Nonzero-sum dynamic games in the managment of biological systems

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Abstract

We argue that the natural objects in many biological problems are *populations* of similar individuals, such as species. This gives a very concrete meaning to (the equivalent of) mixed strategies. Furthermore, the concepts of ESS and limit points of evolutionary dynamics give a new jusification for the investigation of Nash equilibria. But if we turn to interactions between individuals arising over time, i.e. dynamic games as paradigm of one against one conflict, many new problems arise in the investigation of the Nash equilibria of the particular type of systems we are led to consider. We show examples of new discoveries made in this context, but more importantly, we point to open problems, and to our insufficient knowledge of the simplest two-player nonzero-sum differential games, and of the evolutionary dynamics that they could generate.

1 Introduction

In the last few years, much effort has been devoted to the investigation of problems arising in the management of biological entities. An obvious example is the management of agricultural production, both crops and forests, or that of fisheries, an old and difficult topic because of the lack of measurements. Other classical examples include the management of greenhouses and of biological reactors: wastewater treatment plants, vanilin producing reactors, or more recently chemostats used to grow plancton or algaes for biological fuel production. Let us quote also the use of biological pest control, whose history contains many failures and some beautiful success stories, and the growing interest in managing biodiversity.

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The domain so covered is too wide to attempt an exhaustive survey here. Just to give an indication that the managment of biodiversity requires a deep understanding of non trivial biological interactions, we quote the now famous work on "cats protecting birds" [16, 18]. All the recent experience clearly points to a need for a better understanding of mathematical models of the biological content of these artifacts, or of the object of study itself. Our more modest aim here is to draw the attention of the game theoretists interested in management science to new classes of models and new problems that naturally arise in these domains, and more specifically to a large number of open problems.

But first, we want to have a short discussion of the main line of thought underlying much of the quantitative analysis of biologic entities.

On the use of mathematical models in evolutionary biology and behavioural ecology Many biological problems have to do with characteristic traits of species, or populations. A fundamental hypothesis underlying behavioural ecology and many quantitative investigations of biological systems is that evolution has imposed a selective pressure on living organisms such that only the most fit have survived. Hence, if we are able to characterize the determining factors of fitness, optimization theory should let us *explain*, and may be predict, the values of some quantitative traits and the behaviour of natural species.

This is such a fundamental paradigm that it is worth some comments. Some questions to ask are

1. what is "fitness" ?
2. what does *explain* mean in the above sentence ?
3. how do we know that we have isolated the main determinants of fitness ?

These questions are for philosphers and biologists much more than for the mere mathematician, who is only the servant of these scholars. But some dicussion is necessary before we can proceed and use these concepts. Here is an element of answer, very much oriented by our engineering background,

The concept of fitness, in our use of it, is tantamount to that of *utility* in mathematical economy: something never really quantified, but whose existence is postulated, leading to usefull conclusions. But a more pragmatic view is that it will often stand for the relative growth rate of a population. This last definition is a convenient means of getting around an interesting question, but difficult, i.e. the true relationship between fitness and population growth rate.

Concerning the last two questions, the first part of our answer is to reverse the roles of "explanation" and isolation of the determinants of fitness. We construct a mathematical model where we *think* that we have embodied these determinants. Then we investigate, mathematically, the consequences of our model, with an emphasis on measurable quantities. If, in several instances, experience agrees with the "predictions" of our model, then it is a strong indicaton that we have indeed included the determinants of fitness in the said model. Such a validation is in itself an interesting biological result.

In the process, it induces the biologist into looking for regularities predicted by the model —and regularities is what science is about— that he would not have found by mere observation, given the extraordinary complexity of living organisms.

At that stage, two more uses of the model are available. On the one hand, one may try and use it to predict effects difficult to measure directly. This is imporant in such applications as biological pest control for instance. But probably more important, if a model is efficient across several species, when one finds a species where it does not apply, this is a very strong indication that *something* operates differently in this species. A biological fact revealed by the mathematical model.

2 Populations and equilibria

We give here a quick summary of Evolutionary Game Theory, because its concepts are central in biological mathematics, and constitute, to our opinion, the strongest case in favor of mixed straegies and Nash equilibria.

2.1 Mixed strategy

We are interested in populations where each individual has several possible behaviours, or characteristics, called *traits*. A *strategy* of a population is a frequency distribution of the possible traits among its individuals. A pure stratey is a situation where all individuals have the same trait, while a mixed strategy is one where several traits are present in the population.

The average efficiency of a strategy over the population is thus given as the weighted mean of the efficiencies of the various traits, weighted by their frequencies; i.e. a mathematical expectation where the role of the probability is played by the frequency distribution. Indeed, if we decide that an individual is chosen "at random"[1] in that population, the frequency of each trait is the probability that this individual would display that trait.

[1]There is a difficulty here in that on the one hand, to make all this precise and simple, we need to assume that the population is *countably infinite*, and on the other hand, we need to say that "at random" means with uniform probability, an inexistant distribution in a countable infinite population. Resorting to a continuous population raises other technical difficulties. See, e.g. [30, 7]

This gives a concrete and effective meaning to mixed strategies. While, as has often been pointed out, one hardly observes a firm manager, say, use a mixed (randomized) strategy[2], on the contrary, biological populations almost systematically display a polymorphism interpreted here as a "mixed strategy".

In that respect, it may be argued that we only know that a trait is *possible* because we have observed it in the population, albeit seldom, may be. If this is the case, a population strategy is always totally mixed, even if very close to a not totally mixed one. This is akin to the idea underlying the concept of trembling hand perfectness. We shall see that indeed, there is a relationship between ESS and trembling hand perfectness.

2.2 Wardrop equilibrium and ESS

2.2.1 Wardrop equilibrium

In evolutionary game theory, it is assumed that the fitness of any indivudal is a fuction of its own trait, and of the distribution of traits in the overall population. Therefore one is led to investigate the relationship linking the efficiencies of individual traits in a population where each individual is also part of the poulation, or, otherwise stated, the relationship between individual selfishness and collective behaviour.

If several traits are present in a population at equilibrium, and if equilibrium means that sub-populations with differing homogeneous traits have the same growth rate, this implies that the distribution of traits at equilibrium yields the same growth rate for all the traits present in the equilibrium population. A classical equalization property in games. Moreover, if that distribution is to prevent a trait not present in it to spread (outgrow the rest of the population) should it come in, say via a mutation, then the unused traits at equilibrium should not give a better growth rate in that distribution than the common growth rate of the population. These two facts together were first recognized by J.G. Wardrop, a road engineer investigating the behaviour of a population of car drivers striving to choose the fastest route in a road network where the time to travel a given edge is increasing with congestion. (The reference [31] is the proceedings of a meeting held in 1952. This work was therefore done independently of J. Nash's)

We now introduce some notations. We restrict our presentation to the *finite trait space* case. And later we shall also assume that the generating function below is *linear*, leading to the simplest "finite linear" or *matrix* case. See [7] for some extensions to more general cases.

[2] It is precisely to answer that criticism that Harsanyi invented the concept of "purification" and proved his "purification theorem".

Let $N = \{1, \ldots, n\}$ be the set of possible traits. A frequency distribution is therefore an element of the simplex $\Delta(N)$, simply called Δ hereafter, of $\mathbb{R}^n$, a n-vector of positive coordinates summing up to one. The fitness obtained by an individual with the trait i in a population distribution q is given by he *generating function* $G_i(q)$. The notation $G(q)$ means the n-vector with coordinates $G_i(q)$. As a consequence, the average fitness $F(q,p)$ of a subpopulation with frequency distribution q in an overall population of frequency distribution p is given by

$$F(q,p) = \sum_{i=1}^{n} q_i G_i(p) = \langle q, G(p) \rangle . \tag{1}$$

Let $\mathrm{Supp}(q) \subset N$ stand for the *support* of q, the indices of the nonzero coordinates of q, i.e. the traits effectively present in a population with frequency distribution q.

A Wardrop equilibrum as described above is therefore a distribution p such that, for some $f \in \mathbb{R}$,

$$\forall i \in \mathrm{Supp}(p)\,, \quad G_i(p) \quad = \quad f\,, \tag{2}$$

$$\forall j \notin \mathrm{Supp}(p)\,, \quad G_j(p) \quad \le \quad f\,. \tag{3}$$

We may also write this

$$\mathrm{Supp}(p) \subset \mathrm{Argmax}_i \{G_i(p)\}, \tag{4}$$

but, as is well known, this is again equivalent, with definition (1), to

$$F(p, G(p)) = \max_{q \in \Delta} F(q,p)\,. \tag{5}$$

The precise link with the classical Nash equilibrium is that (p,p) is a Nash equilibrium of the game where the first player's payoff is $J^1(q_1,q_2) = F(q_1,q_2)$ and the second player's $J^2(q_1,q_2) = F(q_2,q_1)$.

Let the *best response* multiapplication be defined as

$$\mathbb{BR}(p) = \{r \in \Delta \mid F(r,p) = \max_{q \in \Delta} F(q,p)\}\,. \tag{6}$$

With this notation, Wardrop equlibria can also be characterized as

$$p \in \mathbb{BR}(p)\,. \tag{7}$$

So we now have four ways of writing the Wardrop condition : (2,3) —the original statement by J.G. Wardrop—, (4), (5), and (7). It is worthwhile to point out that the best responses to a distribution q are precisely those distributions whose support is included in $\mathrm{Argmax}_i\{G_i(q)\}$ also denoted $\mathrm{Argmax}(G(q))$.

2.2.2 ESS

In [26], J. Maynard-Smith and G.R. Price introduce the concept of *Evolutionary Stable Strategy*, or ESS, as a distribution $p \in \Delta$ that cannot be invaded by a small enough mutant subpopulation, i.e. that would outgrow this tiny subpopulation should it appear:

$$\forall q \in \Delta \backslash \{p\}\,, \exists \varepsilon_0 : \forall \varepsilon \in (0, \varepsilon_0), \quad F(q, \varepsilon q + (1-\varepsilon)p) < F(p, \varepsilon q + (1-\varepsilon)p). \quad (8)$$

The choice of the *strict* inequality in the above definition is crucial for many results about ESS's.

It is straightforward to see that, if $G(\cdot)$ is continuous, (8) implies (5). But to go further in the analysis, we need now to restrict the attention to the matrix case where there exists a matrix A of *pairwise payoffs* such that

$$\forall q \in \Delta, \quad G(q) = Aq\,, \qquad \text{and hence} \qquad F(q,p) = \langle q, Ap \rangle\,.$$

An important and well known fact is that

Proposition 1 *In the matrix case, a distribution* p *is an ESS if and only if (5) —the* first ESS condition— *holds, and moreover the following* second ESS condition *also holds:*

$$\forall r \in \mathbb{R}(p) - \{p\}\,, \quad F(p,r) > F(r,r)\,. \quad (9)$$

In particular, *any strict (pure) Wardrop equilibrium is an ESS*.

A property equivalent to (9) is that (see [9])

$$\forall r \in \mathbb{R}(p) - \{p\}\,, \quad \langle r-p, A(r-p) \rangle < 0\,.$$

Let A_1 be the restriction of the game matrix to $\mathrm{Argmax}(Ap)$ and A_2 its further restriction to $\mathrm{Supp}(p)$. For any square matrix B, let $\sigma(B)$ be the square matrix with one less dimension obtained by replacing each four adjacent terms by their *symmetric difference*

$$\sigma \begin{pmatrix} b_{k,\ell} & b_{k,\ell+1} \\ b_{k+1,\ell} & b_{k+1,\ell+1} \end{pmatrix} = b_{k,\ell} - b_{k,\ell+1} - b_{k+1,\ell} + b_{k+1,\ell+1}\,,$$

and $\sigma^S(B)$ be its symmetric part $(1/2)(\sigma(B) + \sigma^t(B))$. One can give the following criteria (see [7]):

Proposition 2

1. *A necessary condition for a Wardrop equilibrium* p *to be an ESS is that* $\sigma^S(A_2) < 0$*,*
2. *a sufficient condition is that* $\sigma^S(A_1) < 0$*.*

Thus, in the generic case where $\mathrm{Supp}(p) = \mathrm{Argmax}(Ap)$, and thus $A_1 = A_2$, this gives a necessary and sufficient condition.

2.2.3 Robustness

As already mentioned, we only know that a given trait is possible because we observe its presence, even in very small numbers, in the population. This is exactly the small "errors" that the trembling hand perfectness intends to take into account. Unfortunately, it is *not true* that ESS are necessarily trembling hand perfect, as the following example shows:

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 1.5 & 2 & 0 \end{pmatrix}, \quad p = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix}.$$

The strategy p is an ESS, but not a trembling hand perfect equilibrium. This is shown in appendix.

In that respect, we have two measures of robustness, the first easy:

Proposition 3 *An ESS necessarily puts zero weiht on (weakly) dominated strategies (this is also a feature of trembling hand perfect eqiulibria)*

See a proof in the appendix. It has an interesting consequence:

Corollary 3.1 *Every* pure *ESS is trembling hand perfect.*

This is so because for two player games, pure trembling hand perfect equilibria are exactly those that do not place weight on a weakly dominated srategy.

The second proposition is much more difficult to show, but is classical. See, e.g. [32].See, e.g. [32]. It only holds in the finite case, because the local compacity of the space of strategies is crucial.

Proposition 4 *If p is an ESS, it exists a neighborhood $\mathcal{V}$ of p such that,*

$$\forall q \in \mathcal{V} - p,\ F(p,q) > F(q,q)\,.$$

This property is sometimes called *Evolutionarily Robust Strategy* or ERS. It is an important ingredient of the proof of the stability of the local stability of the replicator dynamics at an ESS.

2.3 Evolutionary dynamics

We have seen heretofore how the consideration of populations lends an effective meaning to mixed equilibria. Evolutionary dynamics are a strong argument in favor of Wardrop equilibria and ESS.

2.3.1 Population evolution

Let a population with n possible traits be described by the frequency distribution $q \in \Delta$. The subpopulation with trait i has a growth rate $G_i(q)q_i$. The effect on the frequency vector is easily calculated to be the *replicator equation* (using notation (1)):

$$\dot{q}_i = q_i[G_i(q) - F(q,q)] \,. \tag{10}$$

This system is thus the most basic model of evolution. It has also been shown to be the asymptotic dynamics of very large populations under various schemes of adaptation, thus becoming an important model in such fields as sociology. See [29]

The main point we want to stress is the following (see [32, 22, 30])

Theorem 1

1. *All limit points of the replicator dynamics are Wardrop equilibria.*
2. *(In the matrix case,) all ESS are locally asymptotically stable eqilibria of the replicator dynamics.*
3. *Their basins of attraction contain a neighborhood of the interior of the lowest dimensional face of Δ they lie on.*

2.4 Population games

It is possible to consider games of two different populations interacting with each other, with no intra-population interaction (in the model). This leads to *population games*. See [29]. The argument in favor of mixed strategies remains the same. If we ccept that the generating function now depends, for each population, of the state of the *other* population, this yields the same replicator dynamics for both populations. Let their distribution frequencies be q^k, $k = 1,2$, we get

$$\dot{q}_i^k = q_i^k[G_i^k(q^{3-k}) - F^k(q^k, q^{3-k})] \,.$$

Rich dynamics may emerge, with limit points and limit cycles. (See, beyound the previously quoted books, [25, 28].) An exhaustive analysis of the two by two case was proposed in [8].

3 Bi-linear two-person nonzero-sum differential games

3.1 Motivations

Thus far, we have assumed that the traits considered are static, and the generating function $G(q)$ was given, and, in this short introduction to evolutionary games, linear in the results of the one against one contests arranged in a game matrix A. Many situations aris where one has to investigate a dynamic contest to determine the "payment" associated with an encounter. Some examples taken in our own works are diet selection in optimal foraging [20], superparasitism in parasitoids [21] for single population problems, seasonwise parental care [19], predators and hiding preys [1] for games between two differing populations.

In each of these cases, the two contestants have two possible strategies, labeled $u_i \in \{0,1\}$, the index $i \in \{1,2\}$ designating the two players. These strategies drive a dynamic system representing resource depletion, or population numbers, or energy accumulation, or number of eggs laid, in any instance a cumulative effect.

Let $x \in \mathbb{R}^n$ be the state of the dynamic system, $a_{k\ell}(x)$, $(k,\ell) \in \{0,1\}\times\{0,1\}$, be the state derivative when $u_1 = k$ and $u_2 = \ell$. Let also

$$\begin{aligned} f_0(x) &= a_{00}(x)\,, \\ f_1(x) &= a_{10}(x) - a_{00}(x)\,, \\ f_2(x) &= a_{01}(x) - a_{00}(x)\,, \\ f_3(x) &= a_{11}(x) - a_{10}(x) - a_{01}(x) + a_{00}(x)\,. \end{aligned}$$

Finally, let

$$f(x,u_1,u_2) := f_0(x) + f_1(x)u_1 + f_2(x)u_2 + f_3(x)u_1u_2\,.$$

Mixed populations lead to mean effects

$$\dot{x} = f(x,u_1,u_2)\,, \quad u_i \in [0,1]\,,\; i = 1,2\,,$$

with u_1 and u_2 the proportion of individuals of each population having the trait 1. Likewise, if integral pay-offs are involved, say, for pure strategies $(u_1,u_2) = (k,\ell)$

$$J^i(x(0);k,\ell) = K^i(x(T)) + \int_0^T b^i_{k,\ell}(x(t))\,\mathrm{d}t\,,$$

the payoff to a mixed population is

$$J^i(x(0);u_1(\cdot),u_2(\cdot)) = K^i(x(T)) + \int_0^T L^i(x(t),u_1(t),u_2(t))\,\mathrm{d}t$$

with the same relationship of the L^i's to the $b^i_{k,\ell}$'s as f to the $a_{k,\ell}$'s.

Adding a state variable equal to time if necessary, we may always assume that the final time T is given via target set $\mathcal{T} \subset \mathbb{R}^n$ by a condition

$$T = \inf\{t \mid x(t) \in \mathcal{T}\}$$

making the game stationary. We also assume that all regularity and growth assumptions hold that insure existence and uniqueness of the state trajectory and payoffs for any (admissible) initial state and any pair of measurable controls $(u_1(\cdot),u_2(\cdot)) : \mathbb{R}_+ \to [0,1]\times[0,1]$.

3.2 Closed Loop Nash equilibrium

According to our previous remarks making Nash equilibria natural outcome of the evolutionary selection process, we sish firs to be able to find the Nash equilibria of this two-person nonzer-sum game. Then, the question may arise as to which select.

3.2.1 Strategies and equilibrium

Unlike the situation for zero-sum games, we do not know how to make a precise theory of closed loop Nash equilibria without using explicitly state feedbacks. This raises the old issue of the choice of admissible sets of such feedbacks linked to the existence of well defined state trajectory and payoffs. This is circumvented in zero-sum differential games via the use of nonanticipative strategies. But here, we do not know how to write the equivalent of Isaacs'equation using only nonanticipative strategies; We regard this fact as a challenge for the theory of Nash equilibria in differential games : we need a more elegant theory.

Until a better theory is available, we propose to use the following variant of our old theory of state-feedback saddle points.[3] Let $\mathcal{U}$, be the set of measurable functions from $\mathbb{R}_+$ to $[0,1]$. Let Φ_1 be the set of state feedbacks : $\varphi_1 : \mathbb{R}^n \to [0,1]$ with the property that the differential equation

$$\dot{x} = f(x, \varphi_1(x), u_2(t)) \,, \quad x(0) = x_0$$

has a unique solution for every (admisible) initial state x_0 and every $u_2(\cdot) \in \mathcal{U}$ leading to a well defined pair of payoffs $J^i(x_0, \varphi_1, u_2(\cdot))$, $i = 1, 2$. Symmetrically, Φ_2 is the set of state feedbacks φ_2 such that the game with controls $(u_1(\cdot), \varphi_2(x(\cdot)))$ is well defined. We define a state-feedback Nash equilibrium as a pair of state feedbacks $(\varphi_1^\star, \varphi_2^\star)$ such that

1. the differential equation

   $$\dot{x} = f(x, \varphi_1^\star(x), \varphi_2^\star(x)) \,, \quad x(0) = x_0$$

   generates for every (admissible) initial state x_0 state trajectories leading to a unique pair of pay-offs $J^i =: V^i(x_0)$,
2. $V^1(x_0) = \sup_{u_1 \in \mathcal{U}} J^1(x_0; u_1(\cdot), \varphi_2^\star)$,
3. $V^2(x_0) = \sup_{u_2 \in \mathcal{U}} J^2(x_0; \varphi_1^\star, u_2(\cdot))$.

Notice that we do not require that the trajectory generated by the pair $(\varphi_1^\star, \varphi_2^\star)$ be unique, which is too restrictive for our purpouse, but only that the payoffs be.

[3] It is a non essential feature of the class of games investigated here that open loop controls of both players live in the same space. In general, we should introduce two sets $\mathcal{U}_1$ and $\mathcal{U}_2$ of open-loop controls.

3.2.2 Hamilton Jacobi Caratheodory Isaacs Bellman Case equations

The two conditions 1. and 2. above are simple optimal control problems. We may thus write the related Hamilton-Jacobi-Caratheodory equations. [4] Let

$$H^i(x,\lambda,u_1,u_2) = L^i(x,u_1,u_2) + \langle \lambda, f(x,u_1,u_2)\rangle \,.$$

Notice also that these hamiltonians have in our case a natural decomposition

$$H^i(x,\lambda^i,u_1,u_2) = H_0(x,\lambda^i) + H_1^i(x,\lambda^i)u_1 + H_2^i(x,\lambda^i)u_2 + H_3^i(x,\lambda^i)u_1u_2\,,$$

with

$$H_j^i(x,\lambda^i) = L_j^i(x) + \langle \lambda^i, f_j(x)\rangle \,.$$

The pair of Hamilton Jacobi equations writes $\forall x \in \mathbb{R}^n$, $\forall u_1 \in \mathcal{U}$, $\forall u_2 \in \mathcal{U}$,

$$\begin{aligned} 0 = H^1(x,\nabla V^1(x),\varphi_1^\star(x),\varphi_2^\star(x)) \ge H^1(x,\nabla V^1(x),u_1,\varphi_2^\star(x))\,,\\ 0 = H^2(x,\nabla V^2(x),\varphi_1^\star(x),\varphi_2^\star(x)) \ge H^2(x,\nabla V^2(x),\varphi_1^\star(x),u_2)\,. \end{aligned} \tag{11}$$

Unfortunately, it is very difficult to state a formal necessity or even sufficiency theorem concerning the relationship of these equations with the Nash equilibrium sought. This is so because we do not want to restrict the equlibrium feedbacks to being continuous. Obviously, our affine-in-the-control problem requires discontinuous feedbacks. But then, in general, it is not possible to prove that the Value functions V^1 and V^2 should be viscosity solutions.

Much work has been devoted to this problem recently see, e.g. [14, 12, 11]. A survey of the linear quadratic literature can be found in [17], and of the non linear-quadratic part of the litterature in [27]. Most of this literature assumes strictly convex cost functions, and a large part is concerned with scalar state. In the case investigated here, there is some hope for a different approach, using the fact that the equilibrium controls are, in regular fields, in piecewise constants, or, in bi-singular fields, given by the particular set of PDE's (12) below which does not countain the equilibrium feedbacks.

Regular fields In the case of linear-in-each-control differential games as considered here, there are at least parts of the state space where there exists a pair of "bang bang" equilibrium strategies, i.e. locally constant controls. In these regions, the optimal feedbacks are smooth, and even constant. Thus the theory of characteristics can be used to find solutions of the Hamilton-Jacobi system, candidate Value functions.

[4] We simply have here two optimal control problems. The use of Hamilton Jacobi equations to state sufficiency conditions in calculus of variations is clearly due to Konstantin Caratheodory[13]. It was re-discovered almost simultaneously and probably independently by Rufus Isaacs in 1951 [23, 24], in the context of zerosum differential games, and then by Richard Bellman in 1953 [2]. Its use in the present form seems to have been published first by Jim Case [15]

3.2.3 Bi-singular field

The inequalities in equations (11) are, for each (x, λ), those of a bi-matrix game, with the bi-matrix formed with the

$$c^i_{k\ell} = b^i_{k\ell} + \langle \lambda^i, a_{k,\ell} \rangle$$

as

c^1_{00}	c^2_{00}	c^1_{01}	c^2_{01}
c^1_{10}	c^2_{10}	c^1_{11}	c^2_{11}

A totally mixed strategy is therefore obtained, if the H^i_3 are non zero, with

$$u_i = -\frac{H^j_j(x, \nabla V^j(x))}{H^j_3(x, \nabla V^j(x))} =: \varphi^\star_i(x)\,, \qquad i = 1, 2\,, \quad j = 3 - i\,.$$

Placing this in the equalities of (11), we obtain two *compatibility conditions*:

$$H^i_1(x, \nabla V^i(x)) H^i_2(x, \nabla V^i(x)) - H^i_0(x, \nabla V^i(x)) H^i_3(x, \nabla V^i(x)) = 0\,, \quad i = 1, 2\,. \tag{12}$$

In contrast with equations (11), these two PDE's

1. are uncoupled,
2. do not involve the $\varphi^\star_i$.

They can be investigated via their characteristics. One should emphasize, though, that these characteristics are *not* strate trajectories under the singular controls.

In terms of optimal control, this corresponds to a situation where both optimization problems are singular. But in optimal control theory, one ususally gets an isolated singular arc. Here we may have a bi-singular field. Curiously, such solutions do not seem to have appeared in the litterature before the reference [19]. In fact, the idea arose in the investigation of [20]. Both problems in behavioural ecology. (Nothing prevents zero-sum differential games to exhibit such bi-singular fields.)

3.3 Singular manifolds

In zero-sum differential games, the investigation of singular manifolds has played a central role. See [24, 4, 6]. Much less is known about the singular manifolds of equilibrium Values and strategies in nonzero sum differential games.

3.3.1 Barriers

Barriers are hypersurfaces along which the Values have a jump discontinuity. A priori, two types of barriers are possible in nonzero-sum Nash equilibria of differential games : *half barriers* where one only of the Value functions has a discontinuity and *full barriers* where both are discontinuous. We shall show with the example below a (rather trivial) half barrier.

Again, two types of *full barriers* are possible: either the jumps of the two Value functions are of the same sign, we shall refer to *cooperative* barriers, or they are of opposite signs, and we shall refer to *competitive* barriers.

The techniques to construct both types of full barriers are the same as in optimal control theory for cooperative barriers, with both players . . . cooperating, and as in zero-sum differential games for competitive barriers. We do not need to describe them here. However, we show that both types of barriers may exist in linear games by showing in the subsection 5.3.5 an example of a Nash equilibrium in a linear D.G. exhibiting both. (Although a bit artificially)

3.3.2 Corners: generalities and the zero-sum case

We want now to investigate how discontinuities in the equilibrium controls appear.

We assume that a smoth hypersurface $\mathcal{S}$, reached by the equilibrium trajectories on one side and left on the other side, bears a *corner*, a slope discontinuity in the field of equilibrium trajectories. Assume that both fields, on both sides of $\mathcal{S}$, are smooth, with smooth Value gradients $\nabla V^i(x)$, hence satisfying adjoint equations. Let the superscripts $-$ and $+$ denote limit values on $\mathcal{S}$ in the fields respectively reaching and leaving it. Thus, $u_i^-(x) := \lim \varphi_i^\star(y)$ as $y \to x \in \mathcal{S}$ in the incoming field and likewise for u_i^+ in the outgoing field.

For $x \in \mathcal{S}$, let $\nu(x)$ denote a normal vector to $\mathcal{S}$, pointing in the region $+$. By definition, on $\mathcal{S}$ one has

$$\langle \nu(x), f(x, u_1^-(x), u_2^-(x)) \rangle \geq 0\,, \quad \langle \nu(x), f(x, u_1^+(x), u_2^+(x)) \rangle \geq 0\,.$$

Permeability condition Introduce the *permeability condition*:

$$\langle \nu(x), f(x, u_1^-(x), u_2^+(x)) \rangle \geq 0\,, \quad \langle \nu(x), f(x, u_1^+(x), u_2^-(x)) \rangle \geq 0\,.$$

The following theorem [4, 5] is the equivalent for zero-sum differential games of the Erdman-Weierstrass condition of the calculus of variations and optimal control theory [10]:

Theorem 2 *In zero-sum differential games, if the permeability condition holds on the corner manifold $\mathcal{S}$, the gradient $\nabla V(x)$ is continuous across $\mathcal{S}$.*

Yet, we do not have a similar result for equilibrium trajectories of nonzero-sum differential games.

Let also $\lambda^{i-}(x)$ and $\lambda^{i+}(x)$ stand for the limits $\nabla V^{i-}(x)$ and $\nabla V^{i+}(x)$ of $\nabla V^i(y)$ as $y \to x$ in the field $-$ and $+$ respectively. Since we assume that the equilibrium trajectories cross the surface $\mathcal{S}$, the Value functions must be continuous across it. It follows:

Proposition 5 *For every x in $\mathcal{S}$, there exists two real numbers $k^i(x)$ such that*

$$\lambda^{i-}(x) = \lambda^{i+}(x) + k^i(x)\nu(x)\,. \tag{13}$$

If $\mathcal{S}$, thus $\nu(x)$, is known, equations (11) let one calculate the k^i's. If to the contrary, $\mathcal{S}$ is not known, but we may find a scalar equation linking x to a $k^i(x)$, this is equivalent to a PDE for $\mathcal{S}$. For zero-sum differential games, this is provided by either theorem 2 above or the *indifference condition* [3, 4], leading to Isaacs *equivocal surfaces* in the linear case and to Breakwell's *switch envelopes* otherwise.

The situation is very far from being as satisfactory for nonzero-sum differential games, this is what we investigate now.

3.3.3 Simple switch hypersurfaces

Assume both fields, $-$ and $+$, are regular. A simple switch manifold is one where one only of the two equilibrium strategies has a discontinuity. Assume that $\varphi_1^\star$ is discontinuous and $\varphi_2^\star$ is continuous. Because of this last fact, the first equation in (11) has a continuous hamiltonian. We should thus expect that V^1 be C^1 in a neighborhood of $\mathcal{S}$. Introduce the switch functions

$$\sigma_i(x) = H_i(x, \nabla V^i(x)) + H_3(x, \nabla V^i(x))\varphi_{3-i}^\star(x).$$

On such a simple switch hypersurface, σ_1 is continuous. A discontinuity of the maximizing control $\varphi_1^\star$ must therefore happen at a place where $\sigma_1(x) = 0$. This is our extra condition to localize such a simple switch surface, and is similar to the case of zero-sum differential games.

3.3.4 Double switch hypersurfaces

A double switch, or "bang bang bang" surface is one where both equilibrium controls switch from an extreme value to the other one. Hence necessarily, both switch functions change sign across $\mathcal{S}$, but they may be discontinuous. We *do not know*, at this time, an extra condition that could help localize such surfaces. Is there a large degree of freedom generating a large non uniqueness of equlibrium ? This question was posed by Andrei Akhmetzhanov during joint work [1]. If yes, extra selection conditions are required. We offer here a tentative such condition.

Assume therefore that both switch functions have non zero limits σ_i^+. Hence, the field $+$ can be extended in a neighborhhod of $\mathcal{S}$, providing switch times earlier or later than on $\mathcal{S}$. Assume that the switch manifold is translated, locally at x, by a small amount $\varepsilon\nu(x)$, displacing the switch point by a small amount δx such that, to first order, $\langle \nu, \delta x\rangle = \varepsilon\|\nu\|^2$. It is easy to see that, on the trajectory through x, the payoffs are modified, to first order, as

$$\delta J^i = \langle \lambda^{i+} - \lambda^{i-}, \delta x\rangle = -k^i\varepsilon\|\nu\|^2 .$$

If both k^i have the same sign, the equilibrium would be dominated by one where te switch surface would be translated, earlier ($\varepsilon < 0$) if $k^i > 0$ and later if $k^i < 0$. This change of switch time should then happen as a result of "Nature's trembling hand" tries: mutations. A stable situation would be found in the case $k^i > 0$ when it is no longer possible to move the switch surface earlier because a change in one of the σ_i's would happen in the field $+$, i.e. with one σ_i^+ null, and in the case $k^i < 0$ when one σ_i^- is null.

Now assue that, say, $k^1 > 0$ and $k^2 < 0$. If player one switches earlier than $\mathcal{S}$, it gets a lower payoff, by definition of an equilibrium. But both players then improve their situations by switching, either player one to come back to the original equilibrium, or player two, to come to the equilibrium with $\mathcal{S}$ translated earlier. If player one comes back to the original equilibrium, nothing has happened. If, to the contrary, it is player 2 who responds, the new equilibrium is more favourable to player one than the original one. Therefore, player one is likely to attempt that modification, and "be patient" and wait for player two to switch in turn.

Is this argument symmetric, with player two trying late switches ? Probably not because the chronology is in favor of player one, who gets the opportunity to act first. We would then conclude that in the long run, player one should force the switching surface to a location where either $k^1 = 0$ or a $\sigma_1^+ = 0$ as previously.

This is by no means a rigorous development, but points to the need for a deeper analysis of evolutionary dynamics when the pay-offs are themselves the result of a Nash equilibrium in a differential game.

3.3.5 Equivocal hypersurfaces ?

We want now to investigate how a bi-singular field can join on a regular field. Indeed, in two examples we analysed [21, 1], we ran into situations where neither a simple switch nor a double switch between extreme controls is possible.

We may assume without loss of generality that in the outgoing, regular field $+$, both equilibrium controls are 0. On the incoming, bi-singular field $-$, we have intermediary controls u_1^- and u_2^-. The geometry imposes that

$$\begin{aligned} \langle \nu, f_0 \rangle &\geq 0\,, \\ \langle \nu, f_0 + f_1 u_i^- + f_2 u_2^- + f_3 u_1^- u_2^- \rangle &\geq 0\,. \end{aligned}$$

We assume further for the present analysis that one of the permeability conditions is violated, say

Assumption A1 $\qquad \langle \nu, f_0 + f_1 u_1^- \rangle < 0\,.$

Furthermore, we also assume

Assumption A2 $\qquad k^1 < 0,$

and, in keeping with the above remark that k^1 and k^2 should be of opposite signs, we should then have $k^2 > 0$.

In that case if, upon reaching $\mathcal{S}$ player one fails to switch to $u_1 = 0$, but keeps its control $u_1 = u_1^-(x)$, player two has a dilemma: if it switches to $u_2 = 0$, the state ends up in region $-$, player one is therefore playing the equlibrium strategy and player two underperforms, if, to the contrary, it keeps its control $u_2 = u_2^-$, the sate drifts into region $+$ where its equilibrium control is 0.

We remark that if $k^2 > 0$, this second situation is also unfavourable to player two, since this amounts to postponing the switch to a later time, creating a true dilemma for it, an incentive to resort to the srategy hereafter. But this is not essential in the analysis.

Indeed, if $k^1 < 0$, player one would do better if none of the players switches upon reaching $\mathcal{S}$. Therefore, for our pair of strategies to be an equilibrium, in case the state does not cross into region $+$, it should specify the Filippov solution, or ‘chatter” arising from the discontinuous field $f(u_1^-, u_2^-)$, $f(u_1^-, 0)$. Everything being linear, this is a very natural solution concept, and equivalent to player two choosing a control $u_2 = \tilde{u}_2(x)$ such that

$$\langle \nu(x), f_0(x) + f_1(x) u_1^-(x) \rangle + \langle \nu(x), f_2(x) + f_3(x) u_1^-(x) \rangle \tilde{u}_2(x) = 0\,.$$

We may notice that, necessarily, there exists such a $\tilde{u}_2(x) \in (0, u_2^-(x))$.

Which type of trajectories results from our pair of strategies: traversing $\mathcal{S}$ or crossing it, depends on the precise behaviour of player one on the hypersurface $\mathcal{S}$ only. For robustness, and according to the classical definition of Filippov solutions, a solution should not depend on the values of the control on a set of (Lebesgue) measure zero. Hence, the two types of trajectories should be considered as possible outcomes of our pair of equilibrium strategies. Both payoffs should however be uniquely defined, which requires that

$$H^i(x, \nabla V^i(x), u_1^-(x), \tilde{u}_2(x)) = 0\,, \quad i = 1, 2\,.$$

These hamiltonians may be evaluated using either λ^{i-} or λ^{i+} for the ∇V^i's, since the dynamics are then orthogonal to their difference. Using λ^{i-}, we see that this is automatically insured for H^2 since the choice $u_1 = u_1^-(x)$ makes it independent on u_2. For H^1 this requires that

$$H_0^1(x, \lambda^{1-}) + H_1^1(x, \lambda^{1-}) u_1^-(x) = 0\,, \quad H_2^1(x, \lambda^{1-}) + H_3^1(x, \lambda^{1-}) u_1^-(x) = 0\,.$$

Thanks to the compatibility condition (12), these two conditions coincide. This provides an equivalent of an equivocal hypersurface for a Nash equilibrium of nonzero-sum bi-linear differential games.

4 Games with an unknown number of players

We offer here another open problem discovered in the investigation of the managment of biological systems. This is a problem of diet selection encountered in [20].

One or several animals deplete a resource according to two possible strategies, called for convenience $u \in \{0, 1\}$. If N animals are present, the resource decreasesaccording to the law

$$\dot{x} = f_0(x) + \sum_{i=1}^{N} f_1(x, u_i)$$

Animal number i arrives at time t_i. The game ends when the state reaches a target set $\mathcal{T}$. Each animal gets a payoff

$$J_i = \int_{t_i}^{T} L(x, u_i)\, \mathrm{d}t\,.$$

Consider the simple problem where an animal is present at initial time $t_1 = 0$, and at most one competitor can arrive, this happening at a time t_2 distributed according to an exponential law:

$$\mathbb{P}(t_2 \geq t) = \exp -\lambda t$$

What is the optimal strategy of player one to maximize its expected payoff ?

It is easy to describe, at least formally, the solution technique for this problem. Solve first the equilibrium problem for two players. It is symmetric. Call $V_2(t_2, x)$ the equilibrium value of that game. Now the problem of player one is to maximize the expectation $\mathbb{E}J_1$ with

$$J_1 = \int_0^{t_2 \wedge T} L(x, u_1)\, \mathrm{d}t + \begin{cases} V_2(t_2, x(t_2)) & \text{if } t_2 < T\,, \\ 0 & \text{if } t_2 \geq T\,. \end{cases}$$

Hence

$$\mathbb{E}J_1 = \int_0^T \lambda \mathrm{e}^{-\lambda t_2} \left(\int_0^{t_2} L(x(t), u_1(t))\,\mathrm{d}t + V_2(t_2, x(t_2)) \right) \mathrm{d}t_2 \\ + e^{-\lambda T} \int_0^T L(x(t), u_1(t))\,\mathrm{d}t\,.$$

Use Fubini's theorem on the first double integral to get

$$\mathbb{E}J_1 = \int_0^T [L(x(t), u_1(t)) + \lambda V_2(t, x(t))] \mathrm{e}^{-\lambda t} \mathrm{d}t\,.$$

This is a standard problem with discounted integral cost $L + \lambda V_2$.

Clearly, if we know that a third player may arrive, still with an exponentially distributed time after t_2, we may perform the same transformation twice. First solve the symmetric game with three players, and call $V_3(t, x)$ its equilibrium Value. Then consider the symmetric two-player game with pay-off, for $i = 1, 2$,

$$G_i = \int_{t_2}^T [L(x(t), u_i(t)) + \lambda V_3(t, x(t))] \mathrm{e}^{-\lambda t}\,\mathrm{d}t\,.$$

Find its Value function $W_2(t_2, x)$, and maximize

$$\mathbb{E}J_1 = \int_0^T [L(x(t), u_1(t)) + \lambda W_2(t, x(t))] \mathrm{e}^{-\lambda t}\,\mathrm{d}t\,.$$

However, what can be done if we only know that new players may arrive according to a Poisson process, but with no known limit to their number ?

5 An example

We show here an equilibrium in a linear two-person nonzero-sum differential game, involving a competitive barrier, a cooperative barrier, a double switch and a simple switch. Unfortunately, we do not have a simple example of a junction of a bi-singular field with a regular one, nor an equivocal surface. Indeed our example cannot involve a bi-singular field, because it is linear but not bi-linear: $f_3 = 0$ and $L_3 = 0$.

5.1 The game

The game state will be denoted $(x, y) \in \mathbb{R}^2$. The playing space is the region $y \geq 0$. The scalar controls u_1 and u_2 both belong to the interval $[-1, 1]$ (rather than $[0, 1]$ in the rest of this note.) The game is defined by

- its dynamics

$$\begin{array}{rll} \dot{x} = & y - 2 + u_1 & u_1 \in [-1,1]\,, \\ \dot{y} = & -x + u_2 & u_2 \in [-1,1]\,. \end{array}$$

(We remark that on any time interval during which the controls are constant, the trajectories are arcs of circle centered at $x = u_2$, $y = 2 - u_1$ and traversed clokwise, with angular velocity one.)

- the terminal manifold $\mathcal{T}$: $\{y = 0\}$,

- two performance indices, one for each player, that hey want to maximize

$$J^1 = \left\{ \begin{array}{ll} x(T) & \text{if } x(T) > 0\,, \\ -\infty & \text{if } x(T) \le 0\,, \end{array} \right. \qquad J^2 = \left\{ \begin{array}{ll} -T & \text{if } x(T) > 0\,, \\ -\infty & \text{if } x(T) \le 0\,. \end{array} \right.$$

We use the notations

$$\nabla V^i(x) = \begin{pmatrix} \lambda^i \\ \mu^i \end{pmatrix}$$

The two hamiltonians are therefore

$$\begin{array}{l} H^1 = \lambda^1(y - 2 + u_1) + \mu^1(-x + u_2)\,, \\ H^2 = -1 + \lambda^2(y - 2 + u_1) + \mu^2(-x + u_2)\,. \end{array}$$

5.2 Regular fields

5.2.1 Primary field

Close to the terminal manifold, the positive x axis, we expect player **1** to play $u_1 = 1$ and player **2** to play $u_2 = -1$. Let us denote $x(T) = s$. We have $V^1(s,0) = s$ and $V^2(s,0) = 0$, it follows that on $\mathcal{T}$, $\lambda^1 = 1$ and $\lambda^2 = 0$. Therefore $u_1 = 1$ is indeed maximizing H^1, and Isaacs-Case equations (11) yield

$$\begin{array}{l} -1 - \mu^1(x + 1) = 0\,, \\ -1 - \mu^2(x + 1) = 0\,, \end{array}$$

so that at final time, $\mu_1 = \mu_2 = -1/(x + 1)$, and μ_2 being negative, $u_2 = -1$ is indeed maximizing H^2.

As long as V^1 and V^2 are C^1, the gradients obey the adjoint equations

$$\begin{array}{rcl} \dot{\lambda}^i & = & \mu^i\,, \\ \dot{\mu}^i & = & -\lambda^i\,. \end{array}$$

(On any time interval over which these equations apply, both gradient vectors are of constant norm, and revolve clockwise with velocity one. Their angle with the trajectory velocity will therefore remain constant, and equal to $\pi/2$ concerning ∇V^1.) It follows that, if we define ρ and φ via

$$\begin{aligned} \rho\cos\varphi &= 1\,, \\ \rho\sin\varphi &= -\frac{1}{s+1}\,, \end{aligned}$$

we get, in the primary field,

$$\begin{array}{ll} \lambda^1 = \rho\cos(\varphi+T-t)\,, & \lambda^2 = \frac{1}{s+1}\sin(T-t)\,, \\ \mu^1 = \rho\sin(\varphi+T-t)\,, & \mu^2 = \frac{1}{s+1}\cos(T-t)\,. \end{array}$$

It is a simple matter that in the primary field,

$$\begin{aligned} x &= -1+(s+1)\rho\cos(\varphi+T-t)\,, \\ y &= 1+(s+1)\rho\sin(\varphi+T-t)\,. \end{aligned}$$

5.2.2 Corner

A consequence of the formulas for the Value gradients, in a retrogressive construction, μ^2 changes sign "before" (closer to final time than) λ^1, at $t_1 = T-\pi/2$, $x(t_1)=0$, $y(t_1)=s+2$. Hence we suspect a corner along $\mathcal{S}=\{x=0, y\geq 2\}$.

Attempt at a simple switch We attempt to construct a field $-$ joining on the primary field as feld $+$, with a switch in u_2 alone, i.e. $(u_1^-, u_2^-) = (1,1)$. Since in that case, u_1 is continuous, we expect ∇V^2 to be continuous. Equation (13) implies that μ_1 is also continuous. And Bellman (Isaacs-Case) equation reads

$$\lambda^{1-}(s+1)+1=0\,,$$

implyng that λ^{1-} is negative. But then $u_1=1$ is not maximizing. Thus the simple switch is impopssible.

Double switch Let us therefore attempt a corner with $(u_1^-, u_2^-)=(-1,1)$. Again, μ^1 and μ^2 are continuous, and the two Isaacs-Case equations yield

$$\lambda^{1-} = \frac{-1}{s-1}\,, \qquad \lambda^{2-} = \frac{1}{s-1}\,.$$

The first equality above shows that an incoming field $-$ can be constructed for $s\geq 1$, and therefore $y\geq 3$. Notice that $k^1 = 2s/(1-s^2) < 0$ and $k^2 = 2/(s^2-1) > 0$ are of opposite signs, and in agreement with our conjecture on evolutionary stability.

For $y\leq 2$, there is no primary reaching the y axis. Between $y=2$ and $y=3$ a field $+$ in the $x>0$ region, but, no incoming field $-$.

Permeability conditions In the region $y \geq 3$, the pemeability conditions are met since $\dot{x} \geq 0$ for all control values.

5.2.3 Secondary field

The secondary field, incoming to the corner at t_1 can be expressed in terms of ρ' and φ' defined as

$$\rho' \cos\varphi' = -1/(s-1)\,,$$
$$\rho' \sin\varphi' = 1\,,$$

as

$$x = 1 + (s-1)\rho' \cos(\varphi' + t_1 - t) = 1 + (s-1)\rho' \sin(\varphi' + T - t)\,,$$
$$y = 3 + (s-1)\rho' \sin(\varphi' + t_1 - t) = 3 - (s-1)\rho' \cos(\varphi' + T - t)\,.$$

It may be useful to notice that $(s-1)\rho' = \sqrt{(s-1)^2+1}$. The gradient vectors are obtained as

$$\lambda^1 = \rho' \cos(\varphi' + t_1 - t) = \rho' \sin(\varphi' + T - t)\,, \qquad \lambda^2 = \tfrac{1}{s-1}\cos(t_1 - t) = \tfrac{1}{s-1}\sin(T-t)$$
$$\mu^1 = \rho' \sin(\varphi' + t_1 - t) = -\rho' \cos(\varphi' + T - t)\,, \quad \mu^2 = \sin(t_1 - t) = -\cos(T-t)\,.$$

The sign reversal prior and closest to t_1 is that of λ^1 at $t_2 = \varphi' + t_1 - 3\pi/2 = \varphi' + T - 2\pi$, and therefore $x(t_2) = 1$ $y(t_2) = 3 - (s-1)\rho'$. This is apparently inside the primary field, but we shall see that we have to discard that region of the primary field, giving room for a tiny part of the tertiary field.

We also notice

$$\lambda^1(t_2) = 0\,, \qquad \lambda^2(t_2) = -\sin\varphi'/(s-1)\,,$$
$$\mu^1(t_2) = -\rho'\,, \qquad \mu^2(t_2) = -\cos\varphi'\,.$$

5.2.4 Tertiary field

The tertiary field ends at $t = t_2$ with $\lambda^1(t_2) = 0$. We look for a corner borne by $x = 1$, thus a possible discontinuity in the λ^i's, and the μ^i's continuous.

As a matter of fact, one easily finds tat a simple switch in u_1 is possible. Since u_2 is then continuous, not surprisingly, the Isaacs-Case equations show that λ^1 is continuous too, and only λ^2 has a discontnuity, with $\lambda^{2-} = 1/(y(t_2) - 1)$.

Looking for a sign change of λ^1 or μ^2 before t_2, we find that the closest such sign change is at t_3 such that $t_2 - t_3 = \pi - \varphi'$, where $\mu^2(t_3) = 0$. One can show the surprising fact that this happens with the state lying on the straight line joining the rotation center of that tertiary field at $(x, y) = (1, 1)$ and the end point of the same trajectory at $(x, y) = (s, 0)$. The only use we really have of the characterization of t_3 is to check that for $s \geq 2$, the time elapsed between t_3 and t_2 is larger or equal to $\pi/4$, because $\tan(t_2 - t_3) = \tan(\pi - \varphi') = -\tan\varphi' = s - 1$.

But much more needs be investigated concerning the interplay of these various fields.

5.3 Singularities and Nash equilibrium strategies

5.3.1 Barrier

Let us concentrate on a pair of strategies that would be $(u_1, u_2) = (1, -1)$ if $x \geq 0$, the primary field, and $(u_1, u_2) = (-1, 1)$ if $x < 0$, the secondary field.

Part of the primary field is in fact not in a Nash equilibrium. This is so because, close to $x = 0$, player **1** may, by playing $u_1 = -1$, prevent the game from terminating "soon", forcing the state to cross the y axis, to subsequently "go around" and terminate at a larger $x(T)$. Of course, this gives a much longer game hurting player **2**. We are therefore led to construct a barrier, a curve along which both value functions have a discontinuity, but of opposite signs. This is therefore completely similar to a zero-sum game barrier, that we may term a *competitive barrier*. We shall see earlier in the game a *cooperative barrier*, where both Value functions have a jump of the same sign, so that in an equilibrium pair of strategies, both players cooperate in attempting not to cross it.

Here, the competitive barrier terminates at the origin. Using classical zero-sum game theory, we find that it is an arc of circle $(u_1, u_2) = (-1, -1)$, hence centered at $(x, y) = (-1, 3)$, extending from a point $\mathsf{V} : (x, y) = (\sqrt{10} - 1, 3)$ to the origin, before that, an arc of circle $(u_1, u_2) = (-1, 1)$, hence centered at $(x, y) = (1, 3)$, extending from a point U at $(x, y) = (1, \sqrt{10} + 1)$ to V where it joins the foowing part smoothly, and prior that again an trajectory with $(u_1, u_2) = (1, 1)$, hence centered on $(x, y) = (1, 1)$, joinig smoothly the arc just constructed at U. We shall see later where is the earliest point of interest on this first arc of the barrier.

The arc UV of the barrier is tangent to a primary trajectory at its $\pi/4$ point T, on the line $y = x + 2$. That primary is interesting, because player **2** can prevent the state from crossing it towards larger x's by playing $u_2 = -1$. We call s_1 the abscissa of its en-point on the x axis, and P the point where it cuts the y axis, at $y = s_1 + 2$.

We need also consider the tertiary trajectory tangent to the barrier. This happens at a point Q at $t_2 - t = \pi/4$. A simple calculation shows that its parameter, the abscissa s_2 of its end point in the primary field, is larger than s_1. It reaches the secondary field, at $x = 1$ at a point W. The corresponding secondary trajectory intersects the barrier on its first arc, at a point R, while the secondary trajectory corresponding to $s = s_1$ cuts the same arc at a point S.

We need to name two closed region : Call $\mathcal{A}$ the curvilinear triangle STP, and $\mathcal{B}$ the area inside the closed contour QRSTQ

5.3.2 Nash equilibrium strategies

We now define a specific strategy pair (O stands for the origin $(0,0)$)

- In the primary field deprived of the region between the arc PTQO and the y axis, $(u_1, u_2) = (1, -1)$,
- in the secondary field deprived of $\mathcal{A} \cup \mathcal{B}$, and of the part of the primary fied described above, $(u_1, u_2) = (-1, 1)$,
- in the tertiary field in the (tiny) region between the arc QW and the barrier QO, $(u_1, u_2) = (1, 1)$,
- in region $\mathcal{A}$, play the *purely competitive* strategies to reach the arc PT in $\max_{u_1} \min_{u_2}$ time,
- in the region $\mathcal{B}$, play the *cooperative* strategies to reach the arc QR in minimum time, —a complex strategy pair, with a cooperative barrier— player **1** making sure that the state does not leave that region by another part, i.e. playing $u_1 = 1$ on the arc RU and $u_1 = -1$ on the arcs UV and V, Q.

Theorem 3 *The above strategy pair is a Nash equilibrium. The equilibrium payoff are characterized by $J^1 = s_1$ in region $\mathcal{A}$, $J^1 = s_2$ in region $\mathcal{B}$, and given by the primary, secondary and tertiary fields elsewhere.*

Proof In region $\mathcal{A}$, the control $u_2 = 1$ insures that the state cannot cross either the arc SP nor the arc SU, it ca therefore only leave it through the arc PT. On that arc, player **2** switches to $u_2 = -1$ making it impossible for player **1** to cross it towards larger $J^1 = s$. Hence player **1** has no incentive to deviate. Conversly, since our strategy pair yields the fastest path to any point in the primary field, and from their to termination for player **2**, he does not have any incentive to deviate either.

A similar argument holds in region $\mathcal{B}$, reversing roles: player **1** insures that the state cannot leave it through any other arc than QR, which yields a larger J^1 than in the other fields adjacent to $\mathcal{B}$. But against $u_2 = 1$, he has no means of reaching a secondary trajectory with larger s.

Finally, in the regular, primary, secondary and tertiary fields, the Value functions are piecewise C^1 and satisfy the Isaacs-Case equations. Moreover, at the double switch corner, the permaebility condition is satisfied. Hence, as long as a comparison trajectory does not leave those fields, the lone deviating player cannot achieve a better result.

It remains to consider deviating trajectories that would penetrate region $\mathcal{A}$ or $\mathcal{B}$.player **1** cannot force the state to enter region $\mathcal{B}$ through the barrier —because it is a barrier—, and has no incentive to do so through the arc QR, and has no incentive to penetrate region $\mathcal{A}$ where J^1 is smaller or equal to that in adjacent regions. Conversely, player **2** has no incentive to penetrate region $\mathcal{B}$ from outside, because J^2 is larger inside, nor to penetrate region $\mathcal{A}$ through the boudary SPT since the state would leave it through T, but the time to reach it from inside $\mathcal{A}$, being $\max\min$ is no less than via the trajectory SPT. He cannot either force the state to penetrate region $\mathcal{A}$ through the barrier —again because it is a barrier to him in that direction. ∎

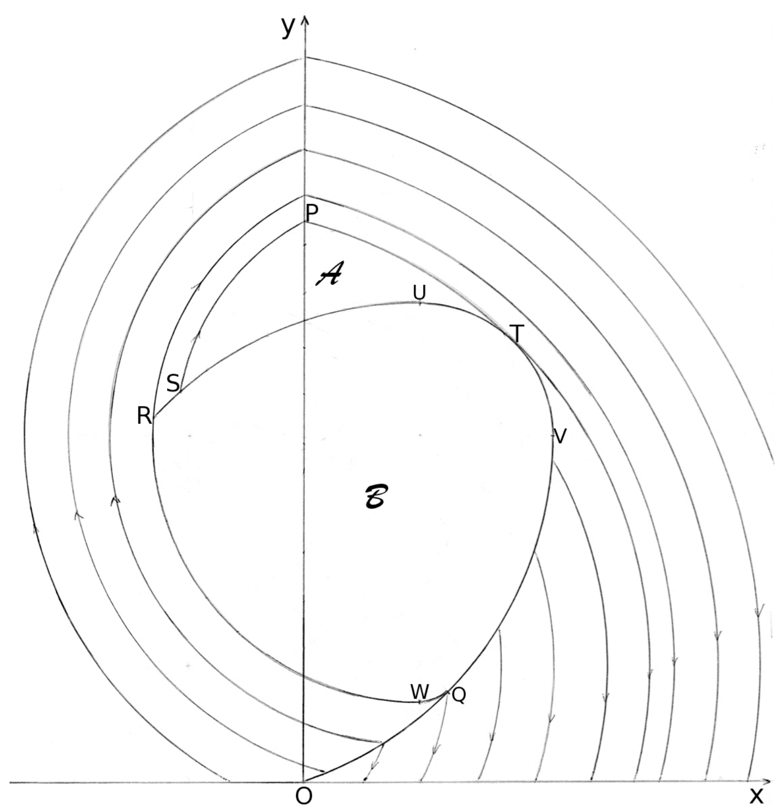

Figure 1: A Nash equilibrium field

5.3.3 Minimax time game within region $\mathcal{A}$

For the sake of completeness, let us describe the minimax time strategy in region $\mathcal{A}$. Te arc PT will be traversed with $u_1 = -1$ and $u_2 = (2x - y - 3)/(y - 1)$. We may easily check that $u_2 \in [-1, 1]$, yet, the left limit requires checking that S is "above" the straight line $y + x = 2$, which is true. Using a classical construction, we assume that this arc will be reached by trajectories $(u_1, u_2) = (-1, 1)$, and we find that the Value gradient upon reaching the arc PT is given by $(\lambda \quad \mu) = ((y-1)^2/(y-3) \quad 0)$. Thus $\lambda > 0$ and just before reaching PT, $\mu > 0$, which is in agreement with our guess concerning the controls. The signs of these gradient coordinates do not change within an arc of circle of $\pi/2$, hence the region $\mathcal{A}$ is filled by this simple field.

5.3.4 Cooperative minimum time strategies within region $\mathcal{B}$

We now turn to the cooperative minimum time strategies within $\mathcal{B}$. The first thing to do is to check that R has an ordinate $y > 3$. We assume that the arc WR will be traversed at maximum speed with $(u_1, u_2) = (-1, 1)$, and reached by trajectories with $(u_1, u_2) = (-1, -1)$, at least in the region $y \leq 3$. A classical construction leads to the fact that there is such a trajectory arriving at $y = 3$ which is a cooperative barrier. At $y < 3$, the final Value gradient is given by $(\lambda \quad \mu) = (1/(y-3) \quad 0)$. Hence λ is negative, as well as μ just before reaching the final arc, which is in agreement with our guess concerning the controls. But before a time $\pi/2$ before reaching this arc, λ is positive, hence $u_1 = 1$. We therefore find a switch line, and the sace "above" is filled by the field before the switch.

There remains to investigate what happens on the arc QW. Here, the fastest way to traverse it is with $u_1 = -1$ and $u_2 = (3 - x - y)/(1 - y)$. Again, it is readily checked that indeed, $u_2 \in [-1, 1]$. If the incoming field is still with $(u_1, u_2) = (-1, -1)$ (as on the adjacent arc WR), then one finds for the Value gradient $(\lambda \quad \mu) = (-1/(3-y) \quad 0)$. Hence, as wished, $\lambda < 0$ and also μ just before reaching the final arc. Thus this smoothly extends the situation on the adjacent arc.

5.3.5 Half barrier

If we replace the cost $J^1 = x(T)$ by the discontinuous $J^1 = x(T)$ if $x(T) < 1$ and $J^1 = 1 + x(T)$ if $x(T) \geq 1$, the trajectory arriving at $s = 1$ becomes a half barrier. Nothing else is modified in the game. Anyyhow, because the payment of player one is purely final, all equilibrium trajectories are non-permeable surfaces for him.

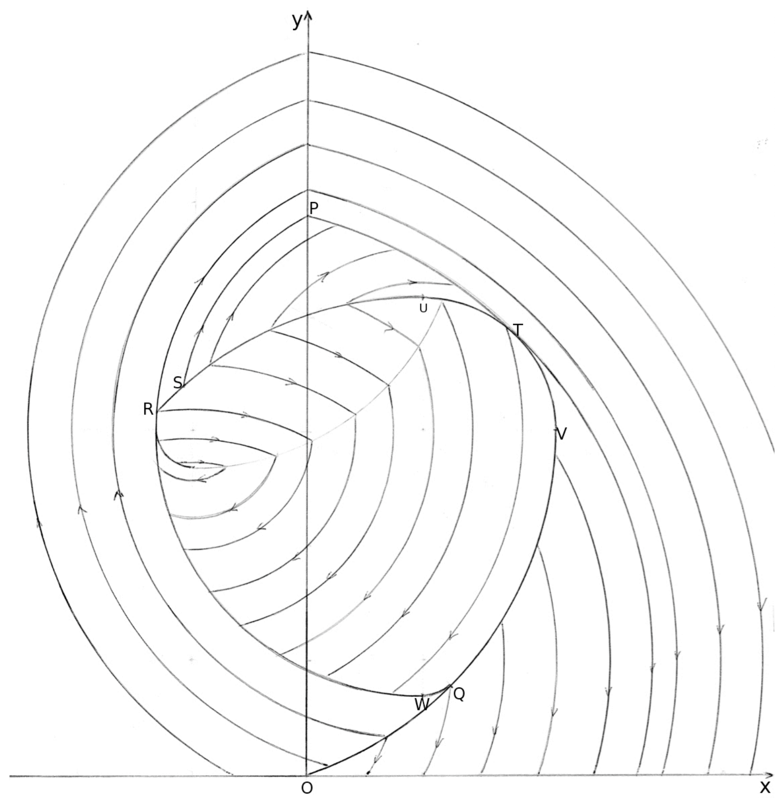

Figure 2: With a competitive field in region $\mathcal{A}$ and cooperative in region $\mathcal{B}$

5.4 Variant

A more elegant, but less interesting, variant is with slighly modified dynamics, as follows:

$$\dot{x} = y - 2 + u_1\,, \qquad u_1 \in [-1,1]\,,$$
$$\dot{y} = -x + 2u_2\,, \qquad u_2 \in [-1,1]\,,$$

the rest of the problem being as in the original example.

The primary field is as in the main example, with $(u_1,u_2) = (1,-1)$, but the double switch happens on the manifold $x = -1$, with $y = s+3$. The secondary field, with $(u_1,u_2) = (-1,1)$ suffices, we shall not need the tertiary field.

The barrier cannot be continued backward beyound its point V where $y = 3$, $x = \sqrt{13} - 2$. The primary trajectory through that point has a parameter $x(T) = s = 2$. It intersects the corner manifold at a point P : $x = -1$, $y = 5$. The secondary trajectory reaching that point originates at the origin O. Thus the region

$\mathcal{A}$ is now the region OPVO.

The purely time-optimal competitive field within the region $\mathcal{A}$ now involves a barrier and an equivocal line. Together with the primary and secondary fields outside of $\mathcal{A}$, it provides a pair of Nash equilibrium strategies.

5.5 Conclusion of the example

As a substitute for a true conclusion, consider the question

> "Why should player **1** cooperate in region $\mathcal{B}$" ?

The answer is

> "why not ?"

We argue that this is an equilibrium strategy, because if player **1** does anything else, any how the state will leave $\mathcal{B}$ through R giving the same payoff to him. Of course, should player **2** deviate, he would loose (increase the time to termination).

Now, one can specify any strategy for player **1** inside that region, provided that it forbids crossing the competitive barrier. Then player **1**'s strategy that minimizes the time to reach R against that particular strategy of player two constitutes an equlibrium pair with it. An averse player **1** may even choose to play a zero-sum game against player **2** with time to reach R as the criterion, as we proposed for region $\mathcal{A}$. The only interest of the choice we make here is to display both a competitive barrier and a cooperative one in the same game.

(Concerning region $\mathcal{A}$, the choice we made of a maximin behaviour is a retaliation strategy needed to remove an incentive by player **2** to play $u_2 = -1$ on, and close to, the arc SP.)

Of course, this only stresses the very high lack of uniqueness of the Nash equilibrium concept.

6 Conclusion

Managing biological systems requires to model them. We have seen that many new problems have arisen in the process. We feel that the arguments in favor of a more thorough analysis of two-player two-pure-strategy games are strong, in view of the motivation provided by evolutionary game theory. Yet, the concept of evolutionary stability will have to be revisited for such games.

Actually, we have shown here more open problems than new results. Our feeling, though, is that these are not completly out of reach. This is a challenge to game theoretists and managers of biological systems alike.

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A Proofs of subsection 2.2.3

A.1 The counterexample

The strategy p is an ESS but is not trembling hand perfect

We have

$$Ap = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 1 & 1 \\ 1.5 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 10/3 \\ 10/3 \\ 10/3 \end{pmatrix}$$

Hence, $\mathrm{Supp}(p) \subset \mathrm{Argmax}(Ap)$ and p is indeed a Wardrop eqilibrium. It is furthermore an ESS since

$$\sigma^S(A) = \begin{pmatrix} -3 & 2.25 \\ 2.25 & -2 \end{pmatrix} < 0\,.$$

But as a Nash equilibrium, p is not trembling hand perfect. If it were, it would be possible to find ε_1 and ε_2 arbitrarily small, but nonzero, such that it would be a best response to the strategy $q = (2/3 - \varepsilon_1,\ 1/3 - \varepsilon_2,\ \varepsilon_1 + \varepsilon_2)$. This requires that

$$\frac{2}{3} - \varepsilon_1 + 3(\frac{1}{3} - \varepsilon_2) = 2(\frac{2}{3} - \varepsilon_1) + \frac{1}{3} - \varepsilon_2 + \varepsilon_1 + \varepsilon_2\,,$$

hence $\varepsilon_2 = 0$.

A.2 Proof of proposition 3

An ESS cannot weight positively a weakly dominated strategy

Assume to the contrary that the first trait is positively weighted by the ESS p, and weakly dominated by a trait not weighted by p, say trait m. The entries of line m in the game matrix A are thus larger or equal to the corresponding ones in line 1.

The entries of line m in columns in $\mathrm{Supp}(p)$ are necessarily *equal* to the corrsesponding ones in line 1, because if any was strictly larger, the m-th pure strategy would perform better, against p, than the first one, which itself performs as the ESS because of the equalization property. As a consequence, the coordinate m belongs to $\mathrm{Argmax}(Ap)$, and any strategy weighting the elements of $\mathrm{Supp}(p)$ and strategy m is a best response to p.

Consider thus the strategy r which differs from p only in the fact that, on the one hand $r_1 = 0$ and on the other hand $r_m = p_1$. It is a strategy, it belongs to $\mathbb{BR}(p)$. Now, against itself, it performs at least as well as p, in contradiction with the strict inequality in condition (9).

If necessary, a simple way to check that statement is the following calculation. Consider the restrictions of the strategies and of the game matrix to $\mathrm{Supp}(p) \cup \{m\}$. Write

$$A = \begin{pmatrix} a_{11} & \ell_1 & a_{1m} \\ c_1 & B & c_m \\ a_{11} & \ell_1 & a_{1m} \end{pmatrix}, \qquad p = \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix}, \qquad r = \begin{pmatrix} 0 \\ p_2 \\ p_1 \end{pmatrix}.$$

Only keep in mind that p_2 is —or may be— a (column) vector, ℓ_1 a line, and c_1 and c_m columns, all of the same dimension $|\mathrm{Supp}(p)| - 1$. Also, B is a square matrix of the same dimension. With these notations, one easily obtains

$$\langle p, Ar \rangle = p_1 \ell_1 p_2 + a_{1m} p_1^2 + \langle p_2, Bp_2 \rangle + \langle p_2 c_m \rangle p_1 \,,$$
$$\langle r, Ar \rangle = \langle p_2, Bp_2 \rangle + \langle p_2, c_m \rangle p_1 + p_1 \ell_1 p_2 + a_{mm} p_1^2 \,.$$

Hence $F(r, r) - F(p, r) = (a_{mm} - a_{1m}) p_1^2$ which is nonnegative if line m dominates line 1.

Assume now that p is mixed, and let $\{1 \;\; 2\} \subset \mathrm{Supp}(p)$. And assume that line 2 dominates weakly line 1. As previously, the first two colomns of lines 1 and 2 must coincide. Consider the strategy r obtained from p by transfering the weight of line 1, which is nonzero, on line 2. (Which is also what we have done, in fact, in the previous case.) It is in $\mathbb{BR}(p)$. But then, $r - p = (-p_1 \;\; p_1 \;\; 0 \ldots 0)$, and as a consequence $\langle r - p, Ar \rangle = 0$, contradicting the second ESS condition.