Robust Control and Dynamic Games

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Abstract

We describe several problems of "robust control" that have a solution using game theoretical tools. This is by no means a general overview of robust control theory beyond that specific purpose, nor a general account of system theory with set-description of uncertaineties.

Keywords Robust control, $\mathcal{H}^\infty$-optimal control, nonlinear $\mathcal{H}^\infty$ control, robust control appraoach to option pricing.

1 Introduction

Control theory is concerned with the design of control mechanisms for dynamic systems that compensate for (a priori) unknown disturbances acting on the system to be controlled. While early servomechanism techniques did not make use of much modeling, neither of the plant to be controlled nor of the disturbances, the advent of multi-input multi-output systems and the drive to more stringent specifications led researchers to use mathematical models of both. In that process, and most prominently with the famous LQG design technique, disturbances were described as unknown inputs to a known plant, and usually high frequency inputs.

"Robust control" refers to control mechanisms of dynamic systems that are designed to counter unknowns in the system equations describing the plant rather than in the inputs. This is also a very old topic. It might be said that a PID controller is a "robust" controller, since it makes little assumption on the controlled plant. Such techniques as gain margin and phase margin guarantees and loop shaping also belong in that class. However, the gap between "classic" and "robust" control designs became larger with the advent of "modern" control designs.

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As soon as the mid seventies, some control design techniques appeared, often based upon some sort of Lyapunov function, to address that concern. A most striking feature is that they made use of bounded uncertainties rather than the prevailing Gaussian-Markovian model. One may cite [24], [23], [14]. This was also true with the introduction of the so-called $\mathcal{H}^\infty$ control design technique in 1981 [34], and will become such a systematic feature that "robust control" became in many cases synonymous to control against bounded uncertainties, usually with no probabilistic structure on the set of possible disturbances.

We stress that this is *not* the topic of this chapter. It is not an account of system theory for systems with set description of the uncertainties. Such comprehensive theories have been developed using set-valued analysis and/or elliptic approximations. See e.g. [2] and [26]. We restrict our scope to *control* problems, thus ignoring pure state estimation such as developed in the above references, or in [7, chap. 7] or [28] for instance, and to game theoretic methods in control design, thus ignoring other approaches of robust control such as the gap metric [33, 21, 27], an input-output approach in the spirit of the original $\mathcal{H}^\infty$ approach of [34], using transfer functions and algebraic methods in the ring of rational matrices, or Linear Matrix Inequality (LMI) approaches [25], and many other approaches found, *e.g.*, in the same Encyclopaedia of Systems and Control [29].

We will first investigate in section 2 the issue of modeling of the disturbances, show how non-probabilistic set descriptions naturally arise, and describe some specific tools we will use to deal with them. Subsection 2.1 is an elementary introduction to these questions, while subsection 2.2 is more technical, introducing typical control theoretic issues and examples in the discussion. From there on, a good control of linear systems theory, optimal control —specifically the Hamilton-Jacobi-Caratheodory-Isaacs-Bellman (HJCIB) theory— and dynamic games theory is required. Then, in section 3, we will give a rather complete account of the linear $\mathcal{H}^\infty$-optimal control design for both continuous time and discrete time systems. Most of the material of these two sections can be found in more detail in [7]. In section 4, we will cover in less detail the so called "nonlinear $\mathcal{H}^\infty$" control design, mainly addressing engineering problems, and a nonlinear example in the very different domain of mathematical finance.

2 Min-max problems in robust control

2.1 Decision theoretic formulation

2.1.1 Worst case design

We start this section with a very general set-up in terms of decision theory. Let U be a decision space, and W a disturbance space. We may think of W as being a bounded set in some ad hoc norm, but this is not necessary at this level of generality. Let $J : U \times W \to \mathbb{R}$ be a performance index, depending on the decision $u \in U$ and on an unknown disturbance $w \in W$. The decision maker, choosing u, wants to make J as small as possible in spite of the a priori unknown disturbance $w \in W$. (One may think of J as a measure, such as the L^2 norm of an error signal.)

The so called *worst case design* method is as follows. Let us first emphasize that there is no "malicious adversary" manipulating the disturbance in the following description, contrary to many accounts of this method.

The basic concept is that of *guaranteed performance* of a given decision $u \in U$: any number g such that

$$\forall w \in W\,, \quad J(u, w) \leq g.$$

Clearly, the best (smallest) guaranteed performance for a given decision u is

$$G(u) = \sup_{w \in W} J(u, w)\,.$$

Hence the phrase "worst case", which has to do with a guarantee, not any malicious adversary.

Now, the problem of finding the best possible decision in this context is to find the smallest guaranteed performance, or

$$\inf_{u \in U} G(u) = \inf_{u \in U} \sup_{w \in W} J(u, w)\,. \tag{1}$$

If the infimum in u is reached, then the minimizing decision $u^\star$ deserves the name of optimal decision in that context.

2.1.2 Robust disturbance rejection or gain control

The following approach is mostly justified in the following linear set-up, but can be extended to a nonlinear one (see [31]). We let U and W be normed vector spaces, Z be an auxiliary normed vector space, $z \in Z$ the output whose norm is to be kept small in spite of the disturbances w. We assume that for each decision $u \in U$,

z depends linearly on w. Therefore, one has a (possibly nonlinear) application $P : U \to \mathcal{L}(W \to Z)$ and

$$z = P(u)w .$$

Clearly, z cannot be kept bounded if w is not. A natural formalization of the problem of keeping it small is to try and make the operator norm of $P(u)$ as small as possible. This may be expressed as follows: let $B(W)$ be the unit ball of W, one seeks

$$\inf_{u \in U} \sup_{w \in B(W)} \|P(u)w\| . \tag{2}$$

But we will often prefer another formulation, again in terms of guaranteed performance, and which can easily be extended to nonlinear systems. Start from the problem of ensuring that $\|P(u)\| \leq \gamma$ for a given positive *attenuation level* γ. This is equivalent to

$$\forall w \in W , \quad \|z\| \leq \gamma \|w\| ,$$

or equivalently (but leading to a smoother problem)

$$\forall w \in W , \quad \|z\|^2 \leq \gamma^2 \|w\|^2 , \tag{3}$$

or equivalently again

$$\sup_{w \in W} [\|P(u)w\|^2 - \gamma^2 \|w\|^2] \leq 0 .$$

Now, given a number γ, this has a solution (there exists a decision $u \in U$ satisfying that inequality) *if* the infimum hereafter is reached or is negative, and *only if*

$$\inf_{u \in U} \sup_{w \in W} [\|P(u)w\|^2 - \gamma^2 \|w\|^2] \leq 0 . \tag{4}$$

This is the method of $\mathcal{H}^\infty$-optimal control. Notice, however, that equation (3) has a meaning even for a nonlinear operator $P(u)$, so that the problem (4) is used in the so-called "nonlinear $\mathcal{H}^\infty$" control.

We will see a particular use of the control of the operator norm in feedback control when using the "small gain theorem". Although this could be set in the abstract context of nonlinear decision theory (see [32]), we will rather show it in the more explicit context of control theory.

Remark 2.1 *The three problems outlined above as equations* (1)*,* (2)*, and* (4) *are all of the form* $\inf_u \sup_w$*, and therefore, in a dynamic context, are amenable to dynamic game machinery. They involve no probabilistic description of the disturbances.*

2.1.3 Set description vs probabilistic

Two main reasons have driven robust control towards set theoretic, rather than probabilistic, descriptions of disturbances. On the one hand, the traditional Gauss-Markov representation is geared towards high frequency disturbances. When the main disturbance is, say, a misevaluated parameter in a dynamic equation, it is the exact opposite of high frequency: zero-frequency. And this is typically the type of unknowns that robust control is meant to deal with.

On the other hand, concerning the coefficients of a differential equation, using a probabilistic description requires that one deals with differential equations with random coefficients, a much more difficult theory than that of differential equations driven by a stochastic process.

Thus, "robust control approaches" to decision problems involve a description of the unknown disturbances in terms of a set containing all possible disturbances, often a bounded set, and hence hard bounds on the possible disturbances. As mentioned in the introduction, systematic theories have been developed to deal with set descriptions of uncertainties under the name of *Viability theory* [2] —particularly well suited to deal with non-smooth problems in engineering, environment, economy, finance [12, part V]—, or *Trajectory Tubes* which, with the use of Hamiltonian formalism and elliptic approximations, recover more regularity, and also leads to a theory of robust control in the presence of corrupted information [26, chap. 10]. We emphasize here somewhat different, and more classical, methods, making explicit use of dynamic game theory.

To practitioners accustomed to Gauss-Markov processes (such as a filtered "white noise"), hard bounds may seem a rather severe limitation, and providing the mathematical model with too much information. Yet, a probability law is an extremely rich information itself; in some sense, much richer than a simple set description. As an example, note that if we model a disturbance function as an ergodic stochastic process, then we tell the mathematical model that the long time average of the disturbance is exactly known. In the case of a Brownian motion, we say that the total quadratic variation of (almost) all realizations is exactly known. And these are only instances of the rich information we provide, that the mathematical machinery will use, may be far beyond what was meant by the modeler.

2.2 Control formulation

We want to apply the above ideas to control problems, i.e. problems where the decision to be taken is a control of a dynamic system. This raises the issues of causality, and open-loop versus closed-loop control. No control theoretician would want to rely on open-loop control to overcome uncertainties and disturbances in a system. But then, the issue of the available information to form one's control becomes crucial.

2.2.1 Closed-loop control

Let a dynamic system be represented by a differential equation in $\mathbb{R}^n$ (we will therefore ignore the existing extensions of these results to infinite-dimensional systems, see [9] for a game theoretic approach):

$$\dot{x} = f(t, x, u, w) \tag{5}$$

and two outputs; y the observed output and z to be controlled:

$$z(t) = g(t, x, u)\,, \tag{6}$$

$$y(t) = h(t, x, w)\,, \tag{7}$$

Here, as in the sequel, we have $x \in \mathbb{R}^n$, $u \in \mathsf{U} \subset \mathbb{R}^m$, $w \in \mathsf{W} \subset \mathbb{R}^\ell$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$, and $u(\cdot) \in \mathcal{U}$, $w(\cdot) \in \mathcal{W}$ the sets of measurable time functions into U and W respectively. Likewise, we let $y(\cdot) \in \mathcal{Y}$ and $z(\cdot) \in \mathcal{Z}$. Moreover, we assume that f enjoys regularity and growth properties that insure existence and uniqueness of the solution to (5) for any initial condition and any pair of controls $(u(\cdot), w(\cdot)) \in \mathcal{U} \times \mathcal{W}$.

Since we are concerned with causality, we introduce the following notation: For any time function $v(\cdot)$, let v^t denote its restriction $[v(s)|s \le t]$. And if $\mathcal{V}$ is the set of functions $v(\cdot)$, let $\mathcal{V}^t$ be the set of their restrictions v^t:

$$\begin{array}{c} \mathcal{V} = \{v(\cdot) : [t_0, t_1] \to \mathsf{V} : s \mapsto v(s)\} \Longrightarrow \\ \forall t \in (t_0, t_1)\,,\ \mathcal{V}^t = \{v^t : [t_0, t] \to \mathsf{V} : s \mapsto v(s)\}\,. \end{array} \tag{8}$$

The control $u(t)$ will be synthesized as a closed-loop control

$$u(t) = \phi(t, y^t)\,, \tag{9}$$

typically by a dynamic compensator driven by the observed output y. Therefore, the operator $\inf_{u \in U}$ in equations (1), (2), and (4) must be replaced by $\inf_{\phi \in \Phi}$. But then, we must specify the class Φ of admissible controllers. However, we postpone that discussion until after we deal with the disturbance w.

The question arises as to whether one should allow closed-loop "disturbance laws", some $w(t) = \psi(t, u^t)$, or be content with open-loop disturbances $w(\cdot) \in \mathcal{W}$. This question is answered by the following lemma, occasionally attributed to Leonard Berkowitz [10]:

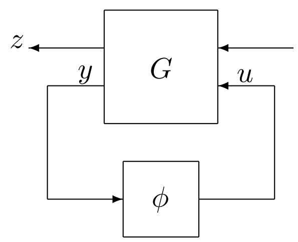

Figure 1: Disturbance rejection problem

Lemma 2.1 *Let a criterion* $J(u(\cdot), w(\cdot)) = K(x(\cdot), u(\cdot), w(\cdot))$ *be given. Let a control function* ϕ *be given, and* Ψ *be a class of closed-loop disturbance strategies* compatible with ϕ *(i.e. such that the differential equations of the dynamic system have a solution when* u *is generated by* ϕ *and* w *by any* $\psi \in \Psi$*). Then, (with a transparent abuse of notation)*

$$\sup_{w(\cdot)\in\mathcal{W}} J(\phi, w(\cdot)) \geq \sup_{\psi\in\Psi} J(\phi, \psi)\,.$$

Therefore, it is never necessary to consider closed-loop disturbance laws. This greatly simplifies the discussion of the class Φ of admissible controllers. It only has to be such that solutions to the differential equations exist against all open-loop measurable disturbances. This is an important difference with "true" (dynamic) game theory, arising from the fact that we are not concerned with the existence of a Value, but only with $\inf\sup$ operators.

The control form of the disturbance rejection problem of paragraph 2.1.2 may therefore be stated, for the system G given by (5), (6), (7) and (9) over a time span $\mathcal{T}$, as follows: let Φ be the class of all causal controllers ensuring the existence in $L^2(\mathcal{T})$ and uniqueness of the solution of the system equations.

Standard problem of $\mathcal{H}^\infty$-optimal control with attenuation level γ: *Does the following inequality hold:*

$$\inf_{\phi\in\Phi} \sup_{w\in L^2(\mathcal{T})} \left(\|z\|_{L^2}^2 - \gamma^2\|w\|_{L^2}^2\right) \leq 0\,? \tag{10}$$

If the infimum is reached or is negative, find an admissible controller ϕ *ensuring the inequality* (3) *in* L^2 *norms.*

2.2.2 Minimax certainty equivalence principle

The standard problem as formulated above amounts to a min-sup differential game with a real time minimizer's information both partial and corrupted by the maximizer. A non-classical problem. In some favorable cases, including that of linear systems, the information problem can be solved via the following certainty equivalence theorem [7, 13]. Let a differential game be specified by (5) and a criterion:

$$J(x_0; u(\cdot), w(\cdot)) = M(x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), u(t), w(t))\,\mathrm{d}t + N(x_0)\,. \qquad (11)$$

Assume that the full state information min-sup problem with $N \equiv 0$ has a state feedback solution

$$u(t) = \varphi^\star(t, x(t))\,, \qquad (12)$$

leading to a Value function $V(t,x)$ of class C^1.

We now define a partial information mechanism. Since we assume that the minimizer does not know $x(t)$, it is consistent to assume that he does not know x_0 either. We therefore allow, in the partial information problem, the added cost $N(x_0)$. Let $\omega \in \Omega = \mathbb{R}^n \times \mathcal{W}$ be the complete disturbance, i.e. the pair $(x_0, w(\cdot))$. Recall notation (8). An *observation process* is a device that, at each time instant t, defines a subset $\Omega_t \subset \Omega$ function of $u(\cdot)$ and ω, which enjoys the following three properties:

1. It is *consistent*: $\forall u \in \mathcal{U}\,,\ \forall \omega \in \Omega\,,\ \forall t\,,\ \omega \in \Omega_t$.
2. It is *perfect recall*: $\forall u \in \mathcal{U}\,,\ \forall \omega \in \Omega\,,\ \forall t' \geq t\,,\ \Omega_{t'} \subset \Omega_t$.
3. It is *nonanticipative*: $\omega^t \in \Omega_t^t \Rightarrow \omega \in \Omega_t$.

In the case of a system (5)-(7),

$$\Omega_t(u(\cdot), \omega) = \{(x_0, w(\cdot)) \mid \forall s \leq t\,,\ h(s, x(s), w(s)) = y(s)\}.$$

We seek a controller

$$u(t) = \phi(t, \Omega_t)\,. \qquad (13)$$

Define the *auxiliary criterion*

$$G_t(u^t, \omega^t) = V(t, x(t)) + \int_{t_0}^{t} L(s, x(s), u(s), w(s))\,\mathrm{d}s + N(x_0)\,,$$

and the *auxiliary problem*:

$$\max_{\omega^t \in \Omega_t^t} G_t(u^t, \omega^t)\,.$$

Note that in this problem, u^t is known, as our own past control.

Theorem 2.1 *If the auxiliary problem admits one or several solutions leading to a* unique *state* $\hat{x}(t)$ *at time t, then, if the controller*

$$u(t) = \varphi^\star(t, \hat{x}(t))$$

is admissible, it is a min-sup controller in partial information.

Conversely, if there exists a time t such that $\sup_{\omega^t \in \Omega^t_t} G_t = \infty$*, then the criterion* (11) *has an infinite supremum in* ω *for any admissible feedback controller* (13).

Remark 2.2 *The state* $\hat{x}(t)$ *may be considered as the* worst possible state *given the available information.*

One way to proceed in the case of an information such as (7) is to solve by forward dynamic programming (forward Hamilton-Jacobi-Caratheodory-Bellman equation) the constrained problem:

$$\max_\omega \left[\int_{t_0}^t L(s, x(s), u(s), w(s))\, \mathrm{d}s + N(x_0) \right]$$

subject to the control constraint $w(s) \in \{w \mid h(s, x(s), w) = y(s)\}$, and the terminal constraint $x(t) = \xi$. It is convenient to call $-W(t, \xi)$ the corresponding Value (or Bellman function). Assuming that, under the strategy $\varphi^\star$, for every t the whole space is reachable by some $\omega \in \Omega$, $\hat{x}(t)$ is obtained via

$$\max_{x \in \mathbb{R}^n} [V(t, x) - W(t, x)] = V(t, \hat{x}(t)) - W(t, \hat{x}(t))\,. \tag{14}$$

If this max is reached at a unique $x = \hat{x}(t)$, then the uniqueness condition of the theorem is satisfied.

Remark 2.3 *In the duality between probability and optimization (see [3, 1]) the function* $-W(t, \cdot)$ *is the* conditional cost measure *of the state for the measure* $\int_{t_0}^t L\mathrm{d}s + N(x_0)$ *knowing* $y(\cdot)$*. The left-hand-side of formula* (14) *is the dual of a mathematical expectation.*

2.2.3 Small gain theorem

We aim to show classical linear control problems that can be cast into a standard problem of $\mathcal{H}^\infty$-optimal control. We develop some preliminary tools.

Linear operators and norms In the case of linear systems, operator norms have concrete forms.

Matrix: A $p \times m$ matrix M represents a linear operator $\mathcal{M} : \mathbb{R}^m \to \mathbb{R}^p$ whose norm $\|\mathcal{M}\|$ is the maximum singular value $\sigma_{\max}(M)$ of the matrix.

Dynamic system: A stable stationary linear dynamic system with input $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^p$ will be considered as a linear operator, say $\mathcal{G}$, from $L^2(\mathbb{R} \to \mathbb{R}^m)$ to $L^2(\mathbb{R} \to \mathbb{R}^p)$. It may be represented by its transfer function $G(s)$, always a rational proper matrix since we confine ourselves to finite-dimensional state systems. Moreover, the system being stable (otherwise it would not map L^2 into L^2), the transfer function has all its poles in the left half complex plane. Thus it belongs to the Hardy space $\mathcal{H}^\infty$ of functions holomorphic in an open set of the complex plane containing the half-plane $\mathrm{Re}(s) \geq 0$. In that case, it follows from Parseval's equality that

$$\|\mathcal{G}\| = \sup_{\omega \in \mathbb{R}} \sigma_{\max} G(\jmath\omega) =: \|G(\cdot)\|_\infty \,.$$

The norm $\|G(\cdot)\|_\infty$ (or $\|G\|_\infty$) is the norm of the transfer function in $\mathcal{H}^\infty$.

Block operator: If the input and output spaces of a linear operator $\mathcal{G}$ are represented as product of two spaces each, the operator takes a block form

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{pmatrix}.$$

We will always consider the norm of product spaces as the Euclidean combination of the norms in the component spaces. In that case, it holds that $\|\mathcal{G}_{ij}\| \leq \|\mathcal{G}\|$. Furthermore, whenever the following operators are defined

$$\left\| \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{pmatrix} \right\| \leq \sqrt{\|\mathcal{G}_1\|^2 + \|\mathcal{G}_2\|^2}\,, \qquad \|(\mathcal{G}_1 \quad \mathcal{G}_2)\| \leq \sqrt{\|\mathcal{G}_1\|^2 + \|\mathcal{G}_2\|^2}\,.$$

Hence

$$\left\| \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} \end{pmatrix} \right\| \leq \sqrt{\|\mathcal{G}_{11}\|^2 + \|\mathcal{G}_{12}\|^2 + \|\mathcal{G}_{21}\|^2 + \|\mathcal{G}_{22}\|^2}\,.$$

We finally notice that for a block operator $\mathcal{G} = (\mathcal{G}_{ij})$, its norm is related to the matrix norm of its matrix of norms $(\|\mathcal{G}_{ij}\|)$ by $\|\mathcal{G}\| \leq \|(\|\mathcal{G}_{ij}\|)\|$.

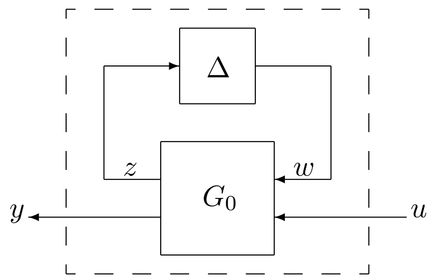

Figure 2: The perturbed system $\mathcal{G}_\Delta$

Small gain theorem Let a two input-two output linear system G_0 be given by

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{10} \\ \mathcal{G}_{01} & \mathcal{G}_{00} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} . \tag{15}$$

Assume that it is connected in feedback by a linear operator Δ according to $w = \Delta z$, leading to a composite system $y = \mathcal{G}_\Delta u$. (See figure 2.) Then, an easy consequence of Banach's fixed point theorem is:

Theorem 2.2 (Small gain theorem) *If* $\|\Delta\mathcal{G}_{11}\| = \alpha < 1$ *or* $\|\mathcal{G}_{11}\Delta\| = \alpha < 1$, *then the combined system*

$$\mathcal{G}_\Delta = \mathcal{G}_{01}(I - \Delta\mathcal{G}_{11})^{-1}\Delta\mathcal{G}_{10} + \mathcal{G}_{00} \quad \textit{or} \quad \mathcal{G}_\Delta = \mathcal{G}_{01}\Delta(I - \mathcal{G}_{11}\Delta)^{-1}\mathcal{G}_{10} + \mathcal{G}_{00}$$

is well defined and is stable. Moreover, it holds that

$$\|\mathcal{G}_\Delta\| \leq \|\mathcal{G}_{00}\| + \|\mathcal{G}_{01}\|\,\|\mathcal{G}_{10}\|\frac{\|\Delta\|}{1-\alpha} .$$

Corollary 2.1 *If* $\|\mathcal{G}_{11}\| < 1$, *then* $\mathcal{G}_\Delta$ *is stable for any* Δ *with* $\|\Delta\| \leq 1$.

This will be a motivation to try and solve the problem of making the norm of an operator smaller than 1, or smaller than a given number γ.

2.3 Robust servo-mechanism problem

2.3.1 Model uncertainty

Assume we deal with a linear system

$$\dot{x} = Ax + Bu\,,$$
$$y = Cx + Du\,.$$

But the system matrices are not precisely known. We have an estimate or "nominal system" (A_0, B_0, C_0, D_0), and we set $A = A_0+\Delta_A$, $B = B_0+\Delta_B$, $C = C_0+\Delta_C$, $D = D_0 + \Delta_D$. The four matrices Δ_A, Δ_B, Δ_C, and Δ_D are unknown, but we know bounds δ_A, δ_B, δ_C, and δ_D on their respective norms. We rewrite the same system as

$$\dot{x} = A_0 x + B_0 u + w_1\,,$$
$$y = C_0 x + D_0 y + w_2\,,$$
$$z = \begin{pmatrix} x \\ u \end{pmatrix}\,,$$

and obviously

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = \Delta z\,.$$

We now have a known system G_0 connected in feedback by the unknown system Δ for which we have a norm bound $\|\Delta\| \le (\delta_A^2 + \delta_B^2 + \delta_C^2 + \delta_D^2)^{1/2}$, to which we can apply the small gain theorem.

2.3.2 Robust stabilization

Indeed, that representation of a partially unknown system can be extended to a dynamic unknown perturbation Δ. We rename w and z as v and ζ respectively, and we add a disturbance input w, an output z to be controlled, and a dynamic compensator $u = Ky$. From now on, to make things simpler, we assume that the signals v and ζ are scalar (See below a short discussion of vector signals). Assume that a frequency dependent bound of the modulus of Δ's transfer function is known as $|\Delta(\jmath\omega)| \le \delta(\omega)$. Then, in a classical "loop shaping" fashion, we devise a dynamic filter $W_1(s)$ such that $|W_1(\jmath\omega)| \ge \delta(\omega)$, and consider the fictitious control system with output $\tilde{\zeta} = W_1\zeta$. Stabilization of the system will be guaranteed if we can keep the transfer function norm from v to $\tilde{\zeta}$ less than one. We may also specify the disturbance rejection objective as keeping a fictitious output $\tilde{z} = W_0 z$ small

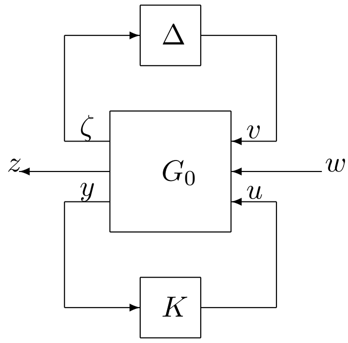

Figure 3: The perturbed system and its compensator

for some prescribed dynamic filter W_0. The control objective of the compensator K will now be to keep the modulus of the transfer function from v to $\tilde{\zeta}$ less than one at all frequencies, while holding the norm of the transfer function from w to $\tilde{z}$ as small as possible. This may be cast into a standard problem for instance by choosing the pair $((1/\gamma)v, w)$ as the input, $(\tilde{\zeta}, \beta\tilde{z})$ as the output, and finding β and γ as small as possible while holding inequality (10) true. But more clever weightings may also be tried.

Some remarks are in order.

Remark 2.4

1. *We might as well have placed the shaping filter W_1 on the input channel v. The two resulting control problems are not equivalent. One may have a solution and the other one none. Furthermore, the weight $\delta(\omega)$ might be divided between two filters, one on the input and one on the output. The same remark applies to the filter W_0 and to the added weights β and γ. There is no known simple way to arbitrate these possibilities.*

2. *In case of vector signals v and ζ, one can use diagonal filters with the weight W_1 on each channel. But this is often inefficient. There exists a more elaborate way to handle that problem, called "μ-synthesis". (See [19, 35].)*

2.3.3 Robust servomechanism

We describe a particular case of the above problem which was at the inception of the $\mathcal{H}^\infty$-optimal control problem in [34]. All signals are scalar.

An uncertain system's transfer function G is described by a multiplicative uncertainty: $G = (I + \Delta)G_0$. We know G_0 and a bound $|\Delta(\jmath\omega)| \leq \delta(\omega)$. However, it holds that as $\omega \to \infty$, $G_0(\jmath\omega) \to 0$, and because of a possible fixed offset in the true plant, $\delta(\omega) \to \infty$. The aim of the control system is to have the output y follow an unpredictable, but low frequency, reference signal w thanks to a dynamic compensator $u = K(w - y)$. We name z the error signal $w - y$, ζ the (fictitious) output of the (fictitious) system G_0, and $v = \Delta\zeta$. (See figure 4.) We can cast

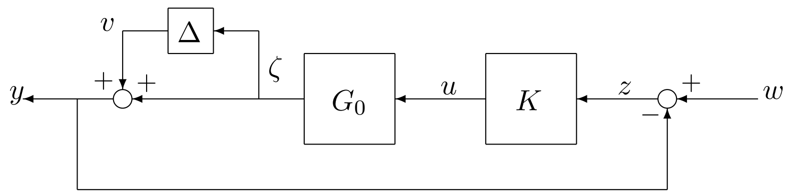

Figure 4: The servomechanism

this problem as a standard problem with input (v, w) and output (ζ, z). However, adding frequency dependent weights is now unavoidable. We have

$$\begin{aligned} z &= -G_0 u - v + w\,, \\ \zeta &= G_0 u\,, \\ u &= Kz\,, \end{aligned}$$

hence

$$\begin{aligned} z &= S(w - v)\,, \qquad & S &= (I + G_0 K)^{-1}\,, \\ \zeta &= T(w - v)\,, \qquad & T &= G_0 K (I + G_0 K)^{-1}\,. \end{aligned}$$

The control aim is to keep the *sensitivity transfer function* S small, while robust stability requires that the *complementary sensitivity transfer function* T be small. However, it holds that $S + T = I$, and hence, both cannot be kept small simultaneously. We need that

$$\forall \omega \in \mathbb{R}\,, \qquad \delta(\omega)|T(\jmath\omega)| \leq 1\,.$$

And if $\delta(\omega) \to \infty$ as $\omega \to \infty$, this imposes that $|T(\jmath\omega)| \to 0$, and therefore $|S(\jmath\omega)| \to 1$. Therefore we cannot follow a high frequency reference input w. However, it may be possible to keep $|S(\jmath\omega)|$ small at low frequencies, where $\delta(\omega)$ is small. The solution is to work with dynamic filters and fictitious outputs $\tilde{z} = W_0 z$, $\tilde{\zeta} = W_1 \zeta$, with $|W_1(\jmath\omega)| \geq \delta(\omega)$ (a generalized differentiator). If we can keep the transfer function from v to $\tilde{\zeta}$ less than one at all frequencies, we ensure stability of the control system. And we choose a weight W_0 large at low frequencies (a generalized integrator), and investigate the standard problem with the output $(\tilde{z}, \tilde{\zeta})$.

If $W_0 G_0$ is strictly proper, as it will usually be, there is no throughput from u to the output $\tilde{z}$. In that case, it is necessary to add a third component $\tilde{u} = W_3 u$

to the regulated output, with W_3 proper but not strictly (may be a multiplicative constant R). This is to satisfy the condition $R > 0$ of the next section.

3 $\mathcal{H}^\infty$-optimal control

Given a linear system with inputs u and w and outputs y and z, and a desired attenuation level γ, we want to know whether there exist causal control laws $u(t) = \phi(t, y^t)$ guaranteeing inequality (3), and if yes, find one. This is the *standard problem* of $\mathcal{H}^\infty$-optimal control. We propose here an approach of this problem based upon dynamic game theory, following [7]. Others exist. See [18, 30, 22].

We denote by an accent the transposition operation.

3.1 Continuous time

We start with a state space description of the system. Here $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^\ell$, $y \in \mathbb{R}^p$, $z \in \mathbb{R}^q$. This prescribes the dimensions of the various matrices in the system equations. In the finite horizon problem, they may all be time dependent, say piecewise continuously. These equations are:

$$\dot{x} = Ax + Bu + Dw\,, \quad x(t_0) = x_0\,, \tag{16}$$

$$y = Cx \qquad\quad + Ew\,, \tag{17}$$

$$z = Hx + Gu\,. \tag{18}$$

We will use the *system matrix*

$$\mathcal{S} = \begin{pmatrix} A & B & D \\ C & 0 & E \\ H & G & 0 \end{pmatrix}. \tag{19}$$

Remark 3.1

1. *The fact that "the same" input w drives the dynamics* (16) *and corrupts the output y in* (17) *is not a restriction. Indeed different components of w may enter the different equations. (Then $DE' = 0$.)*

2. *The fact that we allowed no throughput from u to y is not restrictive. y is the measured output used to control the system. If there were a term $+Fu$ on the r.h.s. of* (17)*, we could always use $\tilde{y} = y - Fu$ as measured output.*

3. *A term $+Fw$ in* (18) *would create cross terms in wx and wu in the linear quadratic differential game to be solved. The problem would remain feasible, but the equations would be more complicated, which we would rather avoid.*

We further let

$$\begin{pmatrix} H' \\ G' \end{pmatrix} (H \quad G) = \begin{pmatrix} Q & P \\ P' & R \end{pmatrix} \text{ and } \begin{pmatrix} D \\ E \end{pmatrix} (D' \quad E') = \begin{pmatrix} M & L' \\ L & N \end{pmatrix}. \qquad (20)$$

All the sequel makes use of the following hypotheses:

Assumption 3.1

1. *G is one to one or, equivalently, $R > 0$,*
2. *E is onto or, equivalently, $N > 0$.*

Duality *Notice that changing $\mathcal{S}$ into its transpose $\mathcal{S}'$ swaps the two block matrices in* (20) *and also the above two hypotheses. This operation (together with reversal of time), that we will encounter again, is called* duality.

3.1.1 Finite horizon

As in subsection 2.2.2, we include initial and terminal costs. For any symmetric nonnegative matrix Z, we write $\|x\|_Z^2 := x'Zx$ and likewise for other vectors. Let Q_0 be a symmetric positive definite matrix, and Q_1 a nonnegative definite one, and

$$J_\gamma(x_0, u(\cdot), w(\cdot)) = \|x(t_1)\|_{Q_1}^2 + \int_{t_0}^{t_1} \left[(x'(t) \quad u'(t)) \begin{pmatrix} Q & P \\ P' & R \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} - \gamma^2 \|w(t)\|^2 \right] \mathrm{d}t - \gamma^2 \|x_0\|_{Q_0}^2 . \quad (21)$$

We introduce two Riccati equations for the symmetric matrices S and Σ, where we use the feedback gain F and the Kalman gain K appearing in the LQG control problem:

$$F = -R^{-1}(B'S + P'), \qquad K = (\Sigma C' + L')N^{-1}, \qquad (22)$$

$$-\dot{S} = SA + A'S - F'RF + \gamma^{-2} SMS + Q, \qquad S(t_1) = Q_1, \qquad (23)$$

$$\dot{\Sigma} = \Sigma A' + A\Sigma - KNK' + \gamma^{-2}\Sigma Q \Sigma + M, \qquad \Sigma(t_0) = Q_0^{-1}. \qquad (24)$$

Theorem 3.1 *For a given γ, if both equations* (23) *and* (24) *have solutions over $[t_0, t_1]$ satisfying*

$$\forall t \in [t_0, t_1], \quad \rho(\Sigma(t)S(t)) \le \gamma^2, \qquad (25)$$

the finite horizon standard problem has a solution for that value of γ and any larger one. In that case, a (nonunique) min-sup controller is given by $u = u^\star$ as defined by either of the following two systems:

$$\dot{\hat{x}} = (A + \gamma^{-2}MS)\hat{x} + Bu^\star + (I - \gamma^{-2}\Sigma S)^{-1} K[y - (C + \gamma^{-2}LS)\hat{x}], \ \hat{x}(t_0) = 0,$$
$$u^\star = F\hat{x},$$

or

$$\dot{\check{x}} = (A + \gamma^{-2}\Sigma Q)\check{x} + (B + \gamma^{-2}\Sigma P)u^\star + K(y - C\check{x}) \,, \qquad \check{x}(t_0) = 0 \,,$$
$$u^\star = F(I - \gamma^{-2}\Sigma S)^{-1}\check{x} \,.$$

If any one the of above conditions fails to hold, then, for any smaller γ, the criterion (21) *has an infinite supremum for any causal controller $u(t) = \phi(t, y^t)$.*

Remark 3.2

1. *The notation $\rho(X)$ stands for the spectral radius of the matrix X. If condition* (25) *holds, then indeed, the matrix $(I - \gamma^{-2}\Sigma S)$ is invertible.*
2. *The two Riccati equations are dual of each other as in the LQG control problem.*
3. *The above formulas coincide with the LQG formulas in the limit as $\gamma \to \infty$.*
4. *The first system above is the "certainty equivalent" form. The second one follows from placing $\check{x} = (I - \gamma^{-2}\Sigma S)\hat{x}$. The symmetry between these two forms seems interesting.*
5. *The solution of the $\mathcal{H}^\infty$-optimal control problem is highly nonunique. The above one is called the* central controller. *But use of Başar's representation theorem [8] yields a wide family of admissible controllers. See [7].*

3.1.2 Infinite horizon stationary problem

In this section, we assume that the matrix $\mathcal{S}$ is a constant. The system is then said to be stationary. The spaces L^2 considered are to be understood as $L^2(\mathbb{R} \to \mathbb{R}^d)$ with the appropriate dimension d. The dynamic equation (16) is to be understood with zero initial condition at $-\infty$, and we will be interested in asymptotically stable solutions. There is no room for the terms in Q_0 and Q_1 of the finite time criterion (21), and its integral is to be taken from $-\infty$ to ∞.

We further invoke a last pair of dual hypotheses:

Assumption 3.2 *The pair (A, D) is stabilizable and the pair (A, H) is detectable.*

The Riccati equations are replaced by their stationary variants, still using (22):

$$SA + A'S - F'RF + \gamma^{-2}SMS + Q = 0 \,, \tag{26}$$
$$\Sigma A' + A\Sigma - KNK' + \gamma^{-2}\Sigma Q\Sigma + M = 0 \,. \tag{27}$$

Theorem 3.2 *Under condition 3.2, if the two algebraic Riccati equations* (26) *and* (27) *have positive definite solutions, the minimal such solutions $S^\star$ and $\Sigma^\star$ can be obtained as the limit of the Riccati equations* (23) *when integrating from $S(0) = 0$ backward and, respectively,* (24) *when integrating from $\Sigma(0) = 0$ forward. If these solutions satisfy the condition $\rho(\Sigma^\star S^\star) < \gamma^2$, then the same formula as in Theorem 3.1 replacing S and Σ by $S^\star$ and $\Sigma^\star$ respectively provide a solution to the stationary standard problem. If moreover the pair (A, B) is stabilizable and the pair (A, C) is detectable, there exists such a solution for sufficiently small γ.*

If the existence or the spectral radius condition fail to hold, there is no solution to the stationary standard problem for any smaller γ.

The condition of positive definiteness of $S^\star$ and Σ^* can be slightly weakened, leading to a slightly more precise theorem (see [7]). But this does not seem to be very useful in practice.

3.2 Discrete time

We consider now the discrete-time system where $t \in \mathbb{N}$:

$$\begin{aligned} x_{t+1} &= A_t x_t + B_t u_t + D_t w_t\,, \quad x_{t_0} = x_0 && (28)\\ y_t &= C_t x_t \qquad\qquad\; + E_t w_t\,, && (29)\\ z_t &= H_t x_t + G_t u_t\,. && (30) \end{aligned}$$

where the system matrices may depend on the time t. We use notation (20), still with assumption 3.1 for all t, and

$$\bar{A}_t = A_t - B_t R_t^{-1} P_t'\,, \qquad \widetilde{A}_t = A_t - L_t' N_t^{-1} C_t\,. \tag{31}$$

We also invoke all along the following two dual hypotheses:

Assumption 3.3

$$\forall t\,, \quad \operatorname{rank}\begin{pmatrix} A_t \\ H_t \end{pmatrix} = n\,, \qquad \operatorname{rank}\begin{pmatrix} A_t & D_t \end{pmatrix} = n\,.$$

We want to control the system with a *strictly causal* controller

$$u_t = \phi_t(t, y^{t-1})\,.$$

(See [7] for a nonstrictly causal controller or delayed information controllers.)

3.2.1 Finite horizon

We introduce two positive definite symmetric $n \times n$ matrices X and Y. The augmented criterion is now

$$J = \|x_{t_1}\|_X^2 + \sum_{t=t_0}^{t_1-1} (\|z_t\|^2 - \gamma^2 \|w_t\|^2) - \gamma^2 \|x_0\|_Y^2 \,. \tag{32}$$

We will not attempt to describe here the (nonlinear) discrete-time certainty equivalence theorem used to solve this problem. (See [11, 7].) We go directly to the solution of the standard problem.

The various equations needed may take quite different forms. We choose one. We need the following notation (note that Γ_t and $\bar{S}_t$ involve S_{t+1}):

$$\begin{aligned} \Gamma_t &= (S_{t+1}^{-1} + B_t R_t^{-1} B_t' - \gamma^{-2} M_t)^{-1} \,, \\ \bar{S}_t &= \bar{A}_t' (S_{t+1} - \gamma^{-2} M_t)^{-1} \bar{A}_t + Q_t - P_t R_t^{-1} P_t' \,, \\ \Delta_t &= (\Sigma_t^{-1} + C_t' N_t^{-1} C_t - \gamma^{-2} Q_t)^{-1} \,. \\ \widetilde{\Sigma}_{t+1} &= \widetilde{A}_t (\Sigma_t^{-1} - \gamma^{-2} Q_t)^{-1} \widetilde{A}_t' + M_t - L_t' N_t^{-1} L_t \,. \end{aligned} \tag{33}$$

The two discrete Riccati equations may be written as

$$S_t = \bar{A}_t' \Gamma_t \bar{A}_t + Q_t - P_t R_t^{-1} P_t' \,, \qquad S_{t_1} = X \,, \tag{34}$$

$$\Sigma_{t+1} = \widetilde{A}_t \Delta_t \widetilde{A}_t' + M_t - L_t' N_t^{-1} L_t \,, \qquad \Sigma_{t_0} = Y^{-1} \,. \tag{35}$$

Theorem 3.3 *Under the hypothesis 3.3, if both discrete Riccati equations* (34) *and* (35) *have solutions satisfying either* $\rho(M_t S_{t+1}) < \gamma^2$ *and* $\rho(\widetilde{\Sigma}_{t+1} S_{t+1}) < \gamma^2$ *or* $\rho(\Sigma_t Q_t) < \gamma^2$ *and* $\rho(\Sigma_t \bar{S}_t) < \gamma^2$*, then the standard problem has a solution for that value of* γ *and any larger one, given by*

$$u_t^\star = -R_t^{-1} (B_t' \Gamma_t \bar{A}_t + P_t') (I - \gamma^{-2} \Sigma_t S_t)^{-1} \check{x}_t \,, \tag{36}$$

$$\check{x}_{t+1} = A_t \check{x}_t + B_t u_t^\star + \gamma^{-2} \widetilde{A}_t \Delta_t (Q_t \check{x}_t + P_t u_t^\star) + (\widetilde{A}_t \Delta_t C_t' + L_t') N_t^{-1} (y_t - C_t \check{x}_t), \tag{37}$$

$$\check{x}_{t_0} = 0 \,. \tag{38}$$

If any one of the above conditions fails to hold, then for any smaller γ*, the criterion* (32) *has an infinite supremum for any strictly causal controller.*

Remark 3.3

1. *Equation* (37) *is a* one-step predictor *allowing one to get the* worst possible state $\hat{x}_t = (I - \gamma^{-2} \Sigma_t S_t)^{-1} \check{x}_t$ *as a function of past* y_s *up to* $s \le t - 1$.

2. *Controller* (36) *is therefore a strictly causal,* certainty-equivalent *controller.*

3.2.2 Infinite horizon stationary problem

We consider the same system as above, with all system matrices constant and with the strengthened hypothesis (as compared to assumption 3.2):

Assumption 3.4 *The pair* (A, D) *is controllable and the pair* (A, H) *is observable.*

The dynamics is to be understood with zero initial condition at $-\infty$ and we seek an asymptotically stable and stabilizing controller. The criterion (32) is replaced by a sum from $-\infty$ to ∞, thus with no initial and terminal terms.

The stationary versions of all the equations in the previous subsection are obtained by removing the index t or $t + 1$ to all matrix-valued symbols.

Theorem 3.4 *Under assumption 3.4, if the stationary versions of* (34) *and* (35) *have positive definite solutions, the smallest such solutions* $S^\star$ *and* $\Sigma^\star$ *are obtained as, respectively, the limit of* S_t *when integrating* (34) *backward from* $S_0 = 0$*, and the limit of* Σ_t *when integrating* (35) *forward from* $\Sigma_0 = 0$*. If these limit values satisfy either of the two spectral conditions in Theorem 3.3, the standard problem has a solution for that value of* γ *and any larger one, given by the stationary versions of equations* (36) *and* (37).

If any one of these conditions fails to hold, the supremum of J *is infinite for all strictly causal controllers.*

4 Nonlinear problems

4.1 Nonlinear $\mathcal{H}^\infty$ control

There are many different ways to extend $\mathcal{H}^\infty$-optimal control theory to nonlinear systems. A driving factor is how much we are willing to parametrize the system and the controller. If one restricts its scope to a parametrized class of system equations (see below) and/or to a more or less restricted class of compensators (e.g. finite dimensional), then some more explicit results may be obtained, e.g. through special Lyapunov functions and LMI approaches (see [15, 20]), or passivity techniques (see [4]).

In this section, partial derivatives are denoted by indices.

4.1.1 A finite horizon problem

As an example, we specialize [17] for a problem slightly more natural than the standard problem (10) in a nonlinear setup (see [31]), i.e. finding a control law such that the controlled system has *finite* L^2*-gain*:

Definition 4.1 *A system $\omega \mapsto z$ is said to have L^2-gain less than or equal to γ if there exists a number β such that,*

$$\forall \omega \in \Omega\,, \quad \|z\|^2 \le \gamma^2 \|\omega\|^2 + \beta^2\,. \tag{39}$$

We consider a nonlinear system defined by

$$\begin{aligned} \dot{x} &= a(t,x,u) + b(t,x,u)w\,, \quad x(0) = x_0\,, \\ z(t) &= g(t,x,u)\,, \\ y(t) &= h(t,x) + v\,. \end{aligned}$$

We work over the time interval $[0,T]$. We use three symmetric matrices, R positive definite, P and Q nonnegative definite, and take as the norm of the disturbance:

$$\|\omega\|^2 = \int_0^T \left(\|w(t)\|_R^2 + \|v(t)\|_P^2 \right) \mathrm{d}t + \|x_0\|_Q^2\,.$$

Let also

$$B(t,x,u) := \frac{1}{4} b(t,x,u) R^{-1} b'(t,x,u)\,.$$

We use the following two HJCIB equations, where $\hat{\varphi} : [0,T] \times \mathbb{R}^n \to \mathsf{U}$ is a state feedback and $\hat{u}(t)$ stands for the control actually used:

$$\begin{aligned} &V_t(t,x) + V_x(t,x)a(t,x,\hat{\varphi}(t,x)) + \gamma^{-2} V_x(t,x) B(t,x,\hat{\varphi}(t,x)) V_x'(t,x) \\ &+ \|g(t,x,\hat{\varphi}(t,x))\|^2 = -p(t,x)\,, \qquad\qquad V(T,x) = 0. \end{aligned} \tag{40}$$

$$\begin{aligned} &W_t(t,x) + W_x(t,x)a(t,x,\hat{u}(t)) + \gamma^{-2} W_x(t,x) B(t,x,\hat{u}(t)) W_x'(t,x) \\ &+ \|g(t,x,\hat{u}(t))\|^2 = \gamma^2 \|y(t) - h(t,x)\|^2\,, \qquad W(0,x) = \gamma^2 \|x\|_Q^2. \end{aligned} \tag{41}$$

Theorem 4.1

1. *If there exists an admissible state feedback $\hat{\varphi}$, a nonnegative C^1 function p and a C^1 function V satisfying equation* (40)*, then the strategy $u(t) = \hat{\varphi}(t,x(t))$ solves the state feedback finite gain problem* (39)*.*

2. *If furthermore, V is C^2, and there exists for all pairs $(\hat{u}(\cdot), y(\cdot))$ a C^2 function W solution of equation* (41)*, and $\hat{x}(t)$ in* (14) *is always unique, then the strategy $u(t) = \hat{\varphi}(t,\hat{x}(t))$ solves the output feedback finite gain problem* (39)*. Moreover, $\hat{x}(t)$ may be computed recursively according to the equation*

$$\begin{aligned} \dot{\hat{x}} &= a(t,\hat{x},\hat{\varphi}(t,\hat{x})) + 2\gamma^{-2} B(t,\hat{x},\hat{\varphi}(t,\hat{x})) V_x(t,\hat{x})' \\ &\quad - [V_{xx}(t,\hat{x}) - W_{xx}(t,\hat{x})]^{-1} \left[2\gamma^2 h_x(t,\hat{x})' P(y - h(t,\hat{x})) - p_x(t,\hat{x})' \right], \\ \hat{x}(0) &= \operatorname{Arg}\min_x \left[V(0,x) - \gamma^2 \|x\|_Q^2 \right]. \end{aligned}$$

Remark 4.1 *Although $\hat{x}$ is given by this ordinary differential equation, yet this is not a finite-dimensional controller since the partial differential equation* (41) *must be integrated in real time.*

4.1.2 Stationary problem

We choose to just show a slight extension of the result of subsection 2.2.2 applied to the investigation of the standard problem (10), for a more general system, and requiring less regularity, than in the previous subsection. Let therefore a system be defined as in (5) with $x(0) = x_0$, (6), and (7), but with f, g and h time-independent. Let also

$$L(x, u, w) = \|g(x, u)\|^2 - \gamma^2 \|w\|^2 \,. \tag{42}$$

We want to find a nonanticipative control law $u(\cdot) = \hat{\phi}(y(\cdot))$ that would guarantee that there exists a number β such that the control $\hat{u}(\cdot)$ and the state trajectory $x(\cdot)$ generated always satisfy

$$\forall (x_0, w(\cdot)) \in \Omega \,, \quad J(\hat{u}(\cdot), w(\cdot)) = \int_0^\infty L(x(t), \hat{u}(t), w(t)) \, \mathrm{d}t \leq \beta^2 \,. \tag{43}$$

We leave it to the reader to specialize the result below to the case (42), and further to such system equations as used, e.g. in [4, 15].

We need the following two HJCIB equations, using a control feedback $\hat{\varphi}(x)$ and the control $\hat{u}(t)$ actually used:

$$\forall x \in \mathbb{R}^n \,, \quad \inf_{w \in \mathsf{W}} [-V_x(x) f(x, \hat{\varphi}(x), w) - L(x, \hat{\varphi}(x), w)] = 0 \,, \quad V(0) = 0 \,, \tag{44}$$

and, $\forall x \in \mathbb{R}^n$,

$$\inf_{w|y(t)} [-W_t(t, x) - W_x(t, x) f(t, \hat{u}(t), w) - L(x, \hat{u}(t), w)] = 0 \,, \quad W(0, x) = 0 \,, \tag{45}$$

(by $\inf_{w|y(t)}$ we mean $\inf_{w|h(t,x,w)=y(t)}$). Let $X(t)$ be the set of reachable states at time t from any x_0 with $\hat{\varphi}$ and a minimizing w in (45), and $\hat{x}(t)$ defined as

$$V(\hat{x}(t)) - W(t, \hat{x}(t)) = \max_{x \in X(t)} [V(x) - W(t, x)] \,. \tag{46}$$

Theorem 4.2

1. *The state feedback problem* (43) *has a solution if and only if there exists an admissible stabilizing state feedback $\hat{\varphi}(x)$ and a BUC viscosity supersolution* [1] *$V(x)$ of equation* (44)*. Then, $u(t) = \hat{\varphi}(x(t))$ is a solution.*

[1] See [6, 5] for a definition and fundamental properties.

2. *If furthermore, there exists for every pair* $(\hat{u}(\cdot), y(\cdot)) \in \mathcal{U} \times \mathcal{Y}$*, a BUC viscosity solution* $W(t,x)$ *of equation* (45) *in* $[0,\infty) \times \mathbb{R}^n$*, and if furthermore, there is for all* $t \geq 0$ *a unique* $\hat{x}(t)$ *satisfying equation* (46)*, then the controller* $u(t) = \hat{\varphi}(\hat{x}(t))$ *solves the output feedback problem* (43).

Remark 4.2

1. *A supersolution of equation* (44) *may make the left hand side strictly positive.*
2. *Equation* (45)*, to be integrated in real time, behaves as an observer. More precisely, it is the counterpart for the conditional cost measure of Kushner's equation of nonlinear filtering (see [16]) for the conditional probability measure.*

If equation (44) has no finite solution, then the full state information $\mathcal{H}^\infty$ control problem has no solution, and, not surprisingly, nor does the standard problem with partial corrupted information considered here. (See [31].) But in sufficiently extreme cases, we may also detect problems where the available information is the reason why there is no solution:

Theorem 4.3 *If equation* (44) *has no finite solution, or, if* V *is a viscosity solution of* (44) *and, for all* $(\hat{u}(\cdot), y(\cdot))$*, either* (45) *has no finite solution, or* $V - W$ *is unbounded by above, then there is no causal controller that can keep the criterion bounded against all* $w(\cdot)$.

It is due to its special structure that in the linear quadratic problem, either of the above two theorems applies.

4.2 Option pricing in Finance

As an example of the use of the ideas of subsection 2.1.1 in a very different dynamic context, we sketch here an application to the emblematic problem of mathematical finance, the problem of option pricing. The present theory is therefore an alternative to Black and Scholes' famous one, using a set description of the disturbances instead of a stochastic one.[2] As compared to the latter, the former allows us to include in a natural fashion transaction costs , and also discrete time trading.

Many types of options are traded on various markets. As an example, we will emphasize here European "vanilla" buy options, or *Call*, and sell options, or *Put*, with closure in kind. Many other types are covered by several similar or related methods in [12].

[2]See, however, in [12, Chapter 2] a probability-free derivation of Black and Scholes' formula.

The treatment here follows [12, Part 3], but it is quite symptomatic that independent works using similar ideas emerged around the year 2000, although they generally appeared in print much later. See the introduction of [12].

4.2.1 The problem

Portfolio For the sake of simplicity, we assume an economy without interest rate nor inflation. The contract is signed at time 0, and ends at the *exercise time* T. It bears upon a single financial security whose price $u(t)$ varies with time in an unpredictable manner. We take as the disturbance in this problem its relative rate of growth $\dot{u}/u = \tau$. In the Black and Scholes theory, τ is modelled as a stochastic process, the sum of a deterministic drift and of a "white noise". In keeping with the topic of this chapter, we will not make it a stochastic process. Instead, we assume that two numbers are known: $\tau^- < 0$ and $\tau^+ > 0$, and all we assume is boundedness and measurability:

$$\forall t \in [0,T]\,,\ \tau(t) \in [\tau^-, \tau^+] \quad \text{and} \quad \tau(\cdot) \in \Omega = \mathcal{M}([0,T] \to [\tau^-, \tau^+])\,. \quad (47)$$

A *portfolio* is made of two components: shares of the security for a monetary amount $v(t)$ and an amount $y(t)$ of currency, for a total worth $v + y = w$. Both v and y may be positive, of course, but also negative through *futures* for v, a "short" portfolio, and borrowing for y. All variables may vary continuously.

A trader manages the portfolio in a *self-financed* fashion, meaning that he buys or sells shares of the security, withdrawing the money to buy from the currency part y of the portfolio, or adding to it the proceeds of the sales. In Merton's "continuous trading" fiction, the trader may trade at a continuous rate $\xi(t)$, taken positive for a buy and negative for a sale. But he can also trade a finite block of shares instantly, resulting in jumps in $v(\cdot)$, represented by impulses in $\xi(\cdot)$. We will therefore allow a finite sum

$$\xi(t) = \xi^c(t) + \sum_k \xi_k \delta(t - t_k) \quad \Leftrightarrow \quad \xi(\cdot) \in \Xi$$

with $\xi^c(\cdot)$ a measurable real function and $\{t_k\} \subset [0,T]$ and $\{\xi_k\} \subset \mathbb{R}$ two finite sequences, all chosen by the trader. We call Ξ the set of such distributions.

There are transaction costs incurred in any transaction, that we will assume to be proportional to the amount of the transaction, with proportionality coefficients $C^- < 0$ for a sale of shares of the security, and $C^+ > 0$ for a buy. We will write these transaction costs as $C^\varepsilon\langle\xi\rangle$ with the convention that this means that $\varepsilon = \mathrm{sign}(\xi)$. The same convention holds for such notation as $\tau^\varepsilon\langle X\rangle$, or later $q^\varepsilon\langle X\rangle$.

This results in the following control system, where the disturbance is τ and the control ξ:

$$\dot{u} = \tau u\,, \qquad \tau(\cdot) \in \Omega \tag{48}$$

$$\dot{v} = \tau v + \xi\,, \qquad \xi(\cdot) \in \Xi\,, \tag{49}$$

$$\dot{w} = \tau v - C^\varepsilon \langle \xi \rangle\;, \tag{50}$$

which the trader controls through a nonanticipative strategy $\xi(\cdot) = \phi(u(\cdot))$, which may in practice take the form of a state feedback $\xi(t) = \varphi(t, u(t), v(t))$ with an additional rule saying when to make impulses, *i.e.* jumps in v, and by what amount.

Hedging A terminal payment by the trader is defined in the contract, in reference to an *exercise price* K. Adding to it the closure transactions costs with rates $c^- \in [C^-, 0]$ and $c^+ \in [0, C^+]$, the total terminal payment can be formulated with the help of two auxiliary functions $\hat{v}(T, u)$ and $\hat{w}(T, u)$ depending on the type of option considered according to the following table:

Closure in kind		$u \leq \frac{K}{1+c^+}$	$\frac{K}{1+c^+} \leq u \leq \frac{K}{1+c^-}$	$\frac{K}{1+c^-} \leq u$
Call	$\hat{v}(T, u)$	0	$\frac{(1+c^+)u-K}{c^+-c^-}$	u
	$\hat{w}(T, u)$	0	$-c^-\hat{v}(T, u)$	$u - K$
Put	$\hat{v}(T, u)$	$-u$	$\frac{(1+c^-)u-K}{c^+-c^-}$	0
	$\hat{w}(T, u)$	$K - u$	$-c^+\hat{v}(u)$	0

(51)

And the total terminal payment is $M(u(T), v(T))$, with

$$M(u, v) = \hat{w}(T, u) + c^\varepsilon \langle \hat{v}(T, u) - v \rangle\;. \tag{52}$$

Definition 4.2 *An initial portfolio* $(v(0), w(0))$ *and a trading strategy* ϕ *constitute a* hedge at $u(0)$ *if they ensure*

$$\forall \tau(\cdot) \in \Omega\,, \qquad w(T) \geq M(u(T), v(T))\,, \tag{53}$$

meaning that the final worth of the portfolio is enough to cover the payment owed by the trader according to the contract signed. Il follows from (50) that (53) is equivalent to

$$\forall \tau(\cdot) \in \Omega\,, \quad M(u(T), v(T)) + \int_0^T \Big(-\tau(t)v(t) + C^\varepsilon \langle \xi(t) \rangle \Big) \mathrm{d}t \leq w(0)\,,$$

a typical guaranteed value according to subsection 2.1.1. Moreover, the trader wishes to construct *the cheapest possible hedge*, and hence solve the problem

$$\min_{\phi \in \Phi} \sup_{\tau(\cdot) \in \Omega} \left[M(u(T), v(T)) + \int_0^T \Big(-\tau(t)v(t) + C^\varepsilon \langle \xi(t) \rangle \Big) \mathrm{d}t \right]. \tag{54}$$

Let $V(t,u,v)$ be the Value function associated with the differential game defined by (48), (49) and (54). The *premium* to be charged to the buyer of the contract, if $u(0) = u_0$, is

$$P(u_0) = V(0, u_0, 0)\,.$$

This problem defines the so-called "robust control approach to option pricing". An extensive use of differential game theory yields the following results.

4.2.2 The solution

Because of the impulses allowed in the control, Isaacs' equation is replaced by the following differential quasi-variational inequality (DQVI):

$$\left.\begin{array}{l} \forall (t,u,v) \in [0,T) \times \mathbb{R}_+ \times \mathbb{R}\,, \\ \max\Big\{ -V_t - \tau^\varepsilon \langle V_u u + (V_v - 1)v \rangle \ , \ -(V_v + C^+) \ , \ V_v + C^- \Big\} = 0\,, \\ \forall (u,v) \in \mathbb{R}_+ \times \mathbb{R}\,, \quad V(T,u,v) = M(u,v)\,. \end{array}\right\} \quad (55)$$

Theorem 4.4 *The Value function associated with the differential game defined by equations* (48) (49)*, and* (54) *is the unique Lipshitz continuous viscosity solution of the differential variational inequality* (55)*.*

Solving the DQVI (55) may be done with the help of the following auxiliary functions. We define

$$\begin{aligned} q^-(t) &= \max\{(1+c^-)\exp[\tau^-(T-t)] - 1\,,\, C^-\}\,, \\ q^+(t) &= \min\{(1+c^+)\exp[\tau^+(T-t)] - 1\,,\, C^+\}\,. \end{aligned}$$

Note that, for $\varepsilon \in \{-,+\}$, $q^\varepsilon = C^\varepsilon$ for $t \le t_\varepsilon$ and increases ($\varepsilon = +$) or decreases ($\varepsilon = -$) towards c^ε as $t \to T$, with

$$t_\varepsilon = T - \frac{1}{\tau^\varepsilon} \ln\left(\frac{1+C^\varepsilon}{1+c^\varepsilon}\right)\,. \quad (56)$$

We also introduce the constant matrix $\mathcal{S}$ and the variable matrix $\mathcal{T}(t)$ defined by

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{T} = \frac{1}{q^+ - q^-} \begin{pmatrix} \tau^+ q^+ - \tau^- q^- & \tau^+ - \tau^- \\ -(\tau^+ - \tau^-) q^+ q^- & \tau^- q^+ - \tau^+ q^- \end{pmatrix}.$$

Finally, we name collectively two functions:

$$W(t,u) = \begin{pmatrix} \hat{v}(t,u) \\ \hat{w}(t,u) \end{pmatrix}$$

involved in the pair of coupled linear partial differential equations

$$W_t + \mathcal{T}(W_u u - \mathcal{S}W) = 0\,. \tag{57}$$

with the boundary conditions (51).

Theorem 4.5

- *The partial differential equation* (57) *with boundary conditions* (51) *has a unique solution.*
- *The Value function of the game* (48)*,* (49)*,* (54) *(i.e. the unique Lipshitz continuous viscosity solution of* (55)*) is given by*

$$V(t,u,v) = \hat{w}(t,u) + q^{\varepsilon}(t)\langle \hat{v}(t,u) - v\rangle\ . \tag{58}$$

- *The optimal hedging strategy, starting with an initial wealth $w(0) = P(u(0))$, is to make an initial jump to $v = \hat{v}(0,u(0))$ and keep $v(t) = \hat{v}(t,u(t))$ as long as $t < t_\varepsilon$ as given by* (56)*, and do nothing for $t \geq t_\varepsilon$, with $\varepsilon = \mathrm{sign}[\hat{v}(t,u(t)) - v(t)]$.*

Remark 4.3

- *In practice, for classic options, $T - t_\varepsilon$ is very small (typically less than one day) so that a simplified trading strategy is obtained by choosing $q^\varepsilon = C^\varepsilon$, and if T is not extremely small, $P(u_0) = \hat{w}(0,u_0) + C^\varepsilon\langle\hat{v}(0,u_0)\rangle$.*
- *The curve $P(u)$ for realistic $[\tau^-,\tau^+]$ is qualitatively similar to that of the Black and Scholes theory, usually larger because of the trading costs not accounted for in the classic theory.*
- *The larger the interval $[\tau^-,\tau^+]$ chosen, the larger $P(u)$. Hence the choice of these bounds is a critical step in applying this theory. The hedge has been found to be very robust against occasional violations of the bounds in* (47).

4.2.3 Discrete time trading

One of the advantages to relinquish the traditional "geometric diffusion" stochastic model for the disturbance $\tau(\cdot)$ is to allow for a coherent theory of discrete-time trading. Let therefore $h = T/N$ be a time step, with $N \in \mathbb{N}$. Assume the trader is allowed to do some trading only at instants $t_k = kh$, $k \in \mathbb{N}$. This means that we keep only the impulsive part of $\xi(\cdot)$, and fix the impulse instants t_k. We therefore have restricted the available trader's choices, thus we will end up with a larger premium.

We need now the parameters

$$\tau_h^\varepsilon = \mathrm{e}^{h\tau^\varepsilon} - 1\,, \quad \varepsilon \in \{-,+\}\,.$$

We write u_k, v_k, w_k for $u(t_k), v(t_k), w(t_k)$. An exact discretization of our system is therefore as follows, with $\tau_k \in [\tau_h^-, \tau_h^+]$:

$$u_{k+1} = (1+\tau_k)u_k\,, \tag{59}$$

$$v_{k+1} = (1+\tau_k)(v_k+\xi_k)\,, \tag{60}$$

$$w_{k+1} = w_k + \tau_k(v_k+\xi_k) - C^\varepsilon\langle\xi_k\rangle\;. \tag{61}$$

The Value function of the restricted game is denoted by $V_k^h(u_k, v_k)$.

Theorem 4.6 *The Value function $\{V_k^h\}$ satisfies the Isaacs recurrence equation*

$$\begin{aligned} &V_k^h(u,v) = \min_\xi \max_{\tau\in[\tau_h^-,\tau_h^+]} \Big[V_{k+1}^h((1+\tau)u,(1+\tau)(v+\xi)) - \tau(v+\xi) + C^\varepsilon\langle\xi\rangle\Big],\\ &\forall (u,v)\in\mathbb{R}_+\times\mathbb{R}\,, \quad V_N^h(u,v) = M(u,v)\,. \end{aligned} \tag{62}$$

Moreover, if one defines $V^h(t,u,v)$ as the Value of the game where the trader (maximizer) is allowed to make one jump at initial time, and then only at times t_k as above, we have:

Theorem 4.7 *The function V^h interpolates the sequence $\{V_k^h\}$ in the sense that, for all (u,v), $V^h(t_k,u,v) = V_k^h(u,v)$. As the step size is subdivided and goes to zero (e.g. $h = T/2^d$, $d\to\infty$), the function $V^h(t,u,v)$ decreases and converges to the function $V(t,u,v)$ uniformly on any compact in (u,v).*

Finally, one may extend the representation formula (58), just replacing $\hat{v}$ and $\hat{w}$ by $\hat{v}_k^h$ and $\hat{w}_k^h$ given collectively by a carefully chosen finite difference approximation of equation (57) (but the representation formula is then *exact*) as follows:

Let $q_k^\varepsilon = q^\varepsilon(t_k)$ be alternatively given by $q_N^\varepsilon = c^\varepsilon$ and the recursion

$$q_{k+\frac12}^\varepsilon = (1+\tau_h^\varepsilon)q_{k+1}^\varepsilon + \tau_h^\varepsilon\,,$$

$$q_k^- = \max\{q_{k+\frac12}^-, C^-\}\,, \qquad q_k^+ = \min\{q_{k+\frac12}^+, C^+\}\,.$$

Also, let

$$Q_k^\varepsilon = (\,q_k^\varepsilon \quad 1\,)\,, \qquad W_k^h(u) = \begin{pmatrix}\hat{v}_k^h(u)\\ \hat{w}_k^h(u)\end{pmatrix}\,.$$

The following algorithm is derived from a detailed analysis of equation (62):

$$
\begin{aligned}
&W_k^h(u) = \frac{1}{q^+_{k+\frac{1}{2}} - q^-_{k+\frac{1}{2}}} \begin{pmatrix} 1 & -1 \\ -q^-_{k+\frac{1}{2}} & q^+_{k+\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Q^+_{k+1} W^h_{k+1}((1+\tau_h^+)u) \\ Q^-_{k+1} W^h_{k+1}((1+\tau_h^-)u) \end{pmatrix}, \\
&W_N^h(u) = W(T,u)\,.
\end{aligned}
$$

And as previously,

$$
V_k^h(u,v) = \hat{w}_k^h(u) + q_k^\varepsilon \Big\langle \hat{v}_k^h(u) - v \Big\rangle,
$$

and for all practical purposes, $P(u_0) = \hat{w}_0^h(u_0) + C^\varepsilon \big\langle \hat{v}_0^h(u_0) \big\rangle$.

5 Conclusion

As stated in the introduction, game theoretic methods are only one part, may be a prominent one, of modern robust control. They typically cover a wide spectrum of potential applications, their limitations being in the difficulty to solve a nonlinear dynamic game, often with imperfect information. Any advances in this area would instantly translate into advances in robust control.

The powerful theory of linear-quadratic games coupled with the min-max certainty equivalence principle makes it possible to efficiently solve the linear $\mathcal{H}^\infty$-optimal control problem, while the last example above shows that some very nonlinear problems may also receive a rather explicit solution via these game theoretic methods.

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