The robust control approach to option pricing and interval models: an overview

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Abstract

We give an overview of our work since 2000 on an alternate theory of option pricing and contingent claim hedging based upon the so-called "interval model" of security prices, which let us develop a consistent theory in discrete and continuous trading within the same model, taking transaction costs into account from the start. The interval model rules out crises on the stock market. While Samuelson's model does not, so does in practice Black and Scholes' theory in its assumption of instantaneous, continuous trading. Our theory does not make use of any probabilistic *knowledge* (or rather *assumption*) on market prices. But we show that Black and Scholes theory does not either.

1 Introduction

If a mathematical model can be termed "good" only relatively to a purpose, the classical Samuelson (geometric diffusion) model of stock prices on the market is not good for the purpose of deriving a hedging method and pricing rule for options in the presence of transaction costs (see [20]), nor in discrete trading. For that reason, new approaches have appeared in the work of various authors ([15, 16, 17, 1, 14, 18, 19]) all based upon the ideas of robust control, and most of them somehow assuming some bounds on how fast prices may evolve.

In these approaches, the critical part of the market model is the set Ω of admissible trajectories. In spite of the choice of name, aimed at recalling that there lie the uncertainties, in these approaches there is no need to endow this set with a probabilistic structure, thus greatly simplifying the model. The intuitive notion

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of "‘causal" strategy ("adapted" in the parlance of stochastic theories) will be rendered by the technical apparatus of *non-anticipative strategies*, classical in the theory of differential games and robust control. This lets one formalize the fact that future stock prices are not known by the trader. In some sense, we even give the trader less information on future prices than any stochastic theory, since we do not even let it know a probability law governing future prices.

We review here our own work, most of which uses the so called "interval model" of the market, independently introduced around 1999 by Aubin and Pujal, ourselves and Roorda, Engwerda, and Schumacher, the latter being the inventors of the name we use. This work has sometimes been criticized on the basis of the fact that since it does not make use of any probabilistic knowledge (or *assumptions*) on the market, it cannot capture the essence of the problem. Hence we review also here an earlier work of ours which shows that the celebrated Black and Scholes theory does not either.

On our way, and in particular in subsection 4.1 and in the conclusion, we provide some discussions of the relative merits and weaknesses of these two approaches.

2 Dynamics and hedging

2.1 Notations and Dynamics

Our notations will be as follows. All the problems we consider will be on a fixed time interval $[0,T]$. There exists a riskless interest rate ρ per unit time, and we shall use the end-value factor

$$R(t) = \mathrm{e}^{\rho(t-T)}$$

which can equivalently be thought of as the value of a riskless bound worth one at time T. All monetary values will be expressed in end-time value, so that the riskless rate will disappear from most of the theory.

We shall let $S(t)$ be the price at time t of a specified *underlying* security, and let

$$u(t) = \frac{S(t)}{R(t)}\,.$$

We shall consider a *hedging portfolio* containing an amount $v(t)$ (in end-time monetary value) of the underlying security, the rest being $y(t)$ riskless bounds, for a total worth (in end-time value) of $w(t) = v(t) + y(t)$.

Transaction amounts will be denoted ξ (see more details below). When transaction costs are present, we shall take them proportional to the amount of the transaction, the proportionality coefficient being C^+ for a buy of the underlying, and

$-C^-$ for a sale. We shall let ε be the sign of the transaction, so that its cost will be $C^\varepsilon\xi$. Closing transactions at exercize time T will bear costs at possibly different rates c^+ and c^-, of absolute value no more than their counterparts C^+ and C^- respectively.

2.1.1 Continuous trading

In the case of continuous trading, in all of our developments but one (section 3.1.2), $u(\cdot)$ will be absolutely continuous, and we shall let

$$\tau(t) = \frac{\dot u(t)}{u(t)}\,.$$

It is defined for almost all t and measurable, and $u(\cdot)$ is the unique (according to Gronwal's inequality) continuous solution of $u(t) = u(0) + \int_0^t \tau(s)u(s)\,\mathrm{d}s$. So that we may alternatively represent $u(\cdot)$ via $u(0)$ and $\tau(\cdot)$.

We shall let $\xi(t)$ be our rate (in end-time value) of trading, a buy if $\xi(t) > 0$, a sale if $\xi(t) < 0$. We shall also allow for impulsions in $\xi(\cdot)$ at a finite number of time instants t_k, of amplitude ξ_k, all chosen by the trader as part of its control.

We summarize the dynamics as

$$\begin{aligned} \dot u &= \tau u\,, && (1)\\ \dot v &= \tau v + \xi\,, && (2)\\ v(t_k^+) &= v(t_k) + \xi_k\,, && (3)\\ \dot w &= \tau v - C^\varepsilon \xi\,, && (4)\\ w(t_k^+) &= w(t_k) - C^{\varepsilon_k}\xi_k\,. && (5) \end{aligned}$$

We shall use the fact that the last two equations integrate explicitly as

$$w(t) = w(0) + \int_0^t (\tau(s)v(s) - C^\varepsilon \xi(s))\,\mathrm{d}s - \sum_{k|t_k<t} C^{\varepsilon_k}\xi_k\,. \qquad (6)$$

2.1.2 Discrete trading

In the case of discrete trading, a time step h is fixed. The trader is constrained to use impulses only, at fixed time instants $t_k = kh$. We write $u(kh) = u(t_k) = u_k$, and similarly for v and w. Let also τ_k be the relative variation of u during the time step $[t_k, t_{k+1}]$:

$$\tau_k = \frac{u_{k+1} - u_k}{u_k}\,,$$

so that again, u_0 and the sequence $\{\tau_k\}$ together define the sequence $\{u_k\}$. Notice however that a non anticipative strategy must make use of strictly past τ_j's to determine ξ_k ($j < k$).

The dynamics are now

$$
\begin{aligned}
u_{k+1} &= (1+\tau_k)u_k\,, && (7)\\
v_{k+1} &= (1+\tau_k)(v_k+\xi_k)\,, && (8)\\
w_{k+1} &= w_k+\tau_k(v_k+\xi_k)-C^\varepsilon\xi_k\,, && (9)
\end{aligned}
$$

Again, the last equation integrates as

$$
w_k = w_0 + \sum_{\ell=0}^{k-1}[\tau_\ell(v_\ell+\xi_\ell)-C^\varepsilon\xi_\ell]\,. \qquad (10)
$$

2.2 Hedging portfolio

2.2.1 Closure

In most of the paper, except subsection 5.4.1, we consider European claims, valued at exercize time T. Let $M(u(T))$ be the amount due by the writer to the buyer according to the contingent claim. It is known to be $M(u) = \max\{0, u-K\}$ for a Call of strike K, and $M(u) = \max\{K-u, 0\}$ for a Put. But other claims may be considered, such as a digital call worth $M(u) = \Upsilon(u-K)$, the Heaviside echelon. Notice however that convexity of M on the one hand, continuity on the other, do make a difference, so that digital calls or puts yield a more complex theory that we shall only allude to here.

The total expense incurred by the writer may be different from $M(u(T))$ due to transaction costs, themselves function of whether the final transaction is in cash or in kind. We discuss here the case of a call, and we consider only positive v's. (We would consider negative v's for a put.)

The auxiliary functions $\breve{v}(u)$ and $\breve{w}(u)$ that we introduce now will serve in subsection 4.3 as the final values $\breve{v}(T,u)$ and $\breve{w}(T,u)$ of the functions $\breve{v}(t,u)$ and $\breve{w}(t,u)$.

Closure in cash We give some details for a call option. The situation for a put is entirely similar. In the case of a closure in cash, the trader will have to trade in all of its underlying stocks, resulting in an added cost of $-c^-v(T)$. (Recall that $c^- \le 0$.) Hence its total closure costs will be $N(u(T), v(T))$ with

$$
N(u,v) = M(u) - c^-v\,.
$$

It will be useful to write this in terms of two auxiliary functions $\check{v}(u)$ and $\check{w}(u)$, according to the following tables

Call	$u \le K$	$u \ge K$
$\check{v}(u)$	0	$\frac{u}{1+c^-}$
$\check{w}(u)$	0	$\frac{u}{1+c^-} - K$

$N(u,v) = \check{w}(u) + c^-(\check{v}(u) - v)$

Put	$u \le K$	$u \ge K$
$\check{v}(u)$	$-\frac{u}{1+c^+}$	0
$\check{w}(u)$	$K - \frac{u}{1+c^+}$	0

$N(u,v) = \check{w}(u) + c^+(\check{v}(u) - v)$

(11)

Closure in kind Again we discuss the case of a call. In the case of a closure in kind, if the buyer exerts the option, the trader will have to bring its underlying content in its portfolio to $v(T) = u(T)$ before giving one share of it to the buyer in exchange for K in cash. Let $\eta := \text{sign}(u - v)$. Hence we now get

$$N(u,v) = \max\{u(T) - K + c^\eta(u - v)\,, -c^- v\}\,.$$

A simple analysis shows that this can be described as follows. Let

Call	$u \le \frac{K}{1+c^+}$	$\frac{K}{1+c^+} \le u \le \frac{K}{1+c^-}$	$\frac{K}{1+c^-} \le u$
$\check{v}(u)$	0	$\frac{(1+c^+)u - K}{c^+ - c^-}$	u
$\check{w}(u)$	0	$-c^- \check{v}(u)$	$u - K$

(12)

Put	$u \le \frac{K}{1+c^+}$	$\frac{K}{1+c^+} \le u \le \frac{K}{1+c^-}$	$\frac{K}{1+c^-} \le u$
$\check{v}(u)$	$-u$	$\frac{K - (1+c^-)u}{c^- - c^+}$	0
$\check{w}(u)$	$K - u$	$-c^+ \check{v}(u)$	0

(12)

N is now given in both cases by the unique formula

$$N(u,v) = \check{w}(u) + c^\varepsilon(\check{v}(u) - v)\,, \quad \varepsilon = \text{sign}(\check{v}(u) - v)\,. \tag{13}$$

2.2.2 Non-anticipative strategies

Let $\omega \in \Omega$ represent a realization of an uncertain time function, (say the price history of a security), $\omega(t)$ be the information available to the trader from time t on, and $\xi(t)$ be the trader's decision at time t (a transaction level) with $\xi(\cdot) \in \Xi$. A *non-anticipative strategy* is an application from Ω to Ξ such that, if the restrictions of ω_1 and of ω_2 to $[0,t]$ coincide, so do those of $\varphi(\omega_1)$ and of $\varphi(\omega_2)$. In discrete time, we may distinguish strictly non anticipative strategies for which the condition is that $\omega_1(s)$ and $\omega_2(s)$ coincide for $s < t$. In continuous time, where we shall allow Dirac impulses in $\xi(\cdot)$, we must specify that if it contains an impulse at time t, then the impulse is present in the restriction of $\xi(\cdot)$ to $[0, t]$.

Let Ψ be the set of admissible relative rate of change histories $\tau(\cdot)$. We can as well decide that the admissible strategies are non-anticipative maps from $\mathbb{R}^+ \times \Psi$ into Ξ, but now, we must request that if the restrictions of $\tau_1(\cdot)$ and $\tau_2(\cdot)$ to the half-open set $[0, t)$ coincide, then $\varphi(u_0, \tau_1(\cdot))(t) = \varphi(u_0, \tau_2(\cdot))(t)$. This is the right definition both in discrete time, because then $\tau(t)$ is an information on $u(t+h)$, and in continuous time because $\tau(\cdot)$ may be discontinuous, and it can be checked that allowing a strategy $\xi(t) = \varphi(\tau(t))$ is not only not feasible in practice, it is also no longer a non anticipative strategy in $u(\cdot)$. (It does no longer forbid arbitrages.)

We shall call Φ the set of admissible (non-anticipative) strategies, the context will decide whether from Ω into Ξ or from $\mathbb{R}^+ \times \Psi$ into Ξ.

2.2.3 Hedging

The aim of a hedging portfolio is to insure that, for all admissible price trajectories,

$$w(T) \geq N(u(T), v(T)) \,. \tag{14}$$

Equivalently, this can be written as

$$\sup_{u(\cdot)\in\Omega} [N(u(T), v(T)) - w(T)] \leq 0 \,,$$

We reformulate this using (6). Use the representation $(u(0), \tau(\cdot))$ of $u(\cdot)$, and fix $u(0)$. We get

$$\sup_{\tau(\cdot)\in\Psi} \left[N(u(T), v(T)) + \int_0^T (-\tau(s)v(s) + C^\varepsilon \xi(s)) \, \mathrm{d}s + \sum_k C^{\varepsilon_k} \xi_k \right] \leq w(0) \,,$$

or equivalently with the discrete formula (10). We wish $w(0)$ to be as small as possible, so we are lead to the investigation of

$$P(u_0) = \inf_{\varphi\in\Phi} \sup_{\tau\in\Psi} \left[N(u(T), v(T)) + \int_0^T (-\tau(s)v(s) + C^\varepsilon \xi(s)) \, \mathrm{d}s + \sum_k C^{\varepsilon_k} \xi_k \right]. \tag{15}$$

As a matter of fact, the left hand side still depends on $u(0)$. We shall therefore get a price (or *premium*) $w(0) = P(u_0)$, function of $u(0) = u_0$.

In the case of discrete trading, the equivalent formula is (with $Kh = T$)

$$P(u_0) = \min_{\varphi\in\Phi} \sup_{\tau\in\Psi} \left[N(u(T), v(T)) + \sum_{k=0}^{K-1} [-\tau_k(v_k + \xi_k) + C^{\varepsilon_k} \xi_k] \right] \tag{16}$$

3 Continuous trading, no transaction costs

This section is based upon [3, 5].

3.1 A simple theory

The framework here is that of the classical theory. We have $C^+ = C^- = 0$, and closure expenses are just $M(u(T))$. In this case, there is no expense in moving money from stocks (i.e. v) to bounds (y), and we may use v as our control and disregard ξ.

In this framework, we achieve (14) via the classical device of *replication* for two different models Ω. We find a portfolio and a trading strategy insuring

$$\forall u(\cdot) \in \Omega\,, \quad w(T) = M(u(T)).$$

A simple way of doing this is to exhibit a function $W(t,u)$ such that

$$\forall u \in \mathbb{R}^+\,, \quad W(T,u) = M(u)\,, \tag{17}$$

together with a strategy $v = \varphi(t,u)$ such that if one actually implements that strategy with an initial portfolio worth $w(0) = W(0,u(0))$, it results in

$$\forall \tau(\cdot) \in \Psi\,,\ \forall t \in [0,T]\,, \quad w(t) = W(t,u(t))\,.$$

The way to obtain this is to choose φ in such a way that

$$\forall t \in [0,T]\,,\ \forall u \in \mathbb{R}^+\,, \quad \dot{w}(t) = \frac{\mathrm{d}W(t,u(t))}{\mathrm{d}t}\,.$$

3.1.1 Prices with bounded total variation

Let us choose for Ω continuous functions with bounded total variation. We use the notations of the Stieltjes calculus to get

$$\forall u \in \mathbb{R}^+\,,\ \tau v\,\mathrm{d}t = \frac{\partial W}{\partial t}\,\mathrm{d}t + \frac{\partial W}{\partial u}\tau u\,\mathrm{d}t$$

which is achieved choosing

$$\frac{\partial W}{\partial t} = 0\,, \quad v = u\frac{\partial W}{\partial u}\,.$$

Hence, $W(t,u) = W(T,u) = M(u)$.

For a simple call, this means that $v = 0$ if $u < K$, $v = 1$ if $u > K$. This is the so called "stop loss" strategy. In that model, the premium is just the parity value $P(u) = M(u)$. (Recall that interest rates have been factored out via the use of end-time values. In current value, this gives $P(u(0)) = \exp(-\rho T)M(\exp(\rho T)u)$).

3.1.2 Prices with fixed relative quadratic total variation

Let σ be a fixed positive number. We now choose for Ω the set Ω_σ of continuous functions such that

$$\forall t \in [0,T]\,, \quad \lim_{h\to 0} \sum_{k=0}^{t/h-1} \left(\frac{u((k+1)h)-u(kh)}{u(kh)}\right)^2 = \sigma^2 t\,.$$

(It should be noticed that the trajectories of the classical Samuelson geometric diffusion model belong almost surely to that set.) These functions have infinite total variation and are nowhere differentiable. But we make use of the following simple form (independently proved in [5]) of a lemma of Föllmer :

Lemma 1 *Let a twice continuously differentiable function $V(t,u) : [0,T]\times\mathbb{R}^+ \to \mathbb{R}$ and $u \in \Omega_\sigma$ be such that $\forall t \in [0,T]$, it holds that $(\partial V/\partial u)(t,u(t)) = 0$, then $t \mapsto V(t,u(t))$ is differentiable and satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,u(t)) = \frac{\partial V}{\partial t}(t,u(t)) + \frac{\sigma^2}{2}u(t)^2\frac{\partial^2 V}{\partial u^2}(t,u(t))\,.$$

We set $v = xu$, so that $w = xu + y$, and let $V(t,u) = W(t,u) - x(t)u - y(t)$. We insure the condition $\partial V/\partial u = 0$ by choosing $x(t) = (\partial W/\partial u)(t,u(t))$, and use the fact that $\mathrm{d}w = x\,\mathrm{d}u$ thus $\mathrm{d}x\,u + \mathrm{d}y = 0$ (the *self-financing condition*) to obtain that the condition $\mathrm{d}V/\mathrm{d}t = 0$ is equivalent to

$$\frac{\partial W}{\partial t} + \frac{\sigma^2}{2}u^2\frac{\partial^2 W}{\partial u^2} = 0\,.$$

Thus, if W satisfies the above equation and the boundary condition (17), which together form the celebrated "Black and Scholes equation" [10], the strategy $v(t) = u(t)(\partial W/\partial u)(t,u(t))$ insures that the hedging condition (14) be met.

This yields a "light" Black and Scholes theory (see [5] for more details), and mainly serves the purpose of showing that the said theory does not really make use of any probabilistic assumption on the price trajectories, but only on their regularity. Incidentally, it also answers the question "why is the drift absent from the solution?". Our answer is "because it is absent from the problem statement".

4 Interval model: continuous trading

This section is based upon [8, 9].

4.1 The interval model

We now introduce the market model that we shall use from now on, in both the continuous trading and the discrete trading theories, which shall merge in a single one.

Let two numbers $\tau^- < 0$ and $\tau^+ > 0$ be given. We choose Ω as follows:

$$\Omega = \{u(\cdot) \quad \text{absolutely continuous} \mid \\ \forall t_1, t_2 \in [0,T]\,,\ \mathrm{e}^{\tau^-(t_2-t_1)} \leq \frac{u(t_2)}{u(t_1)} \leq \mathrm{e}^{\tau^+(t_2-t_1)}\}\,. \tag{18}$$

An equivalent characterization we shall use in the continuous trading theory is (1) together with the condition $\tau \in [\tau^-, \tau^+]$.

Hence we assume that the relative rate of variation of the underlying's price is bounded by a priori bounds. This is a weakness of our theory, since fast variations of the prices are ruled out from the start by the very model we work with, while we know that they do happen in real life.

The geometric diffusion model used by the theory of Black and Scholes does not have that drawback. But it cannot be used to derive a theory of hedging with proportional transaction costs (see [20]) because its trajectories are of unbounded total variation, yielding infinite transaction costs, nor in discrete time, whether with or without transaction costs, because the possible price variation in any finite time interval is unbounded. The fact that the continuous trajectory be of unbounded total variation may be a desirable feature. Anyhow, it is difficult to avoid within a stochastic theory, as it is a consequence of the fact that one needs a process with independent increments to avoid any arbitrage opportunity. The framework of robust control and non-anticipative strategies avoids that problem since we do not need to endow the set of trajectories with a probability which the trader might use to devise an arbitrage.

It should be further emphasized that Black and Scholes'theory suffers its own shortcomings in that it requires a continuous trading with no delay in information use, an unrealistic portfolio model. This is of little consequence as long as the prices do not change too rapidly, but becomes a fundamental limitation of the usefulness of the theory as a guide to managing a hedging portfolio when these sudden changes do happen. Hence in the situations where our market model is violated, so is Black and Scholes'portfolio model.

4.2 Isaacs'equation

We therefore have the following problem to solve. The dynamics are now (1)(2)(3) with $\tau(\cdot) \in \Psi = \{\text{measurable functions } [0,T] \to [\tau^-, \tau^+]\}$, $\xi(\cdot) \in \Xi$ where it is understood that $\xi(\cdot)$ contains the impulses that cause the jumps (3), that is, Ξ is the set of all sums of measurable functions from $[0, T]$ into $\mathbb{R}$ and of finitely many weighted translated Dirac impulses $\xi_k \delta(t - t_k)$.

The problem is to find, if it exists, the non-anticipative strategy $\varphi^* \in \Phi$ that provides the minimum in (15).

This is a non classical differential game in that it features an impulse control, and the corresponding Isaacs quasi variational inequality is further degenerated, as compared to the (control) theory in [2], because the cost of jumps has a zero infimum. Yet, we may take advantage of that last fact to transform that game into a classical one in an artificial time (see Joshua's transformation in [8]), leading to a differential form of the quasi-variational inequality of the impulse control game, and to the following theorem.

Theorem 1 *The Value function $W(t, u, v)$ of the above game is a viscosity solution of the following differential quasi variational inequality:*

$$
\begin{aligned}
0 = \min\Bigg\{ & \frac{\partial W}{\partial t} + \max_{\tau \in [\tau^-, \tau^+]} \tau \left[\frac{\partial W}{\partial u} u + \left(\frac{\partial W}{\partial v} - 1 \right) v \right], \\
& \qquad\qquad\qquad \frac{\partial W}{\partial v} + C^+ , \ -\left(\frac{\partial W}{\partial v} + C^- \right) \Bigg\} , \qquad (19) \\
W(T, u, v) &= N(u, v) .
\end{aligned}
$$

This function is further characterized by its rates of growth at infinity in u (one) and v ($-C^+$ at $-\infty$ and $-C^-$ at $+\infty$). Yet, a direct uniqueness proof derived from the theory of viscosity solutions of Isaacs equation is still missing. We rely instead on the fact that the solution we shall exhibit with the representation theorem below is sufficiently regular for the Isaacs-Breakwell theory to apply, the viscosity condition being the modern form of Breakwell's "non leaking corners" (or our corner conditions). (See [4, 11].)

In our case, the function N is convex. The following is a consequence of the convergence theorem of the next section (we do not know a direct proof):

Theorem 2 *When the function N is convex, the solution of (19) is convex in (u, v) for all t.*

This is an interesting property in itself. In addition, we discuss in [8] how this saves much time in the computations of a discretized scheme for numerically solving that equation. Yet the real breakthrough in the numerical computation of that function is given by the following subsection, and the corresponding numerical algorithm that we shall show in the next section.

4.3 A representation theorem

We need to introduce the following notations. Let, for a closure in kind

$$\begin{aligned} q^-(t) &= \max\{(1+c^-)\exp[\tau^-(T-t)]-1\,,\ C^-\}\,,\\ q^+(t) &= \ \min\{(1+c^+)\exp[\tau^+(T-t)]-1\,,\ C^+\}\,. \end{aligned} \tag{20}$$

(For a closure in cash, both q^+ and q^- are constructed with c^- for a call and c^+ for a put instead of c^ε above). We shall refer to both at a time as q^ε, where $\varepsilon = \pm$ is usually the sign of $\check{v} - v$ (see below). Notice that $q^\varepsilon = C^\varepsilon$ for $t \le t_\varepsilon = T - (1/\tau^\varepsilon)\ln[(1+C^\varepsilon)/(1+c^\varepsilon)]$, and increases (for q^+) or decreases (for q^-) towards c^ε as $t \to T$. For any realistic data, t_+ and t_- are very close to T, say one day or less.

Let also

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\,, \text{ and } \mathcal{T} = \frac{1}{q^+ - q^-}\begin{pmatrix} \tau^+q^+ - \tau^-q^- & \tau^+ - \tau^- \\ -(\tau^+ - \tau^-)q^+q^- & \tau^-q^+ - \tau^+q^- \end{pmatrix}.$$

We introduce a pair of functions of two variables $\check{v}(t,u)$ and $\check{w}(t,u)$ collectively called

$$\mathcal{V}(t,u) = \begin{pmatrix} \check{v}(t,u) \\ \check{w}(t,u) \end{pmatrix}$$

and defined by the final conditions (11) or (12), depending on which applies, and the following linear P.D.E.:

$$\frac{\partial \mathcal{V}}{\partial t} + \mathcal{T}\left(\frac{\partial \mathcal{V}}{\partial u}u - \mathcal{S}\mathcal{V}\right) = 0\,. \tag{21}$$

It may be noticed that $\check{v}(t,u)$ and $\check{w}(t,u)$ keep the same form as in (11) for $u \notin [K\exp(\tau^+(t-T)), K\exp(\tau^-(t-T))]$ for a closure in cash, or (12) and $u \notin [(K/1{+}c^+)\exp(\tau^+(t-T)), (K/1{+}c^-)\exp(\tau^-(t-T))]$ for a closure in kind.

We prove in [9] the following representation theorem:

Theorem 3 *The Value function of the game is given by the formula*

$$W(t,u,v) = \check{w}(t,u) + q^\varepsilon(\check{v}(t,u) - v)\,, \quad \varepsilon = \operatorname{sign}(\check{v}(t,u) - v)\,. \tag{22}$$

This formula gives a useful hindsight into the shape of the value function and the role of the transaction costs. It also is the basis of a fast algorithm to approximate W, that we shall see in the next section. The main point is that we have to compute for each time instant two functions of one variable instead of one function of two variables, a considerable savings if each variable is discretized in a few hundreds to a few thousand points. Moreover, $\check{v}$ plays the role of an "optimal portfolio composition" as the next result shows:

Theorem 4 *The optimal trading strategy is to jump at initial time to* $v = \check{v}(0, u(0))$ *and stay at* $v(t) = \check{v}(t, u(t))$ *while* $t < t_\varepsilon$*, do nothing if* $t \geq t_\varepsilon$*.*

There, however, lies a difficulty. Staying at $v(t) = \check{v}(t, u(t))$ is not possible within our assumptions, because one easily sees that this would entail a strategy of the form $\xi(t) = \varphi(t, u(t), v(t), \tau(t))$. But we have emphasized the fact that this is *not* an admissible, non-anticipative, strategy. The solution of that dilemma is provided by the convergence result of the next section which shows that the optimal value in the game above can be approximated arbitrarily well with a non-anticipative (discrete) trading strategy. (The minimum is actually *not* reached in (15)).

5 Interval model: discrete trading

This section is based upon the same references [8, 9] as the previous one.

5.1 The model

We turn to the more realistic model where trading is restricted to happen at discrete time instants. Let $h > 0$ be our time step. The trader is restricted to jumps of the form (3) at predetermined time instants, multiples of h. Let therefore $t_k = kh$, $k \in \mathbb{N}$, $u(t_k) = u_k$ and $v(t_k) = v_k$. The market model Ω translates into

$$\mathrm{e}^{\tau^- h} - 1 \leq \frac{u_{k+1} - u_k}{u_k} \leq \mathrm{e}^{\tau^+ h} - 1.$$

We shall let $\tau_h^\varepsilon := \exp(\tau^\varepsilon h) - 1$. It should be noted that τ_h^ε converges to 0 as h (it is equivalent to $\tau^\varepsilon h$), instead of as $\sqrt{h}$ in the limiting Cox Ross and Rubinstein theory. This means that we do keep a single market model Ω, while the step size is decreased towards zero, while the classical (and remarkable) limiting process of the Cox Ross and Rubinstein theory towards the Black and Scholes theory entails a continuous change of model.

With these notations, our model is now (7)(8) with $\tau_k \in [\tau_h^-, \tau_h^+]$, and the non-anticipative strategies are simply of the form $\xi_k = \varphi_k(u_k, u_{k-1}, \ldots)$. Equivalently, we shall find the optimal strategy in the form $\xi_k = \varphi_k(u_k, v_k)$. And our problem is to find the minimum, together with the minimizing strategy φ^*, in (16).

5.2 Isaacs'equation

This problem is now a classical multistage dynamic game, whose value function W^h is given by its Isaacs equation (we again use subscripts for the number of the stage, and $Kh = T$):

$$W_k^h(u,v) = \min_\xi \max_{\tau\in[\tau_h^-,\tau_h^+]} \Big[W_{k+1}^h\Big((1+\tau)u,(1+\tau)(v+\xi)\Big) - \tau(v+\xi) + C^\varepsilon\xi\Big]$$
$$W_K^h(u,v) = N(u,v)\,. \tag{23}$$

It turns out to be useful to notice the following "fractional step" form of the first equation :

$$W_{k+\frac{1}{2}}^h(u,v) = \max_{\tau\in[\tau_h^-,\tau_h^+]} \left[W_{k+1}^h\Big((1+\tau)u,(1+\tau)v\Big) - \tau v\right]\,,$$
$$W_k^h(u,v) = \min_\xi \left[W_{k+\frac{1}{2}}^h(u,v+\xi) + C^\varepsilon\xi\right]\,.$$

This form lets one easily show the following theorem:

Theorem 5 *If the function $v \mapsto N(u,v)$ is convex for all u, so is the function $v \mapsto W_k^h(u,v)$ for all (k,u). If the function $(u,v) \mapsto N(u,v)$ is convex, so is the function $(u,v) \mapsto W_k^h(u,v)$ for all k.*

This theorem in turn is useful to accelerate a numerical algorithm to evaluate the sequence $\{W_k^h\}_k$ and the optimal trading strategy using Isaacs'equation. As a matter off fact, then $\tau \mapsto W_{k+1}((1+\tau)u,(1+\tau)v)$ is convex, hence its max is reached at an end of the interval $[\tau_h^-,\tau_h^+]$. And the minimization in ξ can also benefit from the convexity. See [8]. We do not stress much that fact here because the representation theorem below provides a much faster algorithm when it holds. But an important consequence of the remark concerning the maximum in τ is as follows.

Proposition 1 *For a convex claim $M(u)$, the above theory with no transaction costs coincide with the Cox, Ross, and Rubinstein theory [13].*

A consequence of that proposition is that, for small transaction costs and small time step, for reasonable values of τ_h^- and τ_h^+, the pricing curve given by our theory will resemble that of Black and Scholes.

The main theorem of our discrete trading theory is the following. Define $W^h(t,u,v)$ as being the linear interpolation in time of the sequence $\{W_k^h(u,v)\}$.

Theorem 6 *The function $W^h(t,u,v)$ converges uniformly on any compact to the function $W(t,u,v)$ (value of the continuous trading problem) when h tends to zero as $h = Tn^{-d}$, $n,d \in \mathbb{N}$, $d \to \infty$.*

As a matter of fact, one easily sees first that the W_k^h are non-negative, and using classical ideas of dynamical games, we see that W^h decreases as d above increases.

It therefore has a monotoneous limit. This limit is then shown to be a sufficiently regular viscosity solution of (19) using basically the method of [12], but with many technical details to adapt it to our problem.

The consequence of that theorem is that one can approximate arbitrarily well the value of the continuous trading portfolio with discrete trading, if that trading happens often enough. This is a very desirable feature of any continuous trading theory, since trading has to be discrete in practice, yet it is not enjoyed by the Black and Scholes theory.

5.3 Representation theorem and fast algorithm

Introduce the following recursions. Let

$$q_K^\varepsilon = c^\varepsilon$$

for a closure in kind, or

$$q_K^\varepsilon = c^- \text{ for a Call and } q_K^\varepsilon = c^+ \text{ for a Put}$$

for a closure in cash, and

$$\begin{aligned} q^\varepsilon_{k+\frac{1}{2}} &= (1+\tau_h^\varepsilon)q^\varepsilon_{k+1} + \tau_h^\varepsilon\,, \\ q_k^\varepsilon &= \varepsilon\min\{\varepsilon q^\varepsilon_{k+\frac{1}{2}}\,,\,\varepsilon C^\varepsilon\}\,, \end{aligned} \tag{24}$$

and let, for every integer ℓ:

$$Q_\ell^\varepsilon = \begin{pmatrix} q_\ell^\varepsilon & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{V}_\ell^h(u) = \begin{pmatrix} \check{v}_\ell^h(u) \\ \check{w}_\ell^h(u) \end{pmatrix}. \tag{25}$$

(Notice that then $q_k^\varepsilon = q^\varepsilon(kh)$ as given by (20).) Take $\check{v}_K^h(u) = \check{v}(u)$, $\check{w}_K^h(u) = \check{w}(u)$ as given by (11) or (12) according to which applies, and

$$\mathcal{V}_k^h(u) = \frac{1}{q^+_{k+\frac{1}{2}} - q^-_{k+\frac{1}{2}}}\begin{pmatrix} 1 & -1 \\ -q^-_{k+\frac{1}{2}} & q^+_{k+\frac{1}{2}} \end{pmatrix}\begin{pmatrix} Q^+_{k+1}\mathcal{V}^h_{k+1}((1+\tau_h^+)u) \\ Q^-_{k+1}\mathcal{V}^h_{k+1}((1+\tau_h^-)u) \end{pmatrix}. \tag{26}$$

It can be checked that this is a consistent finite difference scheme for (21).

We claim (proof to appear in [9]):

Theorem 7 *The solution of (23) is given by (24),(25),(26), and (11) or (12), as*

$$W_k^h(u,v) = \check{w}_k^h(u) + q_k^\varepsilon(\check{v}_k^h(u) - v) = Q_k^\varepsilon\mathcal{V}_k^h(u) - q_k^\varepsilon v, \quad \varepsilon = \operatorname{sign}(\check{v}_k^h(u) - v).$$

This theorem is the basis for a fast algorithm to compute the sequence $\{W_k^h\}_k$ and the corresponding minimizing non-anticipative strategy φ^*. And in view of the convergence theorem, it is also an algorithm to approximate the continuous trading limit.

5.4 Extensions

Some extensions of that theory are natural and simple. (Some are difficult such as the case of the digital options which are neither convex nor continuous. We shall report on that case elsewhere.) Let us just show two such extensions.

5.4.1 American options

Dealing with American options requires that one re-introduces the riskless interest rate ρ. Let

$$\widehat{N}(t,u,v) = \mathrm{e}^{\rho(t-T)} N\left(\mathrm{e}^{\rho(T-t)}u, \mathrm{e}^{\rho(T-t)}v\right) ,$$

and as usual $\widehat{N}_k^h(u,v) = \widehat{N}(kh,u,v)$. This is the cost to the writer if the buyer exerts the option at time $t < T$. The minimax control problem is now one with stopping time, since the buyer may stop the problem any time. The criterion (16) should therefore be replaced by

$$P(u_0) = \inf_{\varphi\in\Phi} \sup_{\tau\in\Psi} \sup_{\ell<K} \left[N_\ell(u_\ell, v_\ell) + \sum_{k=0}^{\ell-1} [-\tau_k(v_k+\xi_k) + C^{\varepsilon_k}\xi_k] \right] . \tag{27}$$

It is a classical fact that Isaacs equation is now replaced by a quasi variational inequality

$$\begin{aligned} W_k^h(u,v) &= \max\Big\{ \widehat{N}_k^h(u_k,v_k) , \\ &\qquad \min_{\xi} \max_{\tau\in[\tau_h^-,\tau_h^+]} \Big[W_{k+1}^h\Big((1+\tau)u, (1+\tau)(v+\xi)\Big) - \tau(v+\xi) + C^\varepsilon\xi \Big]\Big\}, \\ W_K^h(u,v) &= N(u,v) . \end{aligned}$$

Numerically implementing that equation requires to add single line of code in the implementation of (23). As a matter of fact, one computes the $\min_\xi \max_\tau$ exactly in the same fashion, and upon writing it in the table holding W_k^h, compare with $\widehat{N}_k^h$ and keep whichever is larger for W_k^h. And because the maximum of convex functions is convex, we preserve the convexity of W_k^h, and hence the fast computation we derived from it.

We have done very little work with that theory, and do not have a representation theorem comparable to the above one. Whether there is one is an open question. What we did check is the following, which is as in the stochastic theory:

Theorem 8 *It is never advantageous for the buyer to exert an American call before exercize time.*

5.4.2 Strictly causal strategies

If the trading instants are to be very closely spaced (small h), it may be more realistic to allow the trader to use only u_{k-1} to choose ξ_k. This strictly causal strategy is the equivalent of a predictable strategy in a stochastic framework, as opposed to measurable (only). This is easily done via the following device. Decide that $\xi_k = \varphi(u_k, v_k)$ can be applied only at time $k+1$. (Hence it is the ξ_{k+1} of the previous theory.) Now, this is achieved by changing our dynamic model into

$$v_{k+1} = (1+\tau)v_k + \xi_k$$

instead of (8).

Isaacs'equation is changed accordingly. However, it does not easily split into two equations as was done here, hence even the convexity of the resulting value function is not as easy to prove. Again, this has not been investigated so far.

6 Conclusion

The approach of robust control together with the interval model for the market yields a rather comprehensive theory of option pricing. This model has its drawbacks. But it lets us build a consistent theory of discrete and continuous trading option hedging with transaction costs, a feat that the classical stochastic approach with Samuelson's market model can not achieve. Moreover, while the main weakness of the model is in the market model, it should be noted that this model is violated under the same circumstances that cause the portfolio model of Black and Scholes theory to fail.

This is an incomplete market model, which is a serious drawback, because we must therefore resort to super replication instead of exact replication. Whether this leads to unacceptably high prices depends on the choice of interval $[\tau_h^-, \tau_h^+]$, and there is therefore a trade off to be made between the realism of the market model and the price to pay for that undesirable feature. Yet, the difference in pricing with the Black and Scholes theory, in the case of a convex claim, is not very large, as the comparison with the theory of Cox Ross and Rubinstein shows. (And since our pricing is obviously a continuous function of the transaction costs, which are ignored in that comparison.)

A rather unexpected representation theorem yields a fast algorithm to numerically implement the theory, thus approaching in that respect the great simplicity and elegance of Black and Scholes theory.

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