On the Singularities of an Impulsive Differential Game Arising in Mathematical Finance

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Abstract

We investigate an impulse control differential game arising in a problem of option pricing in mathematical finance. In a previous paper, it was shown that its Value function in $\mathbb{R}^3$ could be described as a pair of functions affine in one of the variables, joined on a 2D manifold. Depending on the regions of the state space, this manifold is either a dispersal one, an equivocal one or a 2D focal manifold. A pair of PDE's were derived for the focal part. Here we show that irrespective of the nature of this manifold, it has to satisfy this same set of PDE's.

1 Introduction

In [2], we introduced a model of option pricing with transaction costs amenable to a robust control type of analysis. This led us to investigate an impulse control two-person zero-sum differential game. This game has a 3 D (2 D plus time) state space.

In [3], we showed that the Value function of that game must satisfy, in the viscosity sense, an equation which shares the feautures of an Isaacs equation and of an Isaacs quasi-variational inequality [1], (not surprisingly concerning an impulse control problem), which we called a differential quasi-variational inequality, or DQVI. In that paper, we solved that DQVI via a classical, —though non trivial—, approach of constructing a field of optimal trajectories and uncovering the singular manifolds of the game. We used the classical augmented state space where the payoff is made purely terminal, and applied the tools of semipermeability. In that

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process, impulses are seen as producing “jump trajectories” orthogonal to the time axis, but otherwise ordinary.

The result of that analysis is that the game exhibits, in retrogressive order, a dispersal manifold, an equivocal manifold (both according to the terminology of Isaacs [5]) and a focal manifold (according to the terminology of Breakwell and Merz, [4, 6]). It is worth mentioning that this last one is, to the best of our knowledge, the first 2D focal manifold explicitly constructed, and was the starting point of a joint work with Arik Melikyan reported in [7] where a simpler and more general derivation can be found. These three manifolds join smoothly in one differentiable manifold. On each side of this composite manifold, the Value function was found to be affine in the state variable y.

In [3], we derived closed form formulas for the dispersal manifold, a system of ordinary differential equations for the equivocal manifold, and a pair of coupled linear PDE’s for the focal manifold, which we integrated numerically with a second order finite difference scheme. The same paper presents a discrete time version of the problem, and a convergence theorem of the solution of this discrete time game towards that of the continuous time game when the step size vanishes. Therefore, the discrete time game provides a numerical algorithm to directly approach the solution of the DQVI. Thus we were able to compare the two approaches, and they coincide, as should be, with an amazing precision given the extremely different computational procedures. (The second one ignores all the theoretical analysis, singular manifolds and piecewise affine Value function.)

Here comes the new fact, though : it occured to us afterward that the dispersal manifold and the equivocal one also satisfy the same pair of linear PDE’s, derived for the focal manifold using arguments that do not hold for these other two manifolds. Yet, having their analytic expression or differential equations for trajectories traversing them allows one to prove that they satisfy that same PDE, as we shall show here.

The aim of this paper is to explain that coincidence.

We show that *if the Value function is a picewise affine function in* y joining on a smooth manifold, then this manifold necessarily satisfies that pair of linear PDE’s, that thus turns out to deserve the name of fundamental equation of the problem. (It is also independant of the terminal conditions on the DQVI, which characterize the type of option considered.) Of course, we would like to prove directly that indeed the Value function has to have that piecwise affine form. This is still an open problem.

In the second section below we recall the general framework and results from [3]. In the third section we show the theorem claimed in this introduction. The concluding section gives some insight into the use of that fundamental PDE to solve the DQVI of other types of options.

2 The framework

We essentially use the same notations as in [3]

2.1 The problem

We are considering a minimax control pproblem with a 2 dimensional, state (x, y), a minimizing control $u \in \mathbb{R}$ and a maximizing control $v \in [v^-, v^+] \subset \mathbb{R}$ with $v^- < 0 < v^+$, according to the following dynamics

$$\dot{x} = vx\,, \tag{1}$$
$$\dot{y} = vy + u\,. \tag{2}$$

In addition, as u is a priori unbounded, we shall need to allow impulses in u, or equivalently discontinuities in y at the will of the minimizer who chooses a finite number of time instants t_k and of jump amplitudes u_k to produce jumps of the form

$$y(t_k^+) = y(t_k) + u_k\,. \tag{3}$$

The game is being played over a fixed time interval $[0, T]$. The definition of the payoff involves two numbers $C^- < 0 < C^+$, and we shall make use of the notation $C^\varepsilon u$ with $\varepsilon = \mathrm{sign}(u)$ to mean C^+u if $u > 0$ and C^-u if $u < 0$. Furthermore, we have a positive number Z and make use of the function

$$M(x) = \max\{0, x - Z\}\,. \tag{4}$$

The payoff is defined by

$$J = M(x(T)) + \int_0^T (-vy + C^\varepsilon u)\,\mathrm{d}t + \sum_k C^{\varepsilon_k} u_k\,. \tag{5}$$

Alternatively, we shall use the extended state space where the payoff is purely terminal by adding a third state variable

$$\dot{z} = vy - C^\varepsilon u\,, \tag{6}$$
$$z(t_k^+) = z(t_k) - C^{\varepsilon_k} u_k\,, \tag{7}$$

making the payoff simply

$$J = M(x(T)) - (z(T) - z(0))\,.$$

We shall let $W(t, x, y)$ be the Isaacs Value function of this game.

2.2 The underlying financial problem

The underlying financial problem is one of option hedging. $M(x)$ is obviously the final payment of a European Call option of exercize time T and of exercize price, or "strike", Z. All monetary values have been converted to final value at the constant riskless interest rate, so that this rate is now seemingly zero.

The dynamic variables are

- x, the price of the underlying asset, which, in this model, is assumed to have a bounded relative rate of change v, between v^- and v^+,
- y, the amount of the underlying asset in the hedging portfolio, so that u is the rate of trading, the impulsions corresponding to instantaneous trading of a finite amount of asset, while the continuous part $u(\cdot)$ represents the traditional fiction of "continuous trading",
- z, the value of the hedging portfolio.

The constants C^+ and C^- are the rate of the transaction costs for buying and selling respectively, assumed to be linear in the amount traded.

The payoff represents the total expense of the trader who has sold one Call option and manages a self-financed portfolio to hedge the risk he or she has taken. For a given trading strategy, the supremum of the possible expenses is the price that the seller wishes to charge for the option, price which should be minimized to be efficient.

Hence the Value of this game at $t = 0$, the time the contract is agreed upon, at $x = x(0)$ and $y = 0$, is the equilibrium price of the option.

2.3 Results from previous papers

2.3.1 The Differential Quasi-Variational Inequality (DQVI)

In [3], it is shown that the Value function $W(t, x, y)$ of this game is a viscosity solution of the following equation :

$$0 = \min \left\{ \frac{\partial W}{\partial t} + \max_{v \in [v^-, v^+]} v \left[\frac{\partial W}{\partial x} x + \left(\frac{\partial W}{\partial y} - 1 \right) y \right] , \right. \\ \left. \frac{\partial W}{\partial y} + C^+ , \, -\frac{\partial W}{\partial y} - C^- \right\} . \tag{8}$$

The first of the three terms in the r.h.s. of that equation corresponds to the classical Isaacs equation, where the term

$$\min_u \left(\frac{\partial W}{\partial y} + C^\varepsilon \right) u$$

is missing, because as long as $-\partial W/\partial y \in [C^-, C^+]$ it is zero. The other two terms correspond to the possible jumps and one is indeed zero in case of a jump, according to the sign of the jump.

One consequence of that equation is that, for all (t,x,y), one has

$$C^- \le -\frac{\partial W}{\partial y} \le C^+ .$$

We shall use the notation $-\partial W/\partial y =: q$.

2.3.2 The singular manifolds

A detailed analysis of the trajectory field associated to that equation shows that the Value function is given by

$$W(t,x,y) = \check{z}(t,x) + q^\varepsilon(\check{y}(t,x) - y)\,, \quad \varepsilon = \mathrm{sign}(\check{y} - y)\,. \tag{9}$$

with the following definitions.

Let

$$\begin{aligned} q^+ &= \min\{\mathrm{e}^{v^+(T-t)} - 1\,,\, C^+\}\,,\\ q^- &= \max\{\mathrm{e}^{v^-(T-t)} - 1\,,\, C^-\}\,, \end{aligned} \tag{10}$$

Notice that $q^+ = C^+$ for $t < t_+$ and $q^- = C^-$ for $t < t_-$ with

$$t_+ = T - \frac{1}{v^+}\ln(1 + C^+) \quad \text{and} \quad t_- = T - \frac{1}{v^-}\ln(1 + C^-)\,.$$

We shall assume that the numbers are such that $t_- < t_+$. Similar results exist for the other case, but this is the one considered in [3].

As for the singular manifold, we shall let

$$\widehat{X}(t,x) = \begin{pmatrix} \check{y}(t,x) \\ \check{z}(t,x) \end{pmatrix} .$$

Its form depends on the region of the (t,x) plane considered. Two simple regions are as follows :

$$\widehat{X}(t,x) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if} \quad x \le Z\mathrm{e}^{-v^+(T-t)}\,, \\ \begin{pmatrix} x \\ x - Z \end{pmatrix} & \text{if} \quad x \ge Z\mathrm{e}^{-v^-(T-t)}\,. \end{cases} \tag{11}$$

In particular, this yields

$$\widehat{X}(T,x) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } x < Z\,, \\ \begin{pmatrix} x \\ x - Z \end{pmatrix} & \text{if } x \ge Z\,. \end{cases} \tag{12}$$

In the *region of interest*

$$Z\mathrm{e}^{-v^+(T-t)} \le x \le Z\mathrm{e}^{-v^-(T-t)}\,,$$

three subregions occur as follows:

- Dispersal manifold $\mathcal{D}$ for $t \ge t_+$:

$$\begin{aligned}\check{y}(t,x) &= \frac{x\mathrm{e}^{v^+(T-t)} - Z}{\mathrm{e}^{v^+(T-t)} - \mathrm{e}^{v^-(T-t)}}\,,\\ \check{z}(t,x) &= (1-\mathrm{e}^{v^-(T-t)})\check{y}(t,x)\,.\end{aligned}$$

- Equivocal manifold $\mathcal{E}$ for $t_- \le t \le t_+$. It is generated by the following differential equations where $s \in [Z, Z\mathrm{e}^{(v^+-v^-)(T-t_+)}]$:

$$\begin{aligned}&\dot{x} = v^+x\,, && x(t_+) = \tfrac{s}{1+C^+}\,,\\ &\dot{y} = \tfrac{v^+(1+C^+)-v^-\mathrm{e}^{v^-(T-t)}}{1+C^+-\mathrm{e}^{v^-(T-t)}}y\,, && y(t_+) = \tfrac{s-Z}{1+C^+-\mathrm{e}^{v^-(T-t_+)}}\,,\\ &\dot{z} = \tfrac{v^+(1+C^+)(1-\mathrm{e}^{v^-(T-t)})+C^+v^-\mathrm{e}^{v^-(T-t)}}{1+C^+-\mathrm{e}^{v^-(T-t)}}y\,, && z(t_+) = (1-\mathrm{e}^{v^-(T-t_+)})y(t_+).\end{aligned}$$

- Focal manifold $\mathcal{F}$, solution of a pair of coupled linear PDE's. We need the following definitions :

$$\mathcal{S} = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\,,\quad \mathcal{T} = \frac{1}{q^+-q^-}\begin{pmatrix}v^+q^+-v^-q^- & v^+-v^-\\ -(v^+-v^-)q^+q^- & v^-q^+-v^+q^-\end{pmatrix}\,.$$

  The fundamental PDE reads

$$\frac{\partial\widehat{X}}{\partial t} + \mathcal{T}\left(\frac{\partial\widehat{X}}{\partial x}x - \mathcal{S}\widehat{X}\right) = 0\,, \qquad (13)$$

  with boundary conditions on $\mathcal{E}$ at t_- for the region of interest, and by continuity with (11) on the two boundaries $x = Z\mathrm{e}^{-v^\varepsilon(T-t)}$, $\varepsilon = \pm$.

It is useful to notice that the left eigenvectors of $\mathcal{T}$ are $(q^+\ \ 1)$ and $(q^-\ \ 1)$ and the right eigenvectors $(1\quad -q^-)^t$ and $(-1\quad q^+)^t$, with the respective eigenvalues v^+ and v^-.

3 A single PDE for the whole singular manifold

3.1 Extending the linear PDE

We shall prove the following fact:

Theorem 3.1 *The entire singular manifold $\widehat{X}(t,u)$ satisfies the PDE (13).*

The proof of this theorem occupies the rest of this subsection.

Notice first that the definition (11) satisfies the fundamental PDE (13).

3.1.1 The dispersal manifold

Concerning the dispersal manifold $\mathcal{D}$, all we have to do is a straightforward differentiation. We choose to write $\exp[v^\varepsilon(T-t)] = 1+q^\varepsilon$, $\Delta := q^+ - q^-$, and notice that $\dot{q}^\varepsilon = -v^\varepsilon(1+q^\varepsilon)$. We have

$$\begin{aligned}
\frac{\partial \check{y}}{\partial t} &= \frac{v^+(1+q^+) - v^-(1+q^-)}{\Delta}\check{y} - \frac{v^+(1+q^+)}{\Delta}x\,,\\
\frac{\partial \check{z}}{\partial t} &= v^-(1+q^-)\check{y} - q^-\frac{\partial \check{y}}{\partial t}\,,
\end{aligned}$$

and

$$\frac{\partial \widehat{X}}{\partial x}x = \begin{pmatrix} 1 \\ -\,q^- \end{pmatrix}\frac{1+q^+}{\Delta}x\,,$$

so that, recognizing an eigenvector,

$$\mathcal{T}\frac{\partial \widehat{X}}{\partial x}x = v^+\begin{pmatrix} 1 \\ -\,q^- \end{pmatrix}\frac{1+q^+}{\Delta}x\,,$$

and finally

$$-\mathcal{T}\mathcal{S}\widehat{X} = \frac{-1}{\Delta}\begin{pmatrix} v^+(1+q^+) - v^-(1+q^-) \\ -\,v^+q^-(1+q^+) + v^-q^+(1+q^-) \end{pmatrix}\check{y}\,.$$

Placing these forms in $(\partial\widehat{X}/\partial t) + \mathcal{T}[(\partial\widehat{X}/\partial x)x - \mathcal{S}\widehat{X}]$ indeed yields 0.

3.1.2 The equivocal manifold

The verification that the equivocal manifold satisfies the fundamental PDE may go as follows. We are now in a region where $q^+ = C^+$, but $q^- = \exp[v^-(T-t)] - 1$.

Let the solution of the differential system be $(x,y,z) = (\tilde{x}(t,s), \tilde{y}(t,s), \tilde{z}(t,s))$ and use the notation

$$\widetilde{X}(t,s) = \begin{pmatrix} \tilde{y}(t,s) \\ \tilde{z}(t,s) \end{pmatrix}$$

and use subscripts for partial derivatives. From the formula $\forall(t,s)$, $\widetilde{X}(t,s) = \widehat{X}(t,\tilde{x}(t,s))$ follows that

$$\begin{aligned} \widetilde{X}_t &= \widehat{X}_t + \widehat{X}_x \tilde{x}_t \,, \\ \widetilde{X}_s &= \widehat{X}_x \tilde{x}_s \,. \end{aligned}$$

We clearly see that $\tilde{x}(t,s) = [s/(1+C^+)]\exp[-v^+(t_+ - t)]$, so that $\tilde{x}_s = \tilde{x}/s$, while $\tilde{x}_t = v^+\tilde{x}$. Hence inverting the above equations, it comes

$$\begin{aligned} \widehat{X}_t &= \widetilde{X}_t - \widetilde{X}_s s v^+ \,, \\ \widehat{X}_x &= \widetilde{X}_s \frac{s}{\tilde{x}} \,. \end{aligned}$$

As a consequence, we get

$$\widehat{X}_t + \mathcal{T}(\widehat{X}_x x - \mathcal{S}\widehat{X}) = \widetilde{X}_t - \mathcal{T}\mathcal{S}\widetilde{X} + s(\mathcal{T} - v^+ I)\widetilde{X}_s$$

We claim:

Proposition 3.2

1. $\widetilde{X}_t - \mathcal{T}\mathcal{S}\widetilde{X} = 0$,
2. $\widetilde{X}_s$ *remains parallel to* $(1 \quad -q^-)^t$, *so that* $(\mathcal{T} - v^+ I)\widetilde{X}_s = 0$.

Proof of the propositions:

1. It suffices to rewrite the two differential equations defining $\widetilde{X}$ as

$$\begin{aligned} \tilde{y}_t &= \frac{v^+(1+q^+) - v^-(1+q^-)}{q^+ - q^-}\tilde{y} \,, \\ \tilde{z}_t &= \frac{-q^- v^+(1+q^+) + q^+ v^-(1+q^-)}{q^+ - q^-}\tilde{y} \,, \end{aligned}$$

   to recognize from earlier calculations that this is indeed $\mathcal{T}\mathcal{S}\widetilde{X}$.

2. Notice that, according to the previous finding

$$\frac{\partial}{\partial t}\widetilde{X}_s = \frac{\partial}{\partial s}\widetilde{X}_t = \mathcal{T}\mathcal{S}\widetilde{X}_s \,.$$

   At t_+, it follows from the formula $z(t_+) = -q^- y(t_+)$ that

$$(q^- \quad 1)\widetilde{X}_s = 0 \,.$$

   Let us check that this property is kept as we integrate, by differentiating with respect to time :

$$\begin{aligned} \tfrac{\partial}{\partial t}[(q^- \quad 1)\widetilde{X}_s] &= (-v^-(1+q^-) \quad 0)\widetilde{X}_s + (q^- \quad 1)\mathcal{T}\mathcal{S}\widetilde{X}_s \\ &= -v^-(1+q^-)\tilde{y}_s + v^-(q^- \quad 1)\mathcal{S}\widetilde{X}_s = 0. \end{aligned}$$

Therefore $\widetilde{X}_s$ remains orthogonal to $(q^- \quad 1)$, and hence parallel to the eigenvector $(1 \quad -q^-)^t$ associated with the eigenvalue v^+. ∎

Hence the equivocal manifold indeed satisfies the fundamental PDE (13).

3.2 A necessary condition

We seek now to give a explanation of this fact in the form of a necessary condition. We shall prove the following theorem:

Theorem 3.3 *If a viscosity solution of (8) is of the form*

$$W(t,x,y) = \check{y}(t,u) + q^\varepsilon(t,x)(\check{y}(t,u) - v)\,, \quad \varepsilon = \mathrm{sign}(\check{y} - y)\,, \tag{14}$$

with $q^+(t,x) \geq 0$, $q^-(t,x) \leq 0$, *and* $\widehat{X}(t,x) := (\check{y}(t,x) \quad \check{z}(t,x))^t$ *of class* C^1, *then necessarily,* $\widehat{X}$ *satisfies the PDE (13).*

The proof of the theorem takes the rest of this section.

We introduce yet new notations. Let first

$$Q^\varepsilon = (\, q^\varepsilon \quad 1\,)\,, \qquad \mathbb{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so that $Q^\varepsilon \mathbb{1} = 1 + q^\varepsilon$, $\mathcal{S}\widehat{X} = \mathbb{1}\check{y}$, and formula (14) also reads $W(t,x,y) = Q^\varepsilon \widehat{X}(t,x) - q^\varepsilon y$. Let also

$$H^* := W_t + \max_{v \in [v^-, v^+]} v\,[W_x x + (W_y - 1)y]\ ,$$

so that equation (8) also reads

$$\min\,\{H^*, W_y + C^+, -W_y - C^-\} = 0\,, \tag{8}$$

and finally

$$H^\varepsilon(t,x,y) := Q^\varepsilon \widehat{X}_t - q^\varepsilon_t(\check{y} - y) + \max_{v \in [v^-, v^+]} v\left[Q^\varepsilon \widehat{X}_x x - q^\varepsilon_x x(\check{y} - y) - (q^\varepsilon + 1)y\right].$$

Equation (8) implies that $H^* \geq 0$. Thus take $y < \check{y}$, and the limit as $y \uparrow \check{y}$. It comes $H^+(t,x,\check{y}(t,x)) \geq 0$. Similarly taking $v \downarrow \check{y}$, we get $H^-(t,x,\check{y}(t,x)) \geq 0$.

Notice that according to the signs of q^+ and q^-, W is the maximum of two affine functions in y, and has a local minimum in y at $y = \check{y}(t,x)$. Hence it has a nonvoid subdifferential, made of all convex combinations of the left and right gradients $\nabla^+ W$ and $\nabla^- W$ respectively, given by

$$\nabla^\varepsilon W = \begin{pmatrix} Q^\varepsilon \widehat{X}_t \\ Q^\varepsilon \widehat{X}_x \\ -\,q^\varepsilon \end{pmatrix}, \qquad \varepsilon = \pm\,.$$

Hence, due to the affine form of the coordinates, it suffices to replace q^ε by $q(\lambda) := \lambda q^+ + (1-\lambda)q^-$ in H^ε to have the equivalent formula for an element of the subdifferential, that we shall denote H^λ. Now, the viscosity condition here is that, for all $\lambda \in [0,1]$, $H^\lambda(t,x,\check{y}(t,x)) \le 0$. Taking the two extreme values $\lambda = 0$ or 1, it comes $H^+(t,x,\check{y}(t,x)) \le 0$ and $H^-(t,x,\check{y}(t,x)) \le 0$. Comparing to the previous two inequalities, we conclude that

$$H^\varepsilon(t,x,\check{y}(t,x)) = 0\,, \qquad \varepsilon = \pm,.$$

It remains to prove that the maximum in v in H^ε is attained at $v = v^\varepsilon$. Let us consider $H^\lambda(t,x,\check{y})$, but disregarding the maximization operation in v, and letting instead either $v = v^-$ or $v = v^+$. Its graph as a function of λ (or equivalently of $q(\lambda)$) has to look like the underneath figure, where η stands for either ε or $-\varepsilon$:

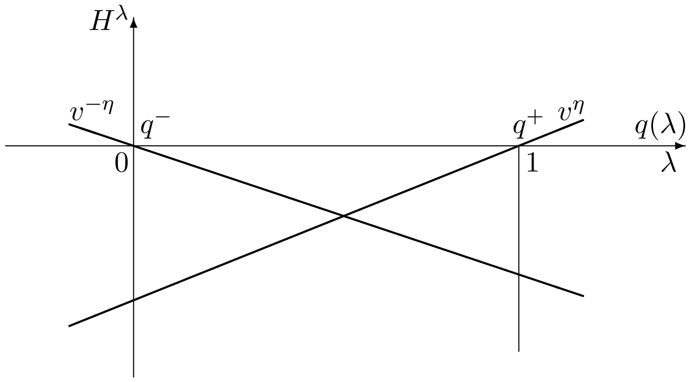

Figure 1: H^λ as a function of λ (or $q = \lambda q^+ + (1-\lambda)q^-$) depending on v

Deciding which v^η is the maximizing one, say for H^-, therefore depends on the sign of $Q^-\widehat{X}_x x - (q^- + 1)y = Q^-(\widehat{X}_x x - \mathbb{1}y)$. We claim

Proposition 3.4

$$Q^-(\widehat{X}_x x - \mathbb{1}y) \le 0\,, \quad \textit{and} \quad Q^+(\widehat{X}_x x - \mathbb{1}y) \ge 0\,.$$

Proof Assume, say, that $Q^-(\widehat{X}_x x - \mathbb{1}y) \ge 0$. Then the maximizing v in $H^-(t,x,\check{y})$ is v^+. It is clear from the game itself that for large y's, that maximizing v in H^- is v^-. Hence there is a $y_0 > \check{y}$ for which $Q^-\widehat{X}_x x - q^-(\check{y} - y_0) - (1 + q^-)y_0 = 0$. Hence, in a neighborhood, W_t would change sign, when it must be always non-positive according to (8). A contradiction.

The proof is similar for $Q^+(\widehat{X}_x x - \mathbb{1}y)$ ∎

Therefore, writing the two equations for H^- and H^+ together, we get

$$\begin{pmatrix} q^- & 1 \\ q^+ & 1 \end{pmatrix} \widehat{X}_t + \begin{pmatrix} v^- & 0 \\ 0 & v^+ \end{pmatrix} \begin{pmatrix} q^- & 1 \\ q^+ & 1 \end{pmatrix} [\widehat{X}_x x - \mathcal{S}\widehat{X}] = 0\,.$$

Multiplying to the left by the matrix

$$\frac{1}{q^+ - q^-} \begin{pmatrix} -1 & 1 \\ q^+ & -q^- \end{pmatrix}$$

yields the desired result (13). And this ends the proof.

4 Conclusion

On the one hand, we have a unified equation for the whole singular manifold. It was shown in [7] that it can be replaced by a scalar second order PDE bearing on $\check{y}$ alone, whose second order terms are

$$\left(\frac{\partial}{\partial t} + v^+ x \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v^- x \frac{\partial}{\partial x}\right)\check{y}\,,$$

displaying its characteristics $\dot{x} = v^\varepsilon x$. (The lower order terms depend on where is t relative to t_ε, i.e. on which of the three types of singular surfaces we are on.) Notice thus that the boundary of the "region of interest" are the characteristics going through the discontinuity of the gradient of the terminal value, along which that discontinuity propagates backwards. But our manifold is continuous.

On the other hand, we understand that it is not an "accident" of the particular option we are looking at. This is a property of any such viscosity solution of the DQVI, regardless of the terminal value, provided that the Value function has this special form. The interesting thing would be, going back to the original problem, to show directly that the Value has to be piecewise affine in y. We did not succeed in doing so as yet.

In a forthcoming paper, we use formula (14) as a convenient representation of the Value, and show that it indeed provides a viscosity solution of the DQVI, and we use it to construct a fast algorithm to actually solve that game, providing an alternative theory of option pricing with transaction costs. Thus this detailed analysis proves usefull in that context. The present paper somehow takes the magic out of that representation formula : it is a necessary consequence of the general form of the solution and of its being a viscosity solution.

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