Robust Optimization in Finance

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Philosophy

"Guessing" (*i.e.* inferring from statistical data) a probability law for unpredictable future prices, interest rates, . . . is adding too much "information" into the model, information that the mathematics will strive to exploit to its ultimate consequences, which were not necessarily meant. A possible lack of robustness to inadequate modelization.

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e.g. the famous Samuelson model

$$\mathrm{d}S/S = \mu \mathrm{d}t + \sigma \mathrm{d}b$$

implies that (inter alia)

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left[\frac{S(2^{-n}(k+1)t) - S(2^{-n}kt)}{S(2^{-n}kt)}\right]^2 = \sigma^2 t$$

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1. Dynamic portfolio optimization
2. Option pricing (*i.e.* risk hedging)

Dynamic portfolio optimization

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- $C = \chi W$: consumption.
  At each step, $C(t) = \chi(t) W(t), \Rightarrow W(t^+) = (1 - \chi(t)) W(t^-)$
  then rearrange portfolio at fixed W choosing new $\varphi(t)$. ($= \varphi(t^+)$.)

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We use u, x, φ vectors in $\mathbb{R}^n$ without the 0 component.

Dynamics

Reminder : Merton's "continuous finance"

If we adopt a stochastic model of prices, one is obliged to choose a model with independent increments to prevent the mathematics from trying to "guess" (infer) future prices based upon past prices. In the continuous trading fiction, this has led, ever since the times of Bachelier (1900) to the adoption of models generating trajectories with unbounded variations.

The undisputed winner in current mathematical finance is "Samuelson's model"

$$\frac{\mathrm{d}S_i}{S_i} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}b$$

σ_i a row of coefficients, b a vector of independent normal brownian motions

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Portfolio

$$w(t+1) = [1 + \varphi^t(t)\tau(t)][1 - \chi(t)]w(t)$$

Utility

Let $\gamma < 1$. ($1 - \gamma$ measures risk aversion.)

Consumption: $U(t,c) = p(t)^{1-\gamma} c^{\gamma}$ *e.g.* $p(t) = \rho \exp[(T-t)/(1-\gamma)]$

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Overall utility: $J = B(w(T)) + \sum_{t=0}^{T-1} U(t, c(t))$

$$J = \Pi^{1-\gamma} w(T)^\gamma + \sum_{t=0}^{T-1} p(t)^{1-\gamma} \chi(t)^\gamma w(t)^\gamma$$

Dynamic programming

$$V(t, w) = \max_{\chi,\varphi} \left[\mathbb{E} V \left(t + 1, (1 + \varphi^t \tau(t))(1 - \chi) w \right) + p(t)^{1-\gamma} \chi^\gamma w^\gamma \right]$$

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$$P(t) = \alpha(t) P(t+1) + p(t)\,, \quad P(T) = \Pi\,, \qquad \chi^\star(t) = \frac{p(t)}{P(t)}\,.$$

Reminder: Merton's problem

With the continuous "Samuelson" market model, $\Sigma = \sigma\sigma^t$

$$\frac{\partial V}{\partial t} + \max_{\varphi,\chi}\left[\frac{\partial V}{\partial w}(\varphi^t(\mu - \mu_0) - \chi)w + \frac{1}{2}\varphi^t \Sigma \varphi \frac{\partial^2 V}{\partial w^2} w^2 + p^{1-\gamma}\chi^\gamma w^\gamma\right] = 0$$

$$V(T, w) = \Pi^{1-\gamma} w^\gamma .$$

Solution

$$V(t, w) = P(t)^{1-\gamma} w^\gamma , \quad \alpha = \frac{\gamma}{2(1-\gamma^2)}(\mu - \mu_0)^t \Sigma^{-1}(\mu - \mu_0) ,$$

$$\dot{P} + \alpha P + p = 0 , \quad \varphi^\star = \frac{1}{1-\gamma}\Sigma^{-1}(\mu - \mu_0) , \quad \chi^\star(t) = \frac{p(t)}{P(t)} .$$

Market model

Let $L(\varphi) = \mathbb{E}[1 + \varphi^t \tau(t)]^\gamma$.

Problem: Solve $\max_\varphi L(\varphi)$

Usually, under the constraint $\varphi_i \geq 0$, $\sum_{i=1}^n \varphi_i \leq 1$ (*i.e.* $\varphi_0 \geq 0$).

Depends on the model for $\tau(t) := \dfrac{u(t+1) - u(t)}{u(t)}$

The empirical market model

Use a known time history $\{\tau(s)\}_{s<t}$ of length ℓ, choose a forget factor $a < 1$ (such that a^ℓ is very small) and set

$$\mathcal{P}\{\tau(t) = \tau(t-k)\} = \frac{1-a}{1-a^\ell}\, a^{k-1}$$

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- Weaknesses: Purely numerical optimization (no analytical help).
  Non stationary, the optimization in φ must be carried out at each step.

The uniform interval model

$$\tau(t) = \mu + \sigma\omega(t),$$

σ a matrix, with $\sum_j |\sigma_{ij}| \leq 1 + \mu_i$. $\omega \in \mathcal{C} = [-1, 1]^n$,

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Let $\psi := \sigma^t\varphi$, $\widehat{\mathcal{C}}$ the verticies of $\mathcal{C}$, and for $\widehat{\omega} \in \widehat{\mathcal{C}}$, $\varsigma(\widehat{\omega}) = \prod_i \widehat{\omega}_i$.

$$L(\varphi) = \frac{1}{2^n \prod_{i=1}^n (\gamma + i)\psi_i} \sum_{\widehat{\omega} \in \widehat{\mathcal{C}}} \varsigma(\widehat{\omega})[1 + \varphi^t(\mu + \sigma\widehat{\omega})]^{\gamma+n}.$$

Whence an (ugly but easy to code) closed form formula for $\nabla L(\varphi)$.

Strength: Easy to optimize, stationary $\Rightarrow$ single computation.
Weakness: Uses an artificial probability law.

Option pricing

A joint work with

Stéphane Thiery and Naïma El Farouq

ENSAM Lille,

and

University Blaise Pascal, Clermont-Ferrand

France

Reminder: an option

A vanilla call (resp put) is a contract by which the seller agrees, if the buyer so requires to sell (resp buy) him a given *underlying* asset (such as a stock) at an agreed *exercise price* or *strike* K at (or whenever the buyer requests no later than) an agreed *exercize time* T.

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Can be seen as a *contingent claim*: a contract according to which the seller will pay the buyer an agreed function $M(\cdot)$ of the underlying's market price $S(t)$ at exercise time t ($= T$ for a *european* option, $\leq T$ for an *american* option.)

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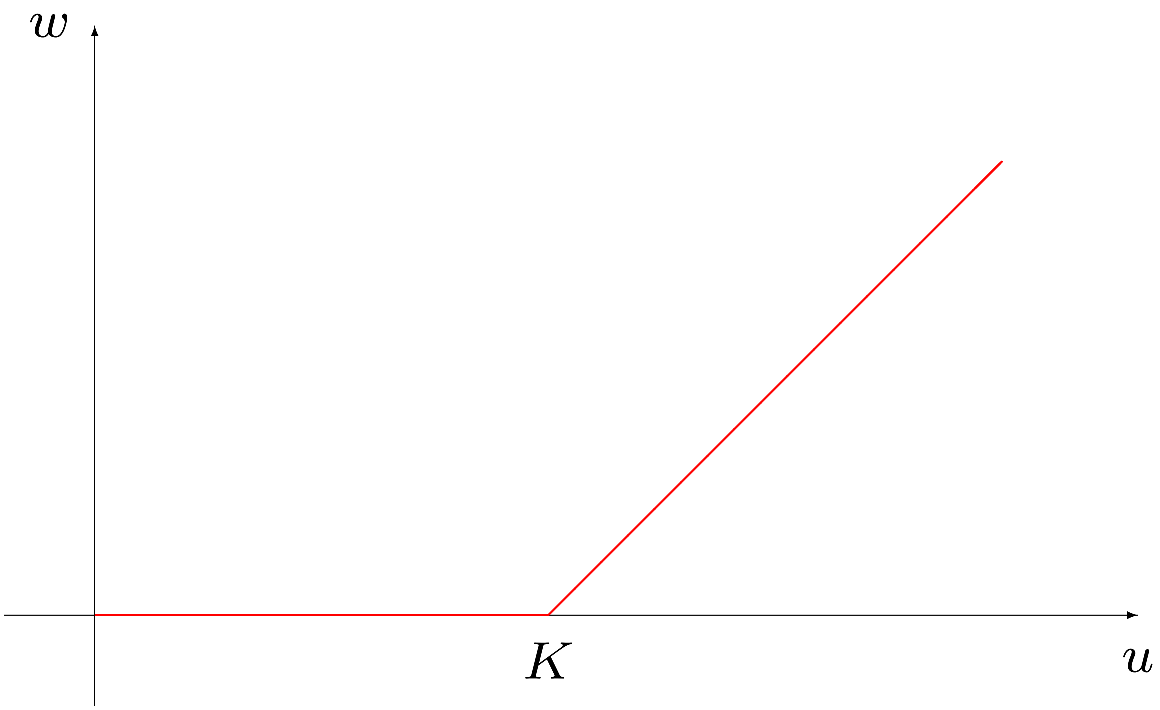
The question is: how to price such a contract ?

The function M



Call

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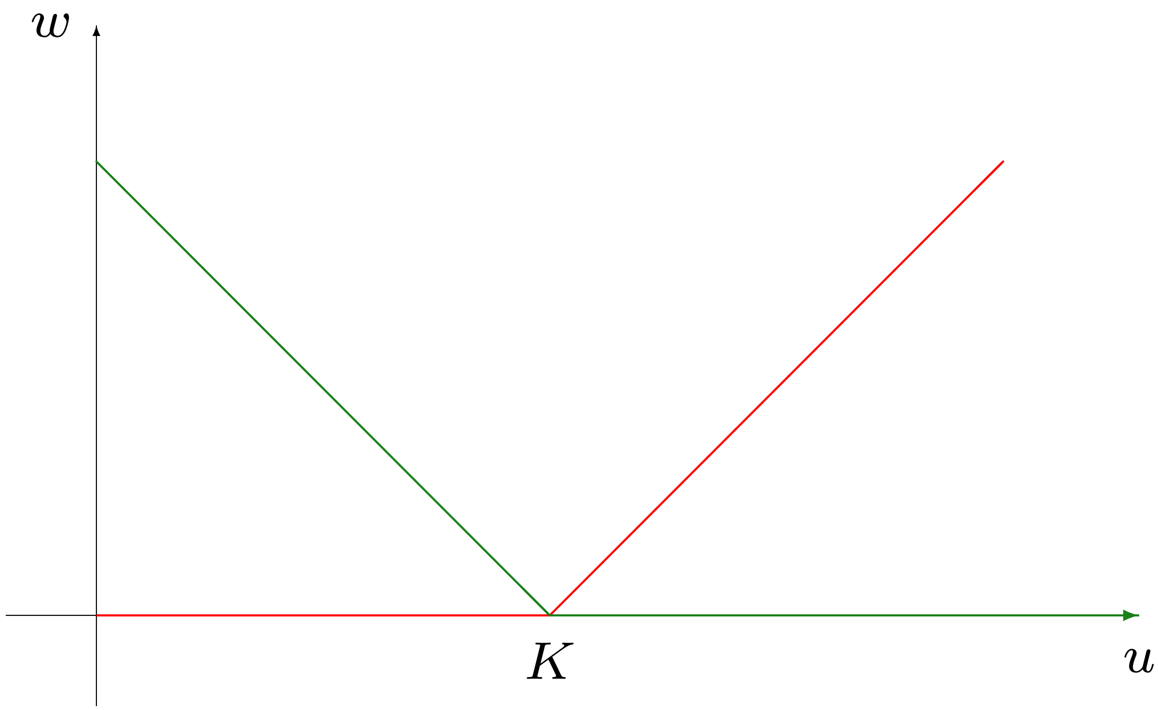

Call, Put

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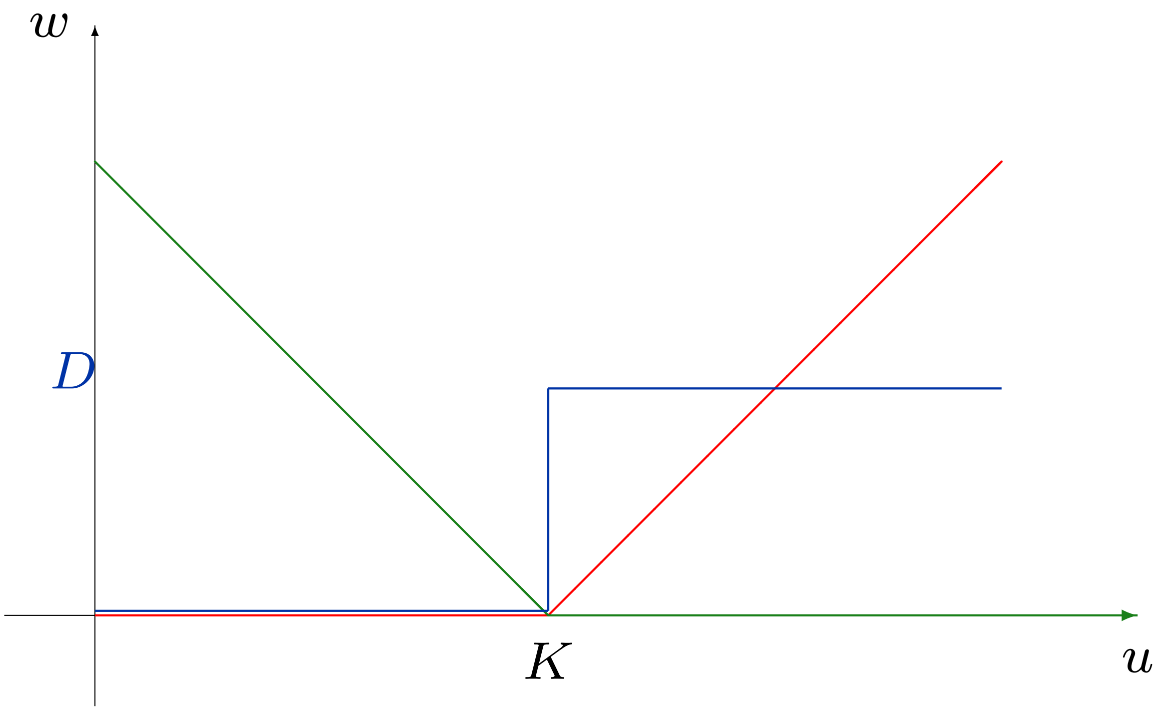

Call, Put, Digital

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Merton's idea: charge a *premium* equal to the cost of a portfolio made of the underlying asset and riskless *bonds*, with return rate μ_0, which, if properly managed in a self-financed way replicates the option.

Black and Scholes solution with Samuelson's model: portfolio worth $W(0, S(0))$ where $W(t, s)$ solves

$$\frac{\partial W}{\partial t} - \mu_0 W + \mu_0 s \frac{\partial W}{\partial s} + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 W}{\partial s^2} = 0 \,, \qquad W(T, s) = M(s) \,.$$

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(Owes nothing to probabilities! Due to the quadratic relative variation)

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- via a minimax control problem.

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In these "constant dollar" prices (or "end-time values"), no discounting on future gains or losses, no interest on riskless lending and borrowing.

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time step h, $u_k := u(kh)$,

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Equivalently

$$\dot{u} = \tau u \,, \quad \tau(\cdot) \text{ measurable}, \quad \tau(t) \in [\tau^-, \tau^+] \,.$$

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Position $v(t)$, (continuous) transaction rate $\xi^c(t)$

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Total gains $\quad \displaystyle\int_0^T \tau(t) v(t)\, \mathrm{d}t\,.$

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$$\dot{v} = \tau v + \xi^c, \qquad\qquad v(t_k^+) = v(t_k) + \xi_k \,.$$

Block buy or sale of an amount ξ_k at a time t_k.

Total gains $\displaystyle\int_0^T \tau(t) v(t)\, \mathrm{d}t \,.$

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$$\dot{v} = \tau v + \xi\,, \qquad \xi(t) = \xi^c(t) + \sum_k \xi_k \delta(t - t_k)$$

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Rates $\quad C^+ > 0, C^- < 0, \quad$ cost $\quad C^\varepsilon \xi, \quad \varepsilon = \mathrm{sign}(\xi)\,.$

$$\text{Total cost} \quad \int_0^T C^\varepsilon \xi^c(t)\, \mathrm{d}t + \sum_k C^{\varepsilon_k} \xi_k = \int_0^T C^\varepsilon \xi(t)\, \mathrm{d}t \,.$$

Portfolio model, closure

Classical closure expense $M(u(T))$ depends on the option type

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$$N(u,v) = \breve{w}(T,u) + c^\varepsilon(\breve{v}(T,u) - v)\,, \quad \varepsilon = \mathrm{sign}(\breve{v}(T,u) - v)\,,$$

where $\breve{v}(T,u)$ and $\breve{w}(T,u)$ depend on option type and the closure mode:
in cash, then $N(u,v) = M(u) + c^\varepsilon(-v)$,
but other considerations lead to choose $(\breve{v}(T,u), \breve{w}(T,u)) \neq (0, M(u))$,
in kind, more complicated, but yields a nicer theory.

Closure modes (vanilla call)

In cash

	$u < K$	$u \geq K$
$\breve{v}(T,u)$	0	$\frac{u}{1+c^-}$
$\breve{w}(T,u)$	0	$\frac{u}{1+c^-} - K$

In kind

	$u \leq \frac{K}{1+c^+}$	$\frac{K}{1+c^+} \leq u \leq \frac{K}{1+c^-}$	$u \geq \frac{K}{1+c^-}$
$\breve{v}(T,u)$	0	$\frac{(1+c^+)u-K}{c^+-c^-}$	u
$\breve{w}(T,u)$	0	$-c^-\breve{v}(T,u)$	$u - K$

Market model

$$\dot{u} = \tau u\,, \qquad u(0) = u_0\,, \;\; \tau \in [\tau^-, \tau^+]\,.$$

Portfolio model

$$\dot{v} = \tau v + \xi\,, \;\; v(0) = v_0\,, \;\; \xi(t) = \xi^c(t) + \textstyle\sum_k \xi_k \delta(t - t_k)\,.$$

Closure

$$N(u, v) = \breve{w}(T, u) + c^{\varepsilon}(\breve{v}(T, u) - v)\,.$$

Total expense

$$J = N(u(T), v(T)) + \int_0^T (-\tau v + C^{\varepsilon}\xi)\,\mathrm{d}t\,.$$

Hedging

Strategies

Admissible strategies : *nonanticipative strategies* $\xi(\cdot) = \varphi(u(\cdot))$.

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Least pricing rule

$$P(u(0)) = \sup_{\tau(\cdot)} J(u(0), \varphi, \tau(\cdot)) \, ,$$

No arbitrage opportunity

$$P(u(0)) = \min_{\varphi} \sup_{\tau(\cdot)} J(u(0), \varphi, \tau(\cdot)) \, .$$

Differential game

Dynamics

$$\dot{u} = \tau u\,, \qquad u(t_0) = u_0\,, \quad \tau \in [\tau^-, \tau^+]\,,$$

$$\dot{v} = \tau v + \xi\,, \quad v(t_0) = v_0\,, \quad \xi(t) = \xi^c(t) + \sum_k \xi_k \delta(t - t_k)\,.$$

Performance index

$$J(t_0, u_0, v_0; \varphi(\tau(\cdot)), \tau(\cdot)) = N(u(T), v(T)) + \int_{t_0}^{T} (-\tau v + C^\varepsilon \xi)\,\mathrm{d}t$$

$$W(t, u, v) = \inf_{\varphi} \sup_{\tau(\cdot)} J(t, u, v; \varphi(\tau(\cdot)), \tau(\cdot))\,.$$

QVI

$$0 = \min\left\{\frac{\partial W}{\partial t} + \max_{\tau\in[\tau^-,\tau^+]} \tau\left[\frac{\partial W}{\partial u}u + \left(\frac{\partial W}{\partial v} - 1\right)v\right]\,, \right.$$
$$\left. \min_{\xi}[W(t,u,v+\xi) - W(t,u,v) + C^{\varepsilon}\xi]\right\}\,.$$

$$W(T,u,v) = N(u,v)\,.$$

QVI & DQVI

$$0 = \min \left\{ \frac{\partial W}{\partial t} + \max_{\tau \in [\tau^-, \tau^+]} \tau \left[\frac{\partial W}{\partial u} u + \left(\frac{\partial W}{\partial v} - 1 \right) v \right] , \right.$$
$$\left. \min_{\xi} [W(t, u, v + \xi) - W(t, u, v) + C^{\varepsilon} \xi] \right\} .$$

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$$\left. \frac{\partial W}{\partial v} + C^+ , - \left(\frac{\partial W}{\partial v} + C^- \right) \right\} .$$

$$W(T, u, v) = N(u, v) .$$

Characterization

Theorem 1 The Value function W is the viscosity solution of the DQVI.

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Proof

W is **a** viscosity solution: Use the “Joshua transform” which transforms the impulse control minimax control problem into a standard minimax control problem of which the DQVI is the Isaacs equation.

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Proof

W is **a** viscosity solution: Use the “Joshua transform” which transforms the impulse control minimax control problem into a standard minimax control problem of which the DQVI is the Isaacs equation.

The unique viscosity solution. A technical (long) uniqueness proof along the lines of typical such proofs. The difficulty arises from the 0 infimum of the impulse costs. (Aknowledgment: Naïma el Farouq and Guy Barles.)

Notation

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\mathcal{T} = \frac{1}{q^+ - q^-} \begin{pmatrix} \tau^+ q^+ - \tau^- q^- & \tau^+ - \tau^- \\ -(\tau^+ - \tau^-) q^+ q^- & \tau^- q^+ - \tau^+ q^- \end{pmatrix},$$

Vanilla call or put, closure in kind

$$q^-(t) = \max\{(1 + c^-) \exp(\tau^-(T - t)) - 1\,,\, C^-\}\,,$$
$$q^+(t) = \min\{(1 + c^+) \exp(\tau^+(T - t)) - 1\,,\, C^+\}\,.$$

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Vanilla call or put, closure in cash

$$q^-(t) = \max\{(1 + c^-)\exp(\tau^-(T - t)) - 1\,,\ C^-\}\,,$$
$$q^+(t) = \min\{(1 + c^-)\exp(\tau^+(T - t)) - 1\,,\ C^+\}\,.$$

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Digital call or put, closure in cash

$$q^-(t) = \max\{(1 + c^-) \exp(\tau^-(T - t)) - 1\,,\, C^-\}\,,$$
$$q^+(t) = \max\{(1 + c^-) K/u - 1\,,\, q^-\}\,.$$

Fundamental PDE

$$\mathcal{V}(t,u) = \begin{pmatrix} \breve{v}(t,u) \\ \breve{w}(t,u) \end{pmatrix} .$$

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Proposition The above P.D.E. has a single solution over $[0, T]$ for evry terminal condition and $\mathcal{T}$ matrix defined above according to the option nature.

Representation

Theorem 2 The function

$$W(t,u,v) = \breve{w}(t,u) + q^{\varepsilon}(\breve{v}(t,u) - v)\,, \qquad \varepsilon = \mathsf{sign}(\breve{v} - v)$$

is a viscosity solution of the DQVI, hence the Value of the game problem.

Proof Long and difficult. Involves a detailed analysis of the field of optimal trajectories and its singularities.

Discrete dynamic game

We consider the same problem, with the same set of possible (maximizing) disturbances, but where the minimizer is restricted to impulses only, and at given time instants $t_k = kh$, $k \in \mathbb{N}$.

Discrete dynamic game

We consider the same problem, with the same set of possible (maximizing) disturbances, but where the minimizer is restricted to impulses only, and at given time instants $t_k = kh$, $k \in \mathbb{N}$. We denote $W_k^h(u,v)$ its Value.

$$\begin{aligned} u_{k+1} &= (1+\tau_k)u_k\,, \quad \tau \in [\tau_h^-, \tau_h^+]\,, \\ v_{k+1} &= (1+\tau_k)(v_k + \xi_k)\,, \end{aligned}$$

Admissible strategies $\xi_k = \varphi_k(u_k, v_k)$, (or $\xi_k = \varphi_k(u_{k-1}, v_{k-1})$)

$$J(0, u_0, v_0; \varphi, \{\tau_k\}) = N(u_K, v_K) + \sum_{k=0}^{K-1} [-\tau_k(v_k + \xi_k) + C^\varepsilon \xi_k]\,.$$

$$W_\ell^h(u,v) = \inf_{\varphi} \sup_{\{\tau_k\}} J(\ell, u, v; \varphi, \{\tau_k\}).$$

Convergence

We interpolate the $W_k^h(u,v)$ with $W^h(t,u,v)$ for all $t \in [0,T]$ defined as the Value of the game where the minimizer is allowed to make an impulse at intital time t, then only a times $t_k = kh$, $k \in \mathbb{N}$, $kh > t$.

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Theorem 3 Take $h = 2^{-d}T$. As $d \to \infty$, W^h converges monotoneously, uniformly on any compact, to the Value W of the continuous time game.

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Theorem 3 Take $h = 2^{-d}T$. As $d \to \infty$, W^h converges monotoneously, uniformly on any compact, to the Value W of the continuous time game.

Proof W^h decreases monotoneously because it is the same game where the set of admissible minimizer's stategies increases. Characterization of its limit is similar to Cappuzzo-Dolcetta's proof for control problems.

Standard algorithm

The natural Isaacs equation of the discrete time game is $W_K^h = N$,

$$W_k^h(u,v) = \\ \min_{\xi} \max_{\tau} \left[W_{k+1}^h((1+\tau)u, (1+\tau)(v+\xi)) - \tau(v+\xi) + C^{\varepsilon}\xi \right]$$

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Standard algorithm

$$W_{k+\frac{1}{2}}^h(u,v) = \max_{\tau\in[\tau^-,\tau^+]} [W_{k+1}^h((1+\tau)u, (1+\tau)v) - \tau v]$$

$$W_k^h(u,v) = \min_{\xi} [W_{k+\frac{1}{2}}^h(u, v+\xi) + C^\varepsilon \xi] \,.$$

Standard algorithm

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$$W^h_k(u,v) = \min_{\xi} \max_{\tau} \left[W^h_{k+1}((1+\tau)u, (1+\tau)(v+\xi)) - \tau(v+\xi) + C^{\varepsilon}\xi \right]$$

Standard algorithm convex

$$W^h_{k+\frac{1}{2}}(u,v) = \max_{\tau \in \{\tau^-, \tau^+\}} [W^h_{k+1}((1+\tau)u, (1+\tau)v) - \tau v]$$

$$W^h_k(u,v) = \min_{\xi} [W^h_{k+\frac{1}{2}}(u, v+\xi) + C^{\varepsilon}\xi] \,.$$

Fast algorithm

Notation

$$Q_\ell^\varepsilon = \begin{pmatrix} q_\ell^\varepsilon & 1 \end{pmatrix}, \qquad \mathcal{V}_\ell^h(u) = \begin{pmatrix} \breve{v}_\ell^h(u) \\ \breve{w}_\ell^h(u) \end{pmatrix},$$

$$\Delta = q^+_{k+\frac{1}{2}} - q^-_{k+\frac{1}{2}}, \qquad \theta^\varepsilon = 1 + \tau_h^\varepsilon.$$

Algorithm

$$q^\varepsilon_{k+\frac{1}{2}} = \theta^\varepsilon q^\varepsilon_{k+1} + \tau_h^\varepsilon, \qquad q_k^\varepsilon = \varepsilon \min\{\varepsilon q^\varepsilon_{k+\frac{1}{2}}, \varepsilon C^\varepsilon\}$$

$$\mathcal{V}_k^h(u) = \frac{1}{\Delta} \begin{pmatrix} 1 & -1 \\ -q^-_{k+\frac{1}{2}} & q^+_{k+\frac{1}{2}} \end{pmatrix} \begin{pmatrix} Q^+_{k+1} \mathcal{V}^h_{k+1}(\theta^+ u) \\ Q^-_{k+1} \mathcal{V}^h_{k+1}(\theta^- u) \end{pmatrix}$$

Representation

Theorem 3 The Value of the discrete dynamical game is given by

$$W_k^h(u) = \breve{w}_k^h(u) + q_k^\varepsilon(\breve{v}_k^h(u) - v)\,, \qquad \varepsilon = \mathsf{sign}(\breve{v}_k^h - v)\,.$$

Proof via a careful, but rather straightforward, analysis of the discrete Isaacs equation.

Thank you

For your attention

Thank you

Phew !

For your attention

Pierre Bernhard
Jean-Pierre Aubin, Patrick Saint-Pierre,
Jacob Engwerda
Vassili Kolokoltsov

The Interval Market Model in Mathematical Finance:
A game theoretic approach

Birkhaüser, 2012 ?

Complements

Joshua's transform

Lemma: the value of the game is unchanged if trader restricted to jumps.

Joshua's transform

Lemma: the value of the game is unchanged if trader restricted to jumps.

J's transform: Let trader's control be $\jmath \in \{-1, 0, 1\}$, and $\bar{\jmath} := 1 - |\jmath|$.
Artificial "time" θ, state variables (t, u, v) , $\mathrm{d}(t, u, v)/\mathrm{d}\theta = (t', u', v')$,

$$\begin{aligned} t' &= \bar{\jmath}, \qquad t(0) = 0, \quad t(\Theta) = T \\ u' &= \bar{\jmath}\tau u, \\ v' &= \bar{\jmath}\tau v + \jmath, \end{aligned}$$

$$J = N(u(\Theta), v(\Theta)) + \int_0^{\Theta} (\bar{\jmath}(-\tau v) + \jmath C^{\jmath})\mathrm{d}\theta.$$

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$$J = N(u(\Theta), v(\Theta)) + \int_0^{\Theta} (\bar{\jmath}(-\tau v) + \jmath C^{\jmath})\mathrm{d}\theta.$$

This is an ordinary, free end-time ($\Rightarrow W_\theta = 0$) game. Isaacs equation is

$$0 = \min_{\jmath \in \{-1,0,1\}} \max_{\tau \in [\tau^-, \tau^+]} \{\bar{\jmath}[W_t + \tau(W_u u + (W_v - 1)v] + \jmath[W_v + C^{\jmath}]\}$$

List the three possibilities for $\jmath$. Yields the DQVI.

American option

A single line of code to add to the standard algorithm:

$$W_k^h(u,v) = \max\Big\{M(u,v)\,,$$
$$\min_{\xi}\max_{\tau}\Big[W_{k+1}^h((1+\tau)u,(1+\tau)(v+\xi)) - \tau(v+\xi) + C^{\varepsilon}\xi\Big]\Big\}$$

Compute the second line as in the standard algorithm, and upon loading the value computed into $W_k(u,v)$, compare with $M(u,v)$ and load the largest.

One step delayed information

If information on u_k only available to act at step $k+1$, replace

$$v_{k+1} = (1+\tau_k)(v_k + \xi_k)\,,$$

by

$$v_{k+1} = (1+\tau_k)v_k + \xi_k\,.$$

The ensuing theory has not been worked out.