersection exists between this blue line and a valid $y = h(p)$ graph, except for the affine one.

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