Reading

Mathematical Understanding of Nature

by Vladimir. I. Arnold

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Foreword

I have the greatest respect for Vladimir. I. Arnold's mathematical achievements and wish I had done as much as one tenth of it. Yet, reading his little popularization book [1], I have strong reservations about several of his problem solutions. This is an account of these reservations.

Chapter 20

Problem 20 is about the stick balancing problem on a cart. The cart moves along

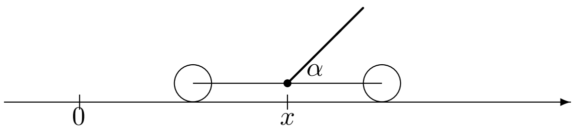

Figure 1: The cart and stick of Problem 20

the horizontal axis according to a law $x = f(t)$.

Vladimir. I. Arnold quotes an incorrect statement by Courant (already an amazing fact!) about the motion of the stick. But he does not dispute the assertion that *obviously, for* $\alpha = 0$ *and* $\alpha = \pi$*, the angle* α *stays constant during the motion*. And then he embarks, as Courant apparently did, to use the continuity of the solution of a differential equation with respect to its initial position to say something about what happens if initially $0 < \alpha < \pi$.

I strongly disagree with everything thereafter. As a matter of fact, if α remains 0 or π when initially there, it is only because of the reaction of the cart bed on the lying stick. And this reaction is not there once the stick is free. Say either that the differential equation obeyed by the stick is not the same, or that there is a discontinuity on the force acting directly on the stick. In either case, nothing can be derived from a continuity argument.

Chapter 21

This problem is about a system of pulleys and a rope, hanging from the ceiling according to Figure 2. The rope goes around the three pulleys and is fixed to the ceiling at one end, and to the axis of the hanging pulley P_1 at the other end. V. I. Arnold neglects all masses, and concludes that pulley P_2 will undergo "free fall".

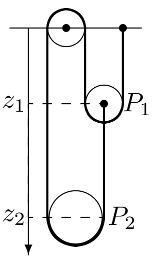

Figure 2

I completely disagree with that statement. There are two ways of stating my objections. The most radical one is to say that if there is no mass, there is no gravity either, no force, no "hanging" and no "free fall". The other way of looking at it is that if V. I. Arnold means that the masses exist but are vanishingly small, then his argument is also wrong. Because he writes equalities that assume an equilibrium state (the algebraic sum of the forces exerted on the pulleys by the rope's tension equal to zero), and then concludes at disiquilibrium.

A correct analysis requires that the masses of the pulleys be taken into account. And at the end of the analysis, one can see if there is a meaningful way to let them go to zero. We shall do that, neglecting the moments of inertia of the pulleys.

The length of the rope is, neglecting the half circumferences of the three pulleys (a constant that does not change the following analysis[1]),

$$\ell = 2z_2 + z_1 \,.$$

Hence the accelerations γ_1 and γ_2 of the centers of the hanging pulleys satisfy

$$\gamma_1 + 2\gamma_2 = 0 \,. \tag{1}$$

Let f be the tension of the rope. The dynamics of the two hanging pulleys of masses m_1 and m_2 (once again, neglecting their moments of inertia) are

$$m_1\gamma_1 = m_1 g - f \,, \qquad m_2\gamma_2 = m_2 g - 2f \,.$$

Using equation (1), and with

$$\frac{m_2}{m_1} =: \rho \,,$$

one easily gets

$$\gamma_2 = \frac{\rho - 2}{\rho + 4} g \,.$$

Therefore, equilibrium is obtained if the (large) pulley 2 is twice as heavy as the (smaller) pulley 1. If their masses are equal, and even if they are "vanishingly small", the lower pulley 2, far from "free falling", will go *up* with an acceleration $g/5$.

Remark 1

1. *If the pulleys are homogeneous, of the same material and the same thickness, pulley 2 actually weighs more than double from pulley 1, more precisely* $\rho = 2.25$*, meaning that it will fall with an acceleration* $\gamma_2 = 0.05g$*.*

2. *It would not be difficult to take into account the radii and the moments of inertia of the pulleys, resulting in four different tensions in the four straight parts of the rope.*

[1] V. I. Arnold draws the fixed pulley as hanging at the end of a string of constant length. This does not change anything in the analysis either

Chapter 29 & 30

V. I. Arnold states that the first digit of the lengths of rivers, or of the height of mountains, is uniformly distributed between integers from 1 to 9. That this fact not be true for the populations of countries he attributes to Malthus' law of population dynamics, and finds a contrieved argument to argue equally of the areas of countries in the world.

It is difficult, to say the least, to believe that populations of current countries obey Malthus' exponential law, and the argument for sizes of countries, following a geometric law because they merge, is even less convincing. Furthermore, I have read in several places (but not checked myself) that the law of freqency of the first digit d_1, as

$$\mathbb{P}(d_1 = k) = p_k = \log\left(1 + \frac{1}{k}\right)$$

is observed in all the physical quantities quoted by V. I. Arnold, and others such as the areas of lakes.

The explanation usually given is that, if such a "natural law" exists —there is little ground to think that one exists, yet the following analysis seems to be substanciated by experiments—, then it must be invariant under a change of units, i.e. under multiplication by a constant. Assume therefore that

$$\mathbb{P}\{a \in [k\,10^m, (k+1)\,10^m)\} = p_k\,, \qquad m \in \mathbb{N}$$

The familiy $\{p_k\}$ must be invariant under multiplication of a by a constant λ. i.e. we want that

$$\mathbb{P}\{\lambda a \in [k\,10^n, (k+1)\,10^n)\} = p_k\,, \qquad n \in \mathbb{N}\,.$$

Using the decimal logarithms, this reads

$$\begin{aligned}\mathbb{P}\{\log a \in [m + \log k, m + \log(k+1))\} &= \\ &\mathbb{P}\{\log a \in [n - \log\lambda + \log k, n - \log\lambda + \log(k+1))\}\,.\end{aligned}$$

But now, $n - \log\lambda$ is not an integer. This has to hold for every λ, i.e., the frequency of a in any interval of length $\log(k+1) - \log(k)$ must be the same. This can only be achieved if $\log a$ is uniformly distributed, leading to

$$p_k = \log(k+1) - \log k = \log\left(1 + \frac{1}{k}\right)\,.$$

References

[1] Vladimir I. Arnold, *Mathematical Understanding of Nature*, AMS, 2014