Discrete time Carathéodory's canonical equations
or
Discrete time Euler equations
for the standard LQ problem

Pierre Bernhard
March 20, 2017

1 Standard LQ optimal control problem

1.1 Problem

Consider the standard dicrete time LQ optimal control problem, with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$:

$$x(t+1) = Ax(t) + Bu(t)\,, \qquad x(0) = x_0\,.$$

Minimize

$$J = \|x(T)\|_K^2 + \sum_{t=0}^{T-1}(\|x(t)\|_Q^2 + \|u(t)\|_R^2)\,.$$

The horizon T may be finite or infinite. In the finite horizon case, all matrices could be time dependent. The matrices K, Q, and R can obviously be chosen symmetric. We assume moreover that K and Q are nonnegative definite, and that R is positive definite.

1.2 Riccati equation

We first recall the closed-loop solution in terms of the Riccati equation. It suffices to try

$$V(t,x) = x'P(t)x$$

in Bellman's equation to find that indeed this is the case, and $P(\cdot)$ is given by the *Riccati equation*:

$$P(T) = K\,,$$
$$P(t) = A'P(t+1)A - A'P(t+1)B(B'P(t+1)B+R)^{-1}B'P(t+1)A + Q\,,$$

Replacing $P(t)$ and $P(t+1)$ by the same P yields the so called *algebraic Riccati equation*. We have the well known result:

Theorem 1

1. *The solution of the finite horizon problem is given by the Riccati equation and*
$$u^\star(t) = -(R + B'P(t+1)B)^{-1}B'P(t+1)Ax(t)\,,$$
$P(t)$ *is nonnegative definite and the optimal value of te criterion is*
$$V(0,x_0) = x_0'P(0)x_0\,,$$

2. *In the time invariant case, if, furthermore, the pair* (A,B) *is stabilizable and the pair* $(Q^{1/2},A)$ *detectable (i.e. the pair* $(A',Q^{1/2})$ *stabilizable), then* $P(t) \to P^\star$ *as* $t \to -\infty$*,* $P^\star$ *is the only positive definite solution of the algebraic Riccati equation, the optimal control and the optimal cost of the infinite horizon problem are given by the above formulas replacing* P *by* $P^\star$*, the closed-loop matrix* $A - B(R + B'P^\star B)^{-1}B'P^\star A$ *is stable (i.e. on the optimal trajectory* $x(t) \to 0$ *as* $t \to \infty$*).*

1.3 Canonical (or Euler's) equations

Let $\lambda(t) = P(t)x(t)$. We show that we also have the canonical equation:

$$\begin{aligned} x(t+1) &= Ax(t) - BR^{-1}B'\lambda(t+1), & x(0) &= x_0\,,\\ \lambda(t) &= Qx(t) + A'\lambda(t+1), & \lambda(T) &= Kx(T)\,, \end{aligned}$$

i.e.
$$u^\star(t) = -R^{-1}B'\lambda(t+1)\,.$$

The above canonical equations may be integrated, either forward or backward, provided that A be invertible, a weakness. Notice also that the equations for x along the optimal path and for $u^\star$ can be obtained by equating to zero the partial derivatives in x and u respectively of the lagrangian

$$\mathcal{L} = \|x(T)\|_K^2 + \sum_{t=0}^{T-1}\left[\|x(t)\|_Q^2 + \|u(t)\|_R^2 - 2\lambda(t+1)'(x(t+1) - Ax(t) - Bu(t))\right].$$

Check first that indeed

$$\begin{aligned} P(t)x(t) &= [A'P(t+1)A - A'P(t+1)B(B'P(t+1)B + R)^{-1}B'P(t+1)A]x(t)\\ &\quad + Qx(t)\\ &= A'P(t+1)[Ax(t) - B(B'P(t+1)B + R)^{-1}B'P(t+1)Ax] + Qx(t)\\ &= A'P(t+1)x(t+1) + Qx(t)\,. \end{aligned}$$

It is hardly more difficult to check that

$$
\begin{aligned}
R^{-1}B'P(t+1)x(t+1) &= -R^{-1}B'P(t+1)\Big[Ax(t) \\
&\quad - B(B'P(t+1)B+R)^{-1}B'P(t+1)Ax(t)\Big] \\
&= -R^{-1}[I - B'P(t+1)B(B'P(t+1)B+R)^{-1}] \\
&\quad B'P(t+1)Ax(t) \\
&= -R^{-1}[R](B'P(t+1)B+R)^{-1}B'P(t+1)Ax(t).
\end{aligned}
$$

QED

2 Simplectic system

2.1 The symplectic (hamiltonian) matrix

The canonical equations can also be written

$$
\begin{pmatrix} A & 0 \\ Q & -I \end{pmatrix} \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \begin{pmatrix} I & BR^{-1}B' \\ 0 & -A' \end{pmatrix} \begin{pmatrix} x(t+1) \\ \lambda(t+1) \end{pmatrix}
$$

If A is invertible, let

$$
\mathcal{H} := \begin{pmatrix} A & 0 \\ Q & -I \end{pmatrix}^{-1} \begin{pmatrix} I & BR^{-1}B' \\ 0 & -A' \end{pmatrix} = \begin{pmatrix} A^{-1} & A^{-1}BR^{-1}B' \\ QA^{-1} & QA^{-1}BR^{-1}B' + A' \end{pmatrix}
$$

The canonical equations yield

$$
\begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} = \mathcal{H} \begin{pmatrix} x(t+1) \\ \lambda(t+1) \end{pmatrix}.
$$

Let

$$
J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},
$$

as is well known, $J^2 = -I$ while $JJ' = I$. It is straightforward to check that

$$
\mathcal{H}J\mathcal{H}' = J \Longleftrightarrow J\mathcal{H}'J' = \mathcal{H}^{-1}.
$$

The matrix $\mathcal{H}$ is said *simplectic*. This has strong implications in relation with the simplectic geometry. We will show one interesting consequence.

2.2 A representation formula for $P(t)$

Let $X(t)$ and $\Lambda(t)$ be two square $n \times n$ matrices, defined by the linear recursion

$$\begin{pmatrix} X(t) \\ \Lambda(t) \end{pmatrix} = \mathcal{H} \begin{pmatrix} X(t+1) \\ \Lambda(t+1) \end{pmatrix}, \quad \begin{pmatrix} X(T) \\ \Lambda(T) \end{pmatrix} = \begin{pmatrix} I \\ K \end{pmatrix}.$$

Theorem 2 *The matrix $X(t)$ is invertible for all $t \leq T$, and it holds that*

$$P(t) = \Lambda(t) X^{-1}(t)\,.$$

The strange fact is that this matrix is therefore symmetric. A consequence of the simplectic character of $\mathcal{H}$.

Proof of the theorem. The calculation showing that $P(t)x(t)$ satisfies Euler's equation for $\lambda(t)$ also holds for X and Λ. Moreover it holds that $P(T)X(T) = K = \Lambda(T)$. Therefore, we have proved that, for all $t \leq T$,

$$\Lambda(t) = P(t)X(t)\,.$$

It remains to show that $X(t)$ is always invertible. Assume that for some time t_1, there exists a n-vector ξ such that $X(t_1)\xi = 0$. Let, for all $t \geq t_1$, $y(t) = X(t)\xi$ and $\mu(t) = \Lambda(t)\xi$. Since $\Lambda(t_1) = P(t_1)X(t_1)$, it follws that $\mu(t_1) = x(t_1) = 0$. But we have seen that $\mathcal{H}$ is invertible. Hence, for all t,

$$\begin{pmatrix} y(t+1) \\ \mu(t+1) \end{pmatrix} = \mathcal{H}^{-1} \begin{pmatrix} y(t) \\ \mu(t) \end{pmatrix}\,.$$

It follows that for all $t \geq t_1$, $y(t) = \mu(t) = 0$. However, $X(T) = I$ so that $y(T) = \xi$. Therefore the null vector is, for all t, the only kernel of $X(t)$. Q.E.D.