Robust control approach to option pricing, including transaction costs

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Abstract

We adopt the robust control, or game theoretic, approach of [5] to option pricing. In this approach, uncertainty is described by a restricted *set* of possible price trajectories, without endowing this set with any probability measure. We seek a hedge against every possible price trajectory.

In the absence of transaction costs, the continuous trading theory leads to a very simple differential game, but to an uninteresting financial result, as the hedging strategy obtained lacks robustness to the unmodeled transaction costs. (A feature avoided by the classical Black and Scholes theory through the use of unbounded variation cost trajectories. See [5].)

We therefore introduce transaction costs into the model. We examine first the continuous time model. Its mathematical complexity makes it beyond a complete solution at this time, but the partial results obtained do point to a robust strategy, and as a matter of fact justify the second part of the paper.

In that second part, we examine the discrete time theory, deemed closer to a realistic trading strategy. We introduce transaction costs into the model from the outset and derive a pricing equation, which can be seen as a discretization of the quasi variational inequality of the continuous time theory. The discrete time theory is well suited to a numerical solution. We give some numerical results. In the particular case where the transaction costs are null, we recover our theory of [5], and in particular the Cox Ross and Rubinstein formula when the contingent claim is a convex function of the terminal price of the underlying security.

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1 Introduction

We consider the classical problem of pricing a contingent claim based upon an underlying stock of curent price $S(t)$, and defined by its terminal value, or *payoff*, $M(S(T))$ at *exercise time* T. In the case of a (European) call, we have $M(s) = [s-K]_+ = \max\{s-K, 0\}$ for a given *striking price* K.

This problem is classically solved by Black and Scholes'theory [6] in the continuous trading framework, and approached by the theory of Cox, Ross and Rubinstein [8] in the limiting vanishing step size case for the discrete trading, discrete time theory. The fundamental device of these theories, due to Merton, is to construct a *replicating* portfolio, made up of the underlying stock and riskless bonds, and a self financed trading strategy, or *hedging strategy*, that together yield the same payoff as the contingent claim to be priced. However, the classical theory of Black and Scholes, based upon the "geometric diffusion" market model, is known to have the major weakness that transaction costs cannot be taken into account in any meaningful way. See [15].

In [5], we proposed a robust control approach to that same idea, that we quickly review hereafter. The distinctive feature of our theory is in our market model. We forgo any stochastic description of the underlying stock price. Instead, we assume that we know hard bounds on the possible (relative) variation rate of the stock price. And we seek to manage our portfolio through self financed trading in such a way as to do at least as well as the option, in terms of final value, on all possible price histories, leading to a minimax control problem.

A very similar approach has been taken independantly and simultaneously with our research by J-P. Aubin, D. Pujal and coworkers, see [13, 2], using their tools of viability theory. A game theoretic approach is also used by [12] in connection with transaction costs, but to investigate a different problem of optimizing these costs from the viewpoint of the banker. Essentially the same market model as ours has been proposed in [14], where they give it the name we shall use of "interval model".

In the absence of transaction costs, the continuous trading theory leads to a simple differential game. However, the solution of that game yields the so called "parity value" for the option, something rather far from observed prices on the market. Correlatively, the bang-bang hedging strategy obtained, that we call the "naive strategy", lacks robustness to the unmodeled transaction costs, in particular if the underlying stock price fluctuates close to the money resulting in a perpetual dilema for the trader. (In [5] we argued that its inherent robustness is the main reason to prefer the Black and Scholes strategy and option price. This is done at the price of adopting, for the underlying stock price, trajectories of unbounded variation and known quadratic relative variation —the volatility. Whether this is realistic in a world where stock prices are updated at discrete time instants is a matter of debate.)

We propose then to include transaction costs into our differential game model. This leads to a three dimensional impulse control game that has up to now resisted our attempts to solve it via classical means.[1] It displays, however, at least one feature of robustness against fluctuating stock prices : the fact that no trading should occur during a final time interval, after a final jump in portfolio composition. This gives a strong hint as how the solution can be approached by a discrete time theory, with a step size function of the transaction costs and of the maximum relative variation rate hypothesized for the underlying stock.

Therefore, in a second part, we investigate the discrete time theory. The theory we obtain is well suited to a numerical solution, particularily so in the convex case where we are able to show that it preserves the convexity of the option price with respect to the current underlying stock's price, yielding a simplification into the computation. Also, not surprisingly, the discrete time pricing equation can be seen as a finite differences approximation of the continuous time equation. But the continuous time theory is far from developed to the point where a rigorous convergence proof would be feasible.

In the case where the transaction costs vanish, we recover our theory of [5]. Hence, if furthermore the contingent claim's payoff, is a convex function of the stock price (e.g. for a simple European call), it is strongly reminiscent of the theory of Cox, Ross and Rubinstein [8], to which it gives a normative value even for a non-vanishing step size. Otherwise, it is shown to give a higher equilibrium price to the option than the previous theory. (If one identifies our $(1-\alpha)$ and $(1+\beta)$ with their d and u.)

2 Continuous time theory

2.1 The models

2.1.1 Market model

In that market, we have a riskless security, called *bonds*, evolving at a known constant rate, which in fact sets the lending and borrowing rate on that market. Let this rate be denoted ρ. The exercise time of the option considered is T. And let

$$R(t) = \mathrm{e}^{\rho(t-T)}$$

be either the value of a bond, or the end-time factor in our market.

[1] At the time of the revision of this paper we are close to a solution in terms of characteristics. It displays an interesting new type of singularity.

We denote $S(t)$ the underlying stock price at time t. We let :

Definition 2.1 *The set Ω of* admissible *price histories is defined by two positive numbers $\tilde{\alpha}$ and $\tilde{\beta}$, and is the set of all absolutely continuous time functions $t \mapsto S(t)$ such that at every instant where it is differentiable, it satisfies the inequalities*

$$-\tilde{\alpha} \le \frac{\dot{S}}{S} \le \tilde{\beta} \tag{1}$$

or, equivalently that between any two instants of time $t_1 < t_2$,

$$e^{-\tilde{\alpha}(t_2-t_1)} S(t_1) \le S(t_2) \le e^{\tilde{\beta}(t_2-t_1)} S(t_1) \, .$$

We choose to represent that hypothesis in a system theoretic fashion :

$$\dot{S} = \tilde{\tau} S \, , \tag{2}$$

where the time function $t \mapsto \tilde{\tau}(t) \in [-\tilde{\alpha}, \tilde{\beta}]$ is assumed to be measurable, and represents an a priori unknown disturbance.

We shall use the notations $\alpha = \tilde{\alpha} + \rho$ and $\beta = \tilde{\beta} - \rho$. The positive numbers ρ, α and β describe the market model and are assumed known.

Although this presentation is meant to emphasize the fact that there are no probabilities involved, and that our "disturbance" $\tilde{\tau}$ is just a mathematical device, a renaming of the quantity that we have assumed to be bounded, it may be usefull to relate this form with one more reminiscent of the classical geometric diffusion model. Let $\mu = (\tilde{\beta} - \tilde{\alpha})/2$ and $\sigma = (\tilde{\beta} + \tilde{\alpha})/2$. Let also $\nu = (\tilde{\tau} - \mu)/\sigma$. Now, all stock price trajectories can be represented by the system

$$\dot{S} = (\mu + \sigma\nu) S \tag{3}$$

where ν is any measurable time function $t \mapsto \nu(t)$ satisfying $|\nu(t)| \le 1, \; \forall t$. In a sense, this is a "normalized" disturbance, and therefore σ is a measure of the volatility of the stock considered.

2.1.2 Portfolio model

We form a portfolio made up of x shares of the underlying stock, and y riskless bonds. The value or *worth* $\tilde{w}$ of this portfolio at any time instant is thus

$$\tilde{w}(t) = x(t)S(t) + y(t)R(t) \, .$$

We aim to precisely define what is a self-financed hedging strategy.

Let us investigate how behaves a self financed trading strategy in the presence of transaction costs, since this is our objective. Assume these costs are proportional to the amount traded, not necessarily with the same proportionality ratio for the two comodities considered. Let us call c_0 the trading cost ratio for the riskless bond, and c_1 for the underlying stock. Each transaction should finance those costs. Therefore, let $\mathrm{d}x$ be the variation in x at a stock price of S, and $\mathrm{d}y$ the variation in y at a bond value R, we should have

$$\mathrm{d}xS + c_1|\mathrm{d}x|S + \mathrm{d}yR + c_0|\mathrm{d}y|R = 0\,. \tag{4}$$

We therefore let :

Definition 2.2

a. *A* dynamic portfolio *(or simply* portfolio*) is a pair of bounded variation time functions* $(x(\cdot), y(\cdot))$ *defined over* $[0, T]$.

b. *A dynamic portfolio is said to be* self financed *if it satisfies (4) (in the sense of Stieltjes calculus).*

The costs c_0 and c_1 are assumed small, of the order of a few percent may be. Introduce

$$\varepsilon = \text{sign}(\mathrm{d}x)\,, \qquad C_\varepsilon := \varepsilon\frac{c_0 + c_1}{1 - \varepsilon c_0} \tag{5}$$

Proposition 2.1 *A self financed dynamic portfolio is entirely defined by its intial composition* $(x(0), y(0))$ *and the bounded variation time function* $x(\cdot)$. *The time function* $y(\cdot)$ *and its worth* $\tilde{w}(\cdot)$ *can be reconstructed through integration of the differential relations*

$$dy = -\frac{(1 + \varepsilon c_1)}{(1 - \varepsilon c_0)}\frac{S}{R}dx \tag{6}$$

and

$$d\tilde{w} = \rho\tilde{w}dt + (\tilde{\tau} - \rho)xSdt - C_\varepsilon Sdx\,. \tag{7}$$

Proof Because c_0 and c_1 are smaller than one, it easily follows that $\mathrm{d}x$ and $\mathrm{d}y$ should have opposite signs. Recall that $\varepsilon = \text{sign}(\mathrm{d}x)$. It therefore comes

$$(1 + \varepsilon c_1)\mathrm{d}xS + (1 - \varepsilon c_0)\mathrm{d}yR = 0\,,$$

hence (6). Further more, the classical fact that $y\mathrm{d}R = \rho yR\mathrm{d}t = \rho(\tilde{w} - xS)\mathrm{d}t$, it comes (7). ▮

Notice that in (7), the last term is always negative, and represents the loss in portfolio value due to the trading costs. Of course, the case without transactions costs can be recovered by letting $c_0 = c_1 = C_\varepsilon = 0$ in the above theory.

We shall let, for short, $C_{+1} = C^+$ and $C_{-1} = C^-$ (a negative number). It is worthwile to examine two extreme cases :

The case $c_0 = 0$**.** If the riskless bond is, say money, and trading in that comodity is free, then we simply have $C_\varepsilon = \varepsilon c$, the transaction cost on the stock.

The case $c_0 = c_1 = c$**.** If both transaction costs are equal, an interesting feature that shows up is that then

$$1 + C^+ = \frac{1+c}{1-c} = \frac{1}{1+C^-}\,.$$

More generally, we may notice the folowing fact :

Proposition 2.2 *Whenever* $c_0 \le c_1$*, one has*

$$1 < 1 + C^+ \le \frac{1}{1+C^-}$$

2.2 Hedging strategies

2.2.1 Variations of x

We would like to let $x(\cdot)$ be our control. But there are costs associated to its *variations*. Hence, in a classical system theoretic fashion, we are led to consider its derivative as the control. However, another difficulty shows up, since we want also to allow discontinuities in x. We would therefore need a theory of impulse control in differential games. It is worthwhile to write the Isaacs quasi-variational inequality that this formally leads to. But the theory of this inequation is not available at this time.

We shall in some respect get around that difficulty with the following approximating device. We shall let the monetary flux be our control :

$$\dot{x}S = \tilde{\xi}\,, \qquad \text{or} \qquad S\mathrm{d}x = \tilde{\xi}\mathrm{d}t\,, \qquad \tilde{\xi} \in [-\widetilde{X}, +\widetilde{X}]\,, \tag{8}$$

where we shall take $\widetilde{X}$ to be a (very) large positive number, and investigate the limit of the solution found as $\widetilde{X} \to \infty$. Whether this limit is the solution of an impulsive problem is a rather technical question, the more so here that we shall deal with a differential game, not a mere control problem. The tools introduced in a recent article [1] seem appropriate to attempt an extension to game problems. We shall not be concerned with that problem here.

2.2.2 Trading strategies

We shall let our control $\tilde{\xi}$ be a function of time, $S(t)$, and $x(t)$ if necessary. We need to chose this function in such a way that the induced differential equations have a solution. We therefore let

Definition 2.3 *An admissible* trading strategy *is a function* $\tilde{\varphi} : [0,T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *such that the differential equation (8) with* $\tilde{\xi} = \tilde{\varphi}(t, S(t), x(t))$ *has a unique solution* $x(\cdot)$ *for every* $x(0)$ *and admissible time function* $S(\cdot) \in \Omega$.

As a result, an admissible trading strategy, together with an initial portfolio $(x(0), y(0))$ yields a well defined dynamic portfolio.

Our aim is described by the following definition :

Definition 2.4 *At an initial market price* $S(0)$ *given,*

a. *An initial portfolio and an admissible trading strategy* $\tilde{\varphi}$ *constitute a* hedge *at* $S(0)$ *if they insure that*

$$\forall S(\cdot) \in \Omega \text{ with } S(0) \text{ given}\,, \quad \tilde{w}(T) \geq M(S(T))\,. \tag{9}$$

b. *The strategy* $\tilde{\varphi}$ *is a* hedging strategy *for the initial portfolio* $(x(0), y(0))$ *at* $S(0)$ *if together they constitute a hedge.*

c. *An initial portfolio is said to be* hedging *at* $S(0)$ *if there exists a related hedging strategy for it.*

Finally, the relation with pricing is as follows :

Definition 2.5 *The* equilibrium price *of the contingent claim* $(T, M(\cdot))$ *at* $S(0)$ *is the least worth* $\tilde{w}(0) = y(0)R(0)$ *of all hedging initial portfolios of the form* $(0, y(0))$.

This last definition stems from the following remark. Let an initial hedging portfolio be given as $(x(0), y(0))$. Let $\varepsilon = \text{sign}(x(0))$, and let us assume that $x(0)$ and $y(0)$ have different signs, as will be the case for efficient hedging portfolios in the case of simple european options. The cost of creating it, or its *price* is

$$\begin{aligned} P(x(0), y(0)) &= (1 + \varepsilon c_1)x(0)S(0) + (1 - \varepsilon c_0)y(0)R(0) \\ &= (1 - \varepsilon c_0)[\tilde{w}(0) + C_\varepsilon x(0)S(0)]\,. \end{aligned}$$

In the notations, of the next paragraph, this leads to define the price of the hedge as $P = (1 - \varepsilon c_0)R(0)\min_v[W(0, u(0), v) + C_\varepsilon v]$. On the other hand, if our theory allows for a jump in x and y at initial time, satisfying (4), then it follows from (the same reasoning as that leading to) equation (22) that indeed, $R(0)W(0, u, 0) = R(0)\min_v[W(0, u, v) + C_\varepsilon v]$. Since ε will be the same for all efficient hedging portfolios (+1 or -1 in the case of a call or a put respectively), in comparing the worth of hedging portfolios, we may neglect the factor $(1 - \varepsilon c_0)$, which disapears altogether if we assume that our original wealth was invested in bounds.

2.3 End-time values

It is convenient to transform everything in end-time values. We shall let

$$u = \frac{S}{R}\,, \quad v = \frac{xS}{R}\,, \quad w = \frac{\tilde{w}}{R}\,,$$
$$\tau = \tilde{\tau} - \rho\,, \quad -\alpha = -\tilde{\alpha} - \rho\,, \quad \beta = \tilde{\beta} - \rho\,,$$
$$\xi = \frac{\tilde{\xi}}{R}\,, \quad X = \frac{\widetilde{X}}{R}\,.$$

Notice that S, x and y are readily recovered from u, v, and w with the help of

$$x = \frac{v}{u}\,, \qquad y = w - v\,. \tag{10}$$

With these notations, the dynamics of the market and portfolio become

$$\dot{u} = \tau u\,, \tag{11}$$
$$\dot{v} = \tau v + \xi\,, \tag{12}$$
$$\dot{w} = \tau v - C_\varepsilon \xi\,. \tag{13}$$

$$\tau \in [-\alpha, \beta]\,, \qquad \xi \in [-X, +X]\,. \tag{14}$$

Our objective is to find the cheapest hedging portfolio and corresponding hedging strategy

$$\xi = \varphi(t, u, v) \tag{15}$$

2.4 Mathematical analysis of the problem

The aim (9) of a hedging strategy φ can be written as

$$\forall \tau(\cdot) \in [-\alpha, \beta]\,, \qquad M(u(T)) - w(T) \le 0\,,$$

where one remembers that $u(T)$ is a function of $u(0)$ and τ, and $w(T)$ a function of $v(0)$, $w(0)$, and both $\tau(\cdot)$ and φ. Obviously, this is equivalent to

$$\sup_{\tau(\cdot)} [M(u(T)) - w(T)] \le 0\,.$$

And for a given $u(0)$, $v(0)$, $w(0)$, there exists a hedging strategy if (and only if provided that the min below exists)

$$\min_{\varphi} \sup_{\tau(\cdot)} [M(u(T)) - w(T)] \le 0\,. \tag{16}$$

Hence, in a typical "robust control" fashion, we face a minimax control problem or dynamic game problem.

Now, notice that w does not appear in the right hand side of the dynamics (11),(12),(13). hence, we may integrate (13) in

$$w(t) = w(0) + \int_0^t (\tau(s)v(s) - C_\varepsilon \xi(s))\,\mathrm{d}s\,.$$

The relation (16) can therefore be rewritten

$$\min_\varphi \sup_{\tau(\cdot)} [M(u(T)) - \int_0^T (\tau(t)v(t) - C_\varepsilon \xi(t))\,\mathrm{d}t - w(0)] \leq 0\,.$$

Now, $u(t)$, $v(t)$ and hence $\xi(t) = \varphi(t, u(t), v(t))$ are independent on $w(0)$. Hence, the above relation is satisfied provided that

$$w(0) \geq \min_\varphi \sup_{\tau(\cdot)} [M(u(T)) - \int_0^T (\tau(t)v(t) - C_\varepsilon \xi(t))\,\mathrm{d}t]\,.$$

We are thus led to the investigation of the function

$$W(t, u(t), v(t)) = \min_\varphi \sup_{\tau(\cdot)} [M(u(T)) - \int_t^T (\tau(s)v(s) - C_\varepsilon \xi(s))\,\mathrm{d}s]\,, \qquad (17)$$

and define the *price* of the contingent claim investigated as $W(0, u(0), 0)$.

We introduce the Isaacs equation of this game :

$$\begin{aligned} &\frac{\partial W}{\partial t} + \min_\xi \sup_{\tau \in [-\alpha, \beta]} \left\{ \tau \left[\frac{\partial W}{\partial u} u + \left(\frac{\partial W}{\partial v} - 1 \right) v \right] + (\frac{\partial W}{\partial v} + C_\varepsilon)\xi \right\} = 0\,, \\ &W(T, u, v) = M(u)\,. \end{aligned} \qquad (18)$$

(Notice that the function between braces in the r.h.s. above is not differentiable in ξ because of the definition of C_ε, involving $\varepsilon = \mathrm{sign}(\xi)$.)

If we adopt the approximation device of restricting ξ to a finite interval $[-X, X]$, then the $\min_\xi$ in (18) above should be restricted accordingly. And we get

Theorem 2.3 *If there exists a viscosity solution W of (18), then $\mathrm{e}^{-\rho T} W(0, u(0), 0)$ is the approximated equilibrium price of the contingent claim investigated.*

Proof From standard differential games theory (see [3, 7]), W is indeed the min-sup in (17). Hence the worth of the cheapest hedging initial portfolio with $x(0) = 0$, hence $v(0) = 0$, for a given $u(0)$ is $w(0) = W(0, u(0), 0)$. Going back to the original variables $\tilde{w}(0) = \mathrm{e}^{-\rho T} w(0)$ yields the result. ∎

2.5 No transaction costs

We first consider the case without transaction costs, i.e. $C_\varepsilon = 0$. This was investigated in more detail in [5], but we stress here a game theoretic analysis which was not discussed there.

Here, we need not keep xS or v as a state variable as it can be changed instantly at no cost. We therefore remain with two state variables (plus time) :

$$\begin{aligned} \dot{u} &= \tau u \,, \\ \dot{w} &= \tau v \,. \end{aligned}$$

The control variables are $\tau \in [-\alpha, \beta]$ and $v \in \mathbb{R}$. A trading strategy in this context will be a function $\varphi : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ giving $v(t) = \varphi(t, u(t))$. The problem at hand is to find states controllable by v to the set $w(T) \geq M(u(T))$ against any control of τ.

The above analysis simplifies in

$$w(T) = w(0) + \int_0^T \tau(t) v(t) \mathrm{d}t \,.$$

and

$$w(0) = \min_{\varphi} \sup_{\tau(\cdot)} \left[M(u(T)) - \int_0^T \tau(t) v(t) \mathrm{d}t \right] \,,$$

which thus provides the equilibrium option price sought.

Therefore, our pricing equation is Isaacs'equation for this game :

$$\frac{\partial W}{\partial t} + \min_v \max_\tau \left[\tau \left(\frac{\partial W}{\partial u} u - v \right) \right] = 0 \,, \quad W(T,u) = M(u) \,. \qquad (19)$$

Theorem 2.4 *In the absence of transaction costs, the equilibrium price of the contingent claim investigated is the so-called* parity value

$$e^{-\rho T} M(e^{\rho T} S(0)) \,.$$

The corresonding hedging strategy is given by

$$x(t) = \frac{dM}{ds} (e^{\rho(T-t)} S(t)) \,.$$

Proof For any v, the $\max_\tau$ in (19) is non negative. Hence it is maximized by the choice $v = (\partial W / \partial u) u$, and Isaacs equation is reduced to $\partial W / \partial t = 0$. Hence its solution is $W(t,u) = M(u)$. The results follow. ∎

We see that we recover the classical fact that the optimal x is the sensitivity of the option's value.

2.5.1 European Calls

Let us examine the case of a European call. Then $M = [u - K]_+$ is not differentiable. Yet it is easy to see that $W = M$ is indeed the viscosity solution of Isaacs'equation, since at $u = K$, we do have that for any p between 0 and 1,

$$\min_v \max_\tau [\tau(pu - v)] = 0\,.$$

However, this is not the last word about the nondifferentiability of W.

The corresponding strategy is the "naive strategy" :

$$v = \begin{cases} 0 & \text{if} \quad u \leq K\,, \\ u & \text{if} \quad u \geq K\,. \end{cases}$$

The strategy at $u = K$ is better analyzed in terms of the semipermeability of the manifold $w = [u - K]_+$. It is readily apparent that if we do not want the corner to leak, we need that both $\dot{w} \geq 0$ and $\dot{w} \geq \dot{u}$, which requires that $v = 0$ if $\tau < 0$ (to abide by the first constraint) and $v = 1$ if $\tau > 0$ (to abide by the second.) Hence the hedging strategy is a bang bang function of the *sign of the variation* of the underlying stock price. A very undesirable feature.

Moreover, if there are transaction costs, and if the prices oscillate around the parity value, this will induce constant large buy and sell decisions which will cost much and ruin that hedging strategy.

2.6 Non-zero transaction costs : a partial solution

To be more specific, we consider the case of a european call with striking price K, where $M(s) = \max\{0, s - K\}$.

2.6.1 Three-D impulsive control formulation

In this paragraph, we investigate the problem allowing for instantaneous trading of a finite amount of securities, hence jumps in x and y at times chosen by the trader, but still in accordance with (4).

We let the trader choose instants of time t_k and trading amounts ξ_k with signs ε_k, and augment the dynamic equations with the jump conditions

$$v(t_k^+) = v(t_k^-) + \xi_k\,, \qquad (20)$$

$$w(t_k^+) = w(t_k^-) - C_{\varepsilon_k} \xi_k\,. \qquad (21)$$

The definition of a trading strategy for this paragraph is therefore as follows:

Definition 2.6 *A trading strategy is defined by* *a. a measurable function* $\xi(\cdot):[0,T]\to\mathbb{R}$ *called the* continuous part,

b. an impulsive part *made of*

- *a finite increasing sequence of time instants* $\{t_k\}$
- *a sequence of corresponding numbers* $\{\xi_k\}$,

The corresponding dynamic portfolio is given by the equations (12) (20), and the worth of the portfolio can be computed from (13)(21).

We denote symbolically by φ a feedback rule that let one decide whether to make a jump and of how much, and also compute the continuous part of the trading strategy, knowing past and present u's and v's.

As previously notice that w does not enter the right hand side. Using the equality

$$w(T)=w(0)+\int_0^T(\tau v(t)-C_\varepsilon\xi(t))\mathrm{d}t-\sum_k C_{\varepsilon_k}\xi_k\,,$$

We therefore end up with the dynamics (11)(12)(20) and the problem to find

$$W(0,u(0),v(0))=\min_{\varphi}\sup_{\tau(\cdot)}\left[M(u(T))-\int_0^T(\tau v(t)-C_\varepsilon\xi(t))\mathrm{d}t+\sum_k C_{\varepsilon_k}\xi_k\right].$$

Determination of this impulsive minimax is beyond the scope of the current theory. One should refer to the theory of impulse control, as developed in [4]. However, added difficulties arise. On the one hand, this is a game not a control problem. On the other hand, this is a deterministic problem, so that the corresponding PDE is first order, and one would need to extend to quasivariational inequalities the technique of viscosity solutions. Moreover, the Q.V.I. is further degenerate due to the fact that the second term in the brace in (22) is nonpositive.[2]

Yet, it is interesting to write the quasivariational inequality that is formally associated with this impulsive game :

$$0=\min\left\{\frac{\partial W}{\partial t}+\max_{\tau\in[-\alpha,\beta]}\tau\left[\frac{\partial W}{\partial u}u+\left(\frac{\partial W}{\partial v}-1\right)v\right]\quad,\quad \min_{\xi}[W(t,u,v+\xi)-W(t,u,v)+C_\varepsilon\xi]\right\} \tag{22}$$

with furthermore $\partial W/\partial v\in[-C^+,-C^-]$ whenever the first minimum is obtained by the first term in the brace, in order for $(\partial W/\partial v+C_\varepsilon)\xi$ to have a minimum in ξ, which is then reached at $\xi=0$.

[2]In that respect, we would have a more classical impulse control problem if the transaction costs were chosen affine, with a fixed part added to the proportional part.

It might be possible to construct the solution of this QVI. We *conjecture* that the solution leads to hedging strategy involving, for any realistic initial condition, an initial jump in v followed by a "coasting" period where ξ takes intermediate values depending on the variations of u, followed by a final period with $\xi = 0$ as the next paragraph shows. Hence providing a non trivial hedging strategy for option pricing with transaction costs, a feat known to be impossible with the classical theory, see [15]

2.6.2 Four-D non impulsive analysis

We now turn to the approximation device consisting in bounding $|\xi|$ by a very large number X that we shall let go to infinity.

We also turn back to the 3 state formulation, considering the qualitative problem of driving the final state to the set $\{w(T) - M(u(T)) \geq 0\}$. The barrier of this problem is the graph of the function W of the 2 state variable formulation. But the geometric intuition of semipermeable surfaces will help here. Notice also that, to gain in intuition (we like to think of the hedging strategy as maximizing the value of the portfolio), we have changed the sign of the terminal term. We make use of the Isaacs Breakwell theory. The reader unfamiliar with that theory could as well turn directly to the subsection 2.6.3.

Hamiltonian set up. In terms of differential games, we must construct a "barrier" separating states that can be driven to the desired set at time T against all disturbances from those for which at least one disturbance function exists that will prevent the aim to be reached. Although we shall only sketch the mathematical details, we shall make free use of the theory. See, e.g. [10, 11].

Because we are in fixed end-time, the state space of this game is four dimensional : (t, u, v, w). Let (n, p, q, r) be a semipermeable normal. It satisfies

$$n + \max_{\xi} \min_{\tau} \{[pu + (q + r)v]\tau + (q - C_\varepsilon)\xi\} = 0\,,$$

and the controls on barrier trajectories are given by

$$\tau = \begin{cases} -\alpha & \text{if} \quad pu + (q + r)v > 0\,, \\ \beta & \text{if} \quad pu + (q + r)v < 0\,, \end{cases} \qquad \xi = \begin{cases} X & \text{if} \quad q > C^+\,, \\ 0 & \text{if} \quad C^- < q < C^+\,, \\ -X & \text{if} \quad q < C^-\,. \end{cases}$$

Furthermore, on a smooth part of a barrier, along the barrier trajectories the semipermeable normal satisfies the adjoint equations :

$$\dot{n} = 0\,,$$

$$\begin{aligned} \dot{p} &= -\tau p\,, \\ \dot{q} &= -(1+q)\tau\,, \\ \dot{r} &= 0\,. \end{aligned}$$

Barrier sheet towards $u < K$. Let us construct the natural barrier arriving on the part $u < K$, $w = 0$ of the target set boundary, that we parametrize with $u(T) = s$, $v(T) = \chi$. We get

$$\begin{aligned} u(T) &= s \le K\,, & p(T) &= 0\,, \\ v(T) &= \chi\,, & q(T) &= 0\,, \\ w(T) &= 0\,, & r(T) &= 1\,. \end{aligned}$$

So, at final time, we get $\tau = -\alpha$ and $\xi = 0$. The equations integrate backwards in

$$\begin{aligned} u(t) &= s\mathrm{e}^{\alpha(T-t)}\,, & p(t) &= 0\,, \\ v(t) &= \chi\mathrm{e}^{\alpha(T-t)}\,, & q(t) &= \mathrm{e}^{-\alpha(T-t)} - 1\,, \\ w(t) &= \chi(\mathrm{e}^{\alpha(T-t)} - 1)\,, & r(t) &= 1\,. \end{aligned}$$

This solution is not valid before the time t_α when q crosses the value C^-, i.e.

$$T - t_\alpha = \frac{1}{\alpha}\ln\left(\frac{1}{1+C^-}\right)\,.$$

Prior to t_α, one has $q < C^-$, and therefore, if that trajectory is still part of a barrier, $\xi = -X$. In view of the fact that we are interested in the case $X \to \infty$, this means a negative jump in v, i.e. in x, the underlying stock content of our portfolio.

Regular barrier sheet towards $u(T) > K$. We now consider the natural barrier towards $u(T) > K$, $w(T) = u(T) - K$. We again parametrize that boundary by $u(T) = s$ and $v(t) = \chi$. We get now :

$$\begin{aligned} u(T) &= s \ge K\,, & p(T) &= -1\,, \\ v(T) &= \chi\,, & q(T) &= 0\,, \\ w(T) &= s - K\,, & r(T) &= 1\,. \end{aligned}$$

The corresponding value of τ at time T depends on the sign of $s - \chi$. Let us first consider the case $s > \chi$. We have then at time T and just before $\tau = \beta$, and still $\xi = 0$. The differential equations integrate backwards in

$$\begin{aligned} u(t) &= s\mathrm{e}^{-\beta(T-t)}\,, & p(t) &= -\mathrm{e}^{\beta(T-t)}\,, \\ v(t) &= \chi\mathrm{e}^{-\beta(T-t)}\,, & q(t) &= \mathrm{e}^{\beta(T-t)} - 1\,, \\ w(t) &= \chi(\mathrm{e}^{-\beta(T-t)} - 1) + s - K\,, & r(t) &= 1\,. \end{aligned}$$

This solution is not valid before the time t_β when q crosses the value C^+ :

$$T - t_\beta = \frac{1}{\beta}\ln(1 + C^+)\,. \tag{23}$$

Prior to that time, and again if the trajectories were part of a continuing barrier, we would have $\xi = X$, which indicates a positive jump in v, hence in x.

Singular barrier sheet towards $u(T) > K$. A particular case arises if we consider the case $s = \chi$. Then it is readily apparent that $pu + (q + r)v$ remains null along any time interval $[t, T]$ on which $\xi = 0$. Then, any τ satisfies the semipermeability condition, and so does $\xi = 0$ as long as

$$\ln(1 + C^-) \le \int_t^T \tau(\theta)\,\mathrm{d}\theta \le \ln(1 + C^+)\,.$$

Along these trajectories, $u = v \in [s/(1+C^+), s/(1+C^-)] \cap [s\mathrm{e}^{-\beta(T-t)}, s\mathrm{e}^{\alpha(T-t)}]$, and $w = u - K$.

Thus, for each s we have two free parameters : t and $\int_t^T \tau\,\mathrm{d}\theta$, yet this constitutes only a 2-D manifold, because all are embedded into the 2-D manifold $u = v = w + K$, t arbitrary.

One of these trajectories for each s is obtained with $\tau = \beta$. It is the "last" trajectory of the sheet we would construct with $\chi \le s$. It will come as no surprise to the reader that the trajectories constructed with $\chi > s$, i.e. in our original variables $x > 1$, will play no role in the solution.

Intersection. These two three-dimensional sheets intersect along a 2-D edge, that joins continuously with the above singular 2-D manifold, and that we can parametrize with $h = T - t$ and $u \in [K\exp(-\beta h), K\exp(\alpha h)]$ as

$$v = \hat{v}(h, u) := \frac{u\mathrm{e}^{\beta h} - K}{\mathrm{e}^{\beta h} - \mathrm{e}^{-\alpha h}}\,, \quad w = \hat{w}(h, u) := (1 - \mathrm{e}^{-\alpha h})\hat{v}(h, u)\,.$$

Notice that because of Proposition 2.2, and assuming that $\alpha \le \beta$, (maximum rate of decrease of a stock price not larger than the maximum rate of increase), then $t_\alpha < t_\beta$. Hence the above intersection only holds over the time interval $[t_\beta, T]$, because before, the sheet towards $u > K$ is missing.

A careful study of the intersection shows that it is a τ-dispersal line, i.e. that the trader must watch the evolution of the stock price and adapt to it.

Composite barrier. These semipermeable surfaces define a composite natural barrier that can as usual be described as (the graph of) a function $w = W(t,u,v) = \inf\{w \mid (t,u,v,w)$ is hedgeable$\}$. We get here $h = T - t$ and

$$W(t,u,v) = \begin{cases} (1 - \mathrm{e}^{-\alpha h})v & \text{if} \quad v \geq \hat{v}(h,u)\,, \\ (1 - \mathrm{e}^{\beta h})v + u\mathrm{e}^{\beta h} - K & \text{if} \quad v \leq \hat{v}(h,u)\,. \end{cases}$$

We may further notice that if $u \leq K\exp(-\beta h)$, we always are in the first case above, and if $u \geq K\exp(\alpha h)$, taking into account the fact that we consider only the cases $v \leq u$, we always are in the second case.

2.6.3 Interpretation of the results

As a cue to interpreting the geometry of the barrier in the state space, notice that if a point (t,u,v,w) is "hedgeable", then any point (t,u,v,w') with $w' > w$ is also admissible. Therefore for a given (t,u), say, we should look for a barrier point with those coordinates and the lowest possible w as a limiting admissible state, and therefore an equilibrium price for the call.

The main point we have shown is that the terminal part of the play leads to $\xi = 0$ as an optimal behaviour, i.e. to a constant x, no trading is necessary during the final $T - t_\beta$ time interval. Therefore the main weakness of the "naive" strategy of [5], which was a risk of constant and costly trading, is avoided. The whole idea to include the transaction costs into the model was aimed at that result.

If $u < K\mathrm{e}^{-\beta(T-t)}$, then according to our model, the call is and shall remain out of the money. The value of the call is 0. As a matter of fact, the only relevant barrier is our sheet towards $u < K$. For a given u and t it is intersected at minimum w by $\chi = 0$ and indeed yields $w = 0$.

If $u > K\mathrm{e}^{-\beta(T-t)}$, the call may end up in the money. If moreover $u > K\mathrm{e}^{\alpha(T-t)}$, then it will surely do. The intersection of the two sheets of the composite barrier is at $v = u$, $w = u - K$. One should have (at least) one share of the stock, and may have borrowed an amount $KR(t)$, worth K at exercise time.

In between, and for the last instant of times, the limiting v and w are just such that with no trading, if the stock goes down at maximum rate, we shall end up with $w = 0$, and if it goes up at maximum rate, we shall have just $w = u - K$. The corresponding equilibrium value for the call is a linear function of u (with $h = T - t$ the maturity) :

$$w = \frac{1 - \mathrm{e}^{-\alpha h}}{\mathrm{e}^{\beta h} - \mathrm{e}^{-\alpha h}}(u\mathrm{e}^{\beta h} - K)\,.$$

It coincides with one step of the discrete time theory hereafter (see also [5]), and therefore more prominently of [8]. As a matter of fact, since we have found that the optimal behaviour was to let $T - t_\beta$ time pass without trading, therefore without incurring trading costs, we find one step of the discrete time theory with that step size.

We therefore suggest to exploit this result by using a discrete time strategy with that step size. This is the minimum time it takes for the stock to increase of a relative amount $-C^-$, i.e. of the order of (but slightly less than) $c_0 + c_1$.

2.6.4 Closing costs

Before we investigate the discrete time theory, we must make a digression on closing costs.

If there are trading costs, it is not equivalent to end up with no stock and no debt or with, say s worth of stock, and as much in debt, as there is a cost to selling the stock and using the proceeds to repay the debt.

The target set at exercise time should therefore be changed to reflect that fact. Let $\eta = \text{sign}(v)$. The correct target set for $u(T) < K$ is then

$$w + C_{-\eta} v \geq 0 \,.$$

(If we know that only positive v's will be used, we may simply set $w + C^- v \geq 0$, but the above form is usefull for the theoretical analysis.) In the case $u(T) > K$, there are two possible ways of comparing a portfolio and the option. Either we decide to liquidate any position, and compare situations with $v = 0$, or we want to bring our portfolio to a position similar to that just after exercising the call, i.e. with $v = u$. Both methods do not lead to the same conclusions. The first one has the advantage of leading to a continuous target set for the portfolio. Incidentally, one should then exercise the call only if the net proceeds after liquidating the position is positive, i.e. if $(1 + C^-)u(T) - K > 0$. And the target set then reads

$$w + C_{-\eta} v \geq [(1 + C^-)u - K]_+ \,.$$

We forgo the mathematical analysis of this case, as it is at this time less advanced than the previous one. The analysis seems to point to a delay without trading at the end of length

$$T - t_\beta = \frac{1}{\beta} \ln \frac{1 + C^+}{1 + C^-} \,, \tag{24}$$

hence roughly twice as long as in the case without closure costs.

3 Discrete time

We turn to the discrete time theory. In [5], we argued that this is a more realsitic theory as a trader is likely to pay attention to a given portfolio a finite number of times per day... Here, however, we have another justification, arising from the continuous time theory itself, where we have seen that introducing transaction costs automatically leads to optimal hedging strategies made of jumps in the contents of the portfolio, with no trading at least after the last jump.[3] This suggests to use the characteristic step size (23) or (24) above. This is typicaly of the order of one third to one half day. The present theory can also be exploited for other step sizes.

3.1 The model

3.1.1 Market model

We let now the time t be an integer, i.e. we take the step size Δt as our unit of time, so that t is now an integer ranging from 0 to a given positive integer T. Let also the price of a unit riskless bound be

$$R(t) = (1+\rho)^{(t-T)}$$

so that ρ in this section is $\mathrm{e}^{\rho\Delta t} - 1$ of the continuous time theory. Likewise, concerning the underlying stock price S we let :

Definition 3.1 *The set Ω of* admissible *price histories is defined by two positive numbers $\tilde{\alpha}$ and $\tilde{\beta}$ and is the set of all sequences $\{S(t)\}$, $t \in \{1, \dots, T\}$ such that*

$$(1-\tilde{\alpha})S(t) \le S(t+1) \le (1+\tilde{\beta})S(t) \tag{25}$$

We choose to define

$$\tilde{\tau}_t := \frac{S(t+1) - S(t)}{S(t)}\,,$$

so that the above definition also reads

$$S(t+1) = (1+\tilde{\tau}_t)S(t)\,, \qquad \tilde{\tau}_t \in [-\tilde{\alpha}, \tilde{\beta}]\,. \tag{26}$$

Notice that the $\tilde{\alpha}$ and $\tilde{\beta}$ of this section are related to those of the previous section via the same relation as ρ.

[3] And affine transaction costs would lead to purely impulsive strategies anyhow.

3.1.2 Portfolio model

We call x_t and y_t the number of shares in the portfolio immediately *before* the transactions at time t, and $\tilde{w}_t = x_t S(t) + y_t R(t)$ the corresponding value of the portfolio. We shall make use of $\tilde{w}_t^+ = x_{t+1} S(t) + y_{t+1} R(t)$, the value of the portfolio immediately *after* the trading at time t, and likewise for v and w below. Let also $dx(t) = x_{t+1} - x_t$ and $dy(t) = y_{t+1} - y_t$. We therefore have

Definition 3.2

a. A dynamic portfolio *is a pair of sequences* $(\{x_t\}, \{y_t\})$*, defined over* $t \in \{1, \dots, T\}$.

b. A dynamic portfolio is said to be self financed *if it satisfies (as equation (4)):*

$$dx(t)S(t) + c_1|dx(t)|S(t) + dy(t)R(t) + c_0|dy(t)|R(t) = 0\,. \qquad (27)$$

We choose $\tilde{\xi}_t$, the amount in S traded at time t as our control, so that

$$(x_{t+1} - x_t)S(t) = \tilde{\xi}_t\,.$$

Let $\varepsilon = \operatorname{sign}(\tilde{\xi})$. The same reasoning as in the continuous time case may be used to conclude that self financing of the strategy imposes the following:

Proposition 3.1 *A self financed dynamic portfolio satisfies*

$$(y_{t+1} - y_t)R(t) = -\frac{1 + \varepsilon c_1}{1 - \varepsilon c_0}\tilde{\xi}_t\,.$$

and

$$\tilde{w}_{t+1} = (1+\rho)(\tilde{w}_t - C_\varepsilon \tilde{\xi}_t) + (\tilde{\tau}_t - \rho)(x_t S(t) + \tilde{\xi}_t)\,.$$

Introduce, as in the continuous time theory, the end-time values

$$u_t = \frac{S(t)}{R(t)}\,, \quad v_t = \frac{x_t S(t)}{R(t)}\,, \quad w_t = \frac{\tilde{w}_t}{R(t)}\,, \quad \xi_t = \frac{\tilde{\xi}_t}{R(t)}\,,$$

as above,

$$v_t^+ = \frac{x_{t+1} S(t)}{R(t)}\,, \quad w_t^+ = \frac{\tilde{w}_t^+}{R(t)}\,,$$

and let

$$-\alpha := \frac{-\tilde{\alpha} - \rho}{1+\rho} \le \tau_t := \frac{\tilde{\tau}_t - \rho}{1+\rho} \le \beta := \frac{\tilde{\beta} - \rho}{1+\rho}\,, \qquad (28)$$

After some simple calculations, the discrete time market and portfolio model then read

$$\begin{array}{rclrcl} & & & u_{t+1} & = & (1+\tau_t)u_t\,, \\ v_t^+ & = & v_t+\xi_t\,, & v_{t+1} & = & (1+\tau_t)v_t^+\,, \\ w_t^+ & = & w_t - C_\varepsilon \xi_t\,, & w_{t+1} & = & w_t^+ + \tau_t v_t^+\,. \end{array} \tag{29}$$

We shall consider feedback trading strategies :

Definition 3.3 *A trading strategy is a sequence of functions $\varphi_t : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$. The self financed dynamic portfolio generated from an initial portfolio (x_0, y_0) by a price history $\{S_t\} \in \Omega$ and the trading strategy $\{\varphi_t\}$ is the pair of sequences generated by (29) with, for all $t \in \{0, 1, \ldots, T-1\}$: $\xi_t = \varphi_t(u_t, v_t)$.*

We now state our objective : finding hedging strategies and the equilibrium price of the contingent claim.

Definition 3.4 *At a given market price $S(0)$,*

a. An initial portfolio (x_0, y_0) and a trading strategy constitute a hedge *at $S(0)$ if, for any $\{S_t\} \in \Omega$, with $S(0)$ given, together they yield $w_T \geq M(u_T)$.*

b. The corresponding trading strategy is then called a hedging strategy.

c. An initial portfolio (x_0, y_0) is said hedging *at $S(0)$ if there exists a corresponding hedging stategy.*

And finally

Definition 3.5 *The* equilibrium price *of the contingent claim investigated at $S(0)$ is the least worth $w_0 = y_0 R(0)$ of all hedging portfolios of the form $(0, y_0)$.*

Remark The definition *a)* above may be slightly modified to reflect the preferred notion of hedge in the presence of closing costs. (e.g., judged at T^+ imposing $v_T^+ \geq u_T$, or alternatively $v_T^+ = 0$ and $w_T^+ \geq [(1-c_1)u - K]_+$). See the section on continuous trading for further hindsight into these definitions.

3.2 Dynamic programming

Let $\mathcal{A}_t$ be the set of states (u_t, v_t, w_t) from which there exists a trading strategy $\xi_k = \varphi_k(u_k, v_k)$, $k \geq t$ that, for any possible future sequence $\{\tau_k\}$, drives the portfolio to an admissible state at time T, i.e. such that $w_T \geq M(u_T)$. It is clear that if $(u, v, w) \in \mathcal{A}_t$, then any (u, v, w') with $w' > w$ will also be in $\mathcal{A}_t$. We may thus characterize the set $\mathcal{A}_t$ as the epigraph of its *floor function*

$$W_t(u,v) = \min \left\{ w \,\middle|\, \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{A}_t \right\} \quad \text{so that} \quad \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{A}_t \Leftrightarrow w \geq W_t(u,v)\,.$$

It is convenient to perform the classical dynamic programming construction, leading to the Isaacs equation, in two steps. Let first $\mathcal{A}_t^+$ be the set of states (u_t, v_t^+, w_t^+) at t^+ that will be driven to $\mathcal{A}_{t+1}$ by any τ, and W_t^+ the corresponding floor function. Thus,

$$\begin{pmatrix} u \\ v^+ \\ w^+ \end{pmatrix} \in \mathcal{A}_t^+ \Longleftrightarrow \forall \tau \in [-\alpha, \beta]\,, \quad \begin{pmatrix} (1+\tau)u \\ (1+\tau)v^+ \\ w^+ + \tau v^+ \end{pmatrix} \in \mathcal{A}_{t+1}.$$

or equivalently

$$w^+ \geq W_t^+(u, v^+) \Longleftrightarrow \forall \tau \in [-\alpha, \beta],\, w^+ + \tau v^+ \geq W_{t+1}((1+\tau)u, (1+\tau)v^+)\,.$$

Thus, we have

$$W_t^+(u, v^+) = \max_{\tau \in [-\alpha, \beta]} [W_{t+1}((1+\tau)u, (1+\tau)v^+) - \tau v^+]\,. \tag{30}$$

Now, $\mathcal{A}_t$ is the set of all states (u, v, w) that can be sent by an appropriate control ξ into a (u, v^+, w^+) in $\mathcal{A}_{t+1}$, hence,

$$w \geq W_t(u, v) \Longleftrightarrow \exists \xi : w - C_\varepsilon \xi \geq W_t^+(u, v+\xi)\,.$$

Therefore

$$W_t(u, v) = \min_\xi [W_t^+(u, v+\xi) + C_\varepsilon \xi] \tag{31}$$

It is useful to give the form taken by the recursion merging the two steps from t_{t+1} to t^+ and from t^+ to t into a single Isaacs equation :

$$W_t(u, v) = \min_\xi \max_{\tau \in [-\alpha, \beta]} [W_{t+1}((1+\tau)u, (1+\tau)(v+\xi)) - \tau(v+\xi) + C_\varepsilon \xi]\,. \tag{32}$$

to be initialized with

$$W_T(u, v) = M(u) \tag{33}$$

We have thus proved the following result:

Theorem 3.2 *If equations (32)(33) have a solution $W_t(u, v)$, the equilibrium price of the contingent claim at $S(0)$ is $(1+\rho)^{-T} W_0((1+\rho)^T S(0), 0)$.*

(The coefficients $(1+\rho)^{-T}$ and $(1+\rho)^T$ are there to come back in the original variables, as opposed to their end-time values.)

Equations (30) and (31), or equivalently (32) and (33), also provide a constructive algorithm to numerically compute the equilibrium price. We discuss that matter at the end of subsection 3.4.

3.3 Limiting cases

3.3.1 Zero transaction costs

The case with no transaction costs corresponds here to $C_\varepsilon = 0$. Then in (32) ξ only appears in combination with v as $(v+\xi)$ (i.e. v^+), which can therefore be taken as our mute maximization variable. If moreover the final value W_T does not depend on v, then the r.h.s. above never depends on v either, leading to a function $W_t(u)$:

$$W_t(u) = \min_{v^+} \max_{\tau\in[-\alpha,\beta]} [W_{t+1}((1+\tau)u) - \tau v^+] \, .$$

We shall argue below that in the case of simple European options, the maximum in τ is reached at an end point of the admissible interval. As a consequence, this maximum is minimum when $\tau = -\alpha$ and $\tau = \beta$ yield the same value, i.e.

$$W_{t+1}((1-\alpha)u) + \alpha v^+ = W_{t+1}((1+\beta)u) - \beta v^+$$

leading to

$$v^+(u) = \frac{W_{t+1}((1+\beta)u) - W_{t+1}((1-\alpha)u)}{\alpha+\beta}$$

and thus

$$W_t(u) = \frac{\alpha}{\alpha+\beta} W_{t+1}((1+\beta)u) + \frac{\beta}{\alpha+\beta} W_{t+1}((1-\alpha)u) \, . \tag{34}$$

These are exactly the equations obtained by Cox, Ross and Rubinstein [8]. The reference [5] develops in some more detail the reason why the two theories seem to coincide. The (big ?) difference, though, is that the theory of Cox, Ross, and Rubinstein is based upon a market model which is *not* realistic for finite step sizes, and only meant to be meaningful in the limit as the step size goes to zero. Here we have a normative theory even for finite step sizes. However, with the same historical data, we shall be led to a model with a larger volatility $\sigma = (\alpha+\beta)/2$ than in their approach.

3.3.2 Vanishing step size

It is interesting to investigate the limiting case of our theory, with transaction costs, when the step size goes to zero in the above recursion. Thus we replace the step size "one" of the above theory by h. We choose to modelize stock price histories of *bounded variation*, as opposed to the classical Black and Scholes model. (See [5] for more details.) Thus replace τ by $\tau_h = h\tau \in [-h\alpha, h\beta]$.

Equation (32) now reads

$$W_t(u,v) = \min_{\xi} \max_{\tau\in[-\alpha,\beta]} [W_{t+h}((1+h\tau)u, (1+h\tau)(v+\xi)) - h\tau(v+\xi) + C_\varepsilon \xi]$$

The following analysis is formal. We strongly conjecture that it can be made precise, though at a rather high mathematical price. Rewrite the above equation as

$$0 = \min_{\xi} \max_{\tau\in[-\alpha,\beta]} \Big\{ W_{t+h}((1+h\tau)u, (1+h\tau)v)) - W_t(u,v) - h\tau(v+\xi) + \\ W_{t+h}((1+h\tau)u, (1+h\tau)(v+\xi)) - W_{t+h}((1+h\tau)u, (1+h\tau)v)) + C_\varepsilon \xi \Big\}. \tag{35}$$

The first line in the above display always goes to zero as $h \to 0$. Now, two situations may arise as $h \to 0$.

Either the minimum in ξ is attained for a non zero ξ. Nevertheless, the second line must also go to zero, since the sum does. Therefore, in the limit we must have

$$\min_{\xi} [W_t(u, v+\xi) - W_t(u,v) + C_\varepsilon \xi] = 0\,,$$

where we recognise the second term of (22). In this case, placing $\xi = 0$ will yield a positive r.h.s. But the second line in the display (35) is zero for $\xi = 0$. Thus the first is positive, and the term above is the minimum of the two lines.

Or the minimum in ξ is reached at $\xi = 0$. This means that the second line in the display would be positive for non zero ξ's. And the first line reads, dividing through by the positive h

$$\max_{\tau\in[-\alpha,\beta]} \left[\frac{1}{h} \Big(W_{t+h}((1+h\tau)u, (1+h\tau)v)) - W_t(u,v) \Big) - \tau v \right] = 0\,.$$

In the limit as h goes to zero we recognize the first line of (22).

Altogether, we see that we end up with the quasivariational inequality (QVI) (22) of the continuous time theory. Thus, equation (32) can be seen as an "upwind" finite difference scheme for the continuous QVI, strongly suggesting that the solution of the discrete trading problem converges to that of the continuous trading one as the setp size goes to zero.

3.4 The convex case

As is the case without transaction costs, convexity of the evaluation function M at terminal time is preserved by the recursion and helps in the computations. Let us state the main fact :

Theorem 3.3

a. The functions W_t and W_t^+ generated by the recursion (30),(31),(33) are convex in v for each u.

b. If furthermore the function M is convex, then they are jointly convex in (u,v).

Proof We provide the proof of the second statement. The first one goes along the same lines, just simpler.

Notice first that M being convex (in u), W_T is jointly convex in (u,v). Assume that $W_{t+1}(u,v)$ is convex in (u,v). So is $W_{t+1}((1+\tau)u,(1+\tau)v) - \tau v$. And therefore, according to (30), W_t^+ is the maximum of a family of convex functions, thus convex.

Assume therefore that W_t^+ is convex in (u,v). Introduce the extended function

$$\Gamma(\eta,\xi) = \begin{cases} +\infty & \text{if} \quad \eta \neq 0\,, \\ -C_{-\varepsilon}\xi & \text{if} \quad \eta = 0\,. \end{cases}$$

It is convex in (ξ,η). Now, (31) reads

$$W_t(u,v) = \min_{\eta,\xi}[W_t^+(u-\eta, v-\xi) + \Gamma(\eta,\xi)]\,.$$

Hence W_t is the inf convolution of two convex functions, therefore it is convex. The theorem follows by induction.

Beyond its theoretical significance, —there are deep reasons to expect the value of a call to be convex, at least in u (see, e.g. [9])— this fact has an important computational consequence. Let us first emphasize the following fact :

Corollary 3.3.1 *If M is convex, the function $\tau \mapsto [W_{t+1}((1+\tau)u,(1+\tau)v) - \tau v]$ is convex.*

As a consequence, the maximum in τ in (30) is necessarily reached at an end point of the segment $[-\alpha,\beta]$. Computationally, this means that the maximization is reduced to comparing two values, a significant simplification. The practical consequence is that using the recurrence relation to compute a pricing is very fast for a convex terminal value. We typically had run times of 6 seconds per time step on a 500 Mhz P.C., with u and v discretized in 200 steps each, a golden search in ξ, and $P1$ finite elements interpolation of the function W_t.

This will not be so for a digital call, say. The maximization in τ then has to be done via an exhaustive search. Yet, since we preserve the convexity in v, the minimization in ξ can still be performed via an efficient algorithm.

Remark The various ways of taking closing costs into account usually preserve the convexity of M.

3.5 Partial solution for a European call

If we take $M(s) = \max\{0, s - K\}$, we can perform "by hand" the first steps of the recursion (32). We find the following facts.

- The optimal choice for $\xi(T)$ is always zero : there is no incentive to perform a portfolio readjustment at final time since this has a cost, and buys us nothing in this formulation without closure costs.

- If $u_t \geq (1-\alpha)^{t-T} K$, one finds that the recursion reaches a fixed point

  $$W_t(u, v) = [(1 + C^+)(1 + \beta) - 1](u - v) + u - K\,.$$

  The optimal trading strategy is always to jump to $v = u$, i.e. own one share of the underlying stock.

- If $u_t \leq (1+\beta)^{t-T} K$, the siuation is slightly more subtle, at least for large trading costs. As a matter of fact, if $\alpha + C^- < 0$, for the last time steps, where $(1-\alpha)^{T-t} > 1 + C^-$, the optimal hedging strategy is $\xi_t = 0$, hence do not trade. This is again a feature of the robustness of this theory against small variations in u. For earlier t's, the optimal hedging strategy is to jump to $v = 0$, and the lowest value of a replicating portfolio is

  $$W_t(u, v) = [1 - (1 + C^-)(1 - \alpha)]v\,.$$

  That is, one needs to have at least $(1+C^-)(1-\alpha)v$ worth of riskless bonds to pay for the trading in of the stock at hand at the next time instant. (Remember that v_t is the value of the stock in the portfolio *before* the trading at time t.)

- For u between these two limits, the value is piecewise affine in u and v. We have shown in the previous subsection that the "worst" market evolution, i.e. the one that dimensions the necessary portfolio, is always an extreme value, which makes the numerical solution very fast (only $\tau = -\alpha$ and $\tau = \beta$ have to be compared).

- In the case where $-\alpha = C^-$, which corresponds to the critical time of the continuous time theory, further degeneracies appear in the minimization in ξ in the recursion (32), making $\xi = 0$ a possible optimal hedging strategy for a larger region of the (u, v) space.

- Adding a closure cost does not change much the above results.

4 Conclusions

Our objective was to introduce transaction costs in our non-stochastic theory of option pricing, not so much to have a more realistic theory, although this may be of interest, but mainly because we speculated that doing so would alleviate the bad feature of the previous continuous trading theory, which was found to give a naive srategy much too sensitive to trading costs (which had been neglected) in case the stock price oscillates around the present-value of the striking cost.

This aim seems to be indeed achieved, since at least for a (small) final time interval, no trading occurs. Yet at this time, we do not have a complete solution of the continuous time problem.

So we turn to a discrete time scheme, which may be considered as a more realistic formulation anyway. The corresponding problem can easily be solved numerically. The option value it will yield is reminiscent of that of Cox Ross and Rubinstein in that it is piecewise affine, and coincides with it in the case of simple put or call options with no transaction costs in the theory. We point out that this is now a normative theory even with a finite step size, and not only as the step size vanishes. A particularily meaningful step size is the critical time deduced from the continuous trading strategy. Moreover, we conjecture that the limit of that theory as the stepsize goes to zero is the solution of the continuous time theory, in effect making our algorithm an efficient approximation scheme for the latter.

At this time, detailed numerical comparisons are being made between this theory and classical stochastic theories and between option pricing with or without trading costs. Even more importantly, we want to see whether this theory accounts for observed option prices on the market. Notice that we have more parameters to adjust than Black and Scholes, say.

5 Acknowledgements

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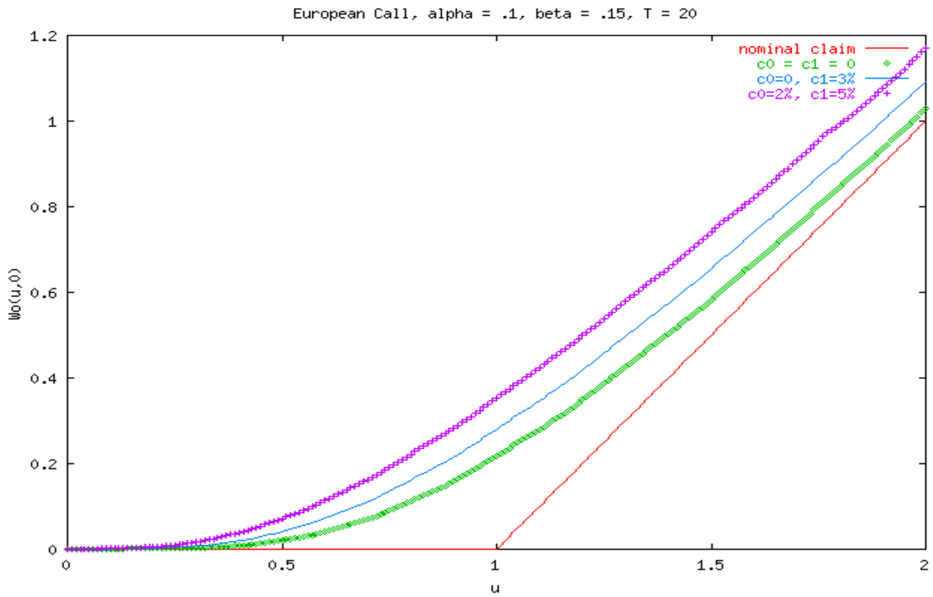

Figure 1: Equilibrium prices for various transaction costs, 20 time steps

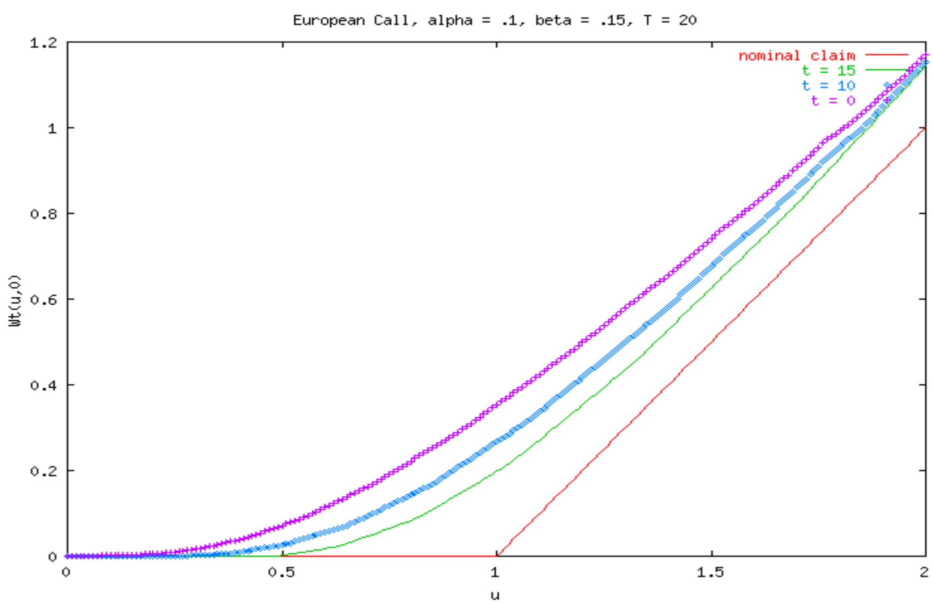

Figure 2: Equilibrium prices for various maturities, $c_0 = 2\%$, $c_1 = 5\%$