Chain differentials with an application to the mathematical fear operator

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1 introduction

Motivated by the need to perform Hamilton Jacobi Caratheodory theory with an infinite dimensional state, we first introduce a concept of derivative in between the Gâteaux derivative, which does not yield the chain rule, and Fréchet derivative, which is too restrictive for our purpouse. While the Gâteaux derivative can be defined over any vector space, the Fréchet derivative requires a normed vector space. The concept of chain derivative we introduce can be defined over a topological vector space (t.v.s.), and seems to be a natural concept in that framework. Not surprisingly, this concept is closely related to epiderivatives [1], which have been extensively used in modern, (but finite dimensional) Hamilton Jacobi theory [2, 3].

In order to illustrate the concept, and as a complement to a couple of earlier papers [4, 5, 6] where the derivation was presented as formal, for lack of a precise framework, we show that the mathematical fear operator is chain differentiable with respect to the cost distribution in the space of continuous functions endowed with the topology of pointwise convergence uniform on every compact subsets.

2 Chain differentials and chain derivatives

In the sequel, we shall use directional derivatives of f at x in the direction ξ denoted $\mathrm{D}f(x;\xi)$, or *differentials*. Everywhere, in the case where the differential is linear and continuous in ξ:

$$\mathrm{D}f(x;\xi) = \mathrm{D}f(x)\cdot\xi\,,$$

then the linear operator $\mathrm{D}f(x)$ is called a *derivative*.

Wile Gâteaux differentiation can be defined over any vector space, Fréchet differentiation requires that the variable lie in a normed space. The following concept is well suited to the intermediary structure of a topological vector space, or t.v.s. Let therefore X and Y be two t.v.s., and $f : X \to Y$.

Definition 1 *The function f has a* chain differential $\mathrm{D}f(x;\xi)$ *at x in the direction ξ if, for any sequence $\xi_n \to \xi \in X$ and any sequence* [1] *of real numbers $\theta_n \to 0$ it holds that*

$$\lim_{n\to\infty}\frac{1}{\theta_n}[f(x+\theta_n\xi_n) - f(x)] = \mathrm{D}f(x;\xi)\,. \tag{1}$$

That this be indeed a concept in between Gâteaux and Fréchet differentiation, and close to an epiderivative, is stated in the following fact:

Proposition 1 i) *If f has a chain differential, it also is a Gâteaux differential,*

ii) *If f has a chain differential, it is also an epiderivative,*

iii) *If X is a normed space, and f has a Fréchet derivative, it also is a chain derivative.*

We also need to emphasize the following fact:

Proposition 2 *The chain differential $\mathrm{D}f(x;\xi)$ is homogeneous of degree one in ξ :*

$$\forall\theta\in\mathbb{R}\,,\quad \mathrm{D}f(x;\theta\xi) = \theta\mathrm{D}f(x;\xi)\,. \tag{2}$$

The main objective of this definition is to yield the following theorem:

Theorem 1 *Let X, Y and Z be three t.v.s., $f : x \to Y$, $g : Y \to Z$ and f and g have chain differentials at x in the direction ξ and at $f(x)$ in the direction $\mathrm{D}f(x;\xi)$ respectively. Let $h = g\circ f$. Then h has a chain differential at x in the direction ξ, given by the chain rule:*

$$\mathrm{D}h(x;\xi) = \mathrm{D}g(f(x);\mathrm{D}f(x;\xi))\,.$$

[1] it would be possible, as in the definition of epiderivatives, to restrict the θ_n's to be positive. However, this does not let us extend the second theorem below to *cones* Ξ rather than vector spaces. Thus it seems of little interest here.

Proof Let $\xi_n \to \xi \in X$ and $\theta_n \to 0 \in \mathbb{R}$. Let

$$\varphi_n := \frac{1}{\theta_n}[f(x+\theta_n\xi_n) - f(x)]\,.$$

On the one hand, we have that $\varphi_n \to \mathrm{D}f(x;\xi)$ by the definition of chain differentiation for f, and on the other hand that $f(x+\theta_n\xi_n) = f(x) + \theta_n\varphi_n$. It follows that

$$\frac{1}{\theta_n}[h(x+\theta_n\xi_n)-h(x)] = \frac{1}{\theta_n}[g(f(x)+\theta_n\varphi_n)-g(f(x))] \to \mathrm{D}g(f(x);\mathrm{D}f(x;\xi))\,,$$

again by the definition of chain differentiation for g. ∎

We shall need to consider the concept of *restricted differential* (and *restricted derivative*):

Definition 2 *Let X and Y be two t.v.s., and $\Xi \subset X$ be a linear subspace of X. Let $f : X \to Y$. We call* differential of f retsricted to Ξ, *or if no ambiguity results* restricted differential *of f at x in a direction $\xi \in \Xi$ the differential at x in the direction ξ of the restriction of f to $x + \Xi$.*

Hence, a restricted differential is obtained by restricting the sequence $\{\xi_n\}$ in (1) to belong to Ξ.

The following theorem allows one to identify a Gâteaux differential with a chain differential, and will be used in the example of section 2.

Theorem 2 *Let X and Y be two t.v.s., $\Xi \subset X$ a subspace of X, and $f : X \to Y$ have a Gâteaux differential at all points in an open set $\Omega \subset X$ in all directions in Ξ. For all $(y,\xi) \in \Omega \times \Xi$, assume that $\mathrm{D}f(y;\xi)$ is continuous in y for every fixed ξ, and for some $x \in X$, that $\mathrm{D}f(y;\xi)$ is jointly continuous in $(y,\xi) \in X \times \Xi$ at $(x,0)$, i.e., that*

$$\forall x_n \to x\,, \forall \xi_n \to 0\, \mathit{in}\, \Xi\,, \quad \mathrm{D}f(x_n;\xi_n) \to 0\,. \tag{3}$$

Then $\mathrm{D}f(x;\xi)$ is a chain differential restricted to Ξ.

A simple way of using this theorem is via the simpler

Corollary 1 *If the Gâteaux differential $\mathrm{D}f(x;\xi)$ is jointly continuous in (x,ξ) over $\Omega \times \Xi$, it also is a continuous restricted chain differential.*

Proof of the theorem Let $\{\xi_n\} \subset \Xi$, and $\xi_n \to \xi \in \Xi$ and $\theta_n \to 0$. Using the finite difference formula for the function $F(t) := f(x + \theta_n\xi + t\theta_n(\xi_n - \xi))$, we have that there exists $\tau_n \in [0, 1]$ such that:

$$f(x + \theta_n\xi_n) = f(x + \theta_n\xi) + \mathrm{D}f(x + \theta_n\xi + \tau_n\theta_n(\xi_n - \xi); \theta_n(\xi_n - \xi)) \,.$$

Hence, using proposition 2,

$$\frac{1}{\theta_n}[f(x+\theta_n\xi_n)-f(x)] = \frac{1}{\theta_n}[f(x+\theta_n\xi)-f(x)]+\mathrm{D}f(x+\theta_n\xi+\tau_n\theta_n(\xi_n-\xi); \xi_n-\xi).$$

The first term in the right hand side converges to the Gâteaux derivative $\mathrm{D}f(x; \xi)$, and the second one to zero thanks to hypothesis (3). ∎

The same proof as for elementary derivatives also yields the following result, which can be adapted to restricted derivatives by replacing everywhere f by its restriction to a set of the form $(x, y) + \Xi \times \Upsilon$ for some linear subspaces Ξ and Υ of X and Y respectively.

Proposition 3 *Let X, Y, and Z be three t.v.s., and let $f : X \times Y \to Z$. Assume that both partial functions $x \mapsto f(x, y)$ and $y \mapsto f(x, y)$ have chain differentials $\mathrm{D}_1 f$ and $\mathrm{D}_2 f$ respectively, for all (x, y) in an open set $\Omega \subset X \times Y$ and in any direction, and assume that $\mathrm{D}_1 f$ is jointly continuous in its three arguments over $\Omega \times X$. Then f admits (in the product topology of $X \times Y$) a chain differential at any (x, y) in that domain, in any direction (ξ, η), given by*

$$\mathrm{D}f(x, y; \xi, \eta) = \mathrm{D}_1 f(x, y; \xi) + \mathrm{D}_2 f(x, y; \eta) \,.$$

Corollary 2 *Let X be a t.v.s., $F : X \to X$ a continuous function, $T \in \mathbb{R}^+$, and $x(\cdot) : [0, T] \to X$ be a differentiable function with derivative $\dot{x}(\cdot)$ satisfying*

$$\forall t \in [0, T] \,, \quad \dot{x}(t) = F(x(t)) \,.$$

Let also $V : \mathbb{R} \times X \to \mathbb{R}$ have a continuous partial derivative V_t in t and have a continuous chain differential $\mathrm{D}_2 V$ in x. Then, for all $t \in \mathbb{R}$,

$$V(t, x(t)) = V(0, x(0)) + \int_0^t [V_t(s, x(s)) + \mathrm{D}_2 V(s, x(s); F(x(s)))] \, ds \,. \quad (4)$$

Proof of the corollary According to theorem 1, the function $t \mapsto V(t, x(t))$ has a continuous derivative

$$\frac{\mathrm{d}V}{\mathrm{d}t}(s, x(s)) = V_t(s, x(s)) + \mathrm{D}_2 V(s, x(s); F(x(s))) \,.$$

And as it is a continuous real function over the compact set $[0, T]$, this derivative is bounded thus $V(t, x(t))$ is absolutely continuous, and the formula (4) follows.

This is the equation needed to perform Hamilton-Jacobi-Caratheodory (or Isaacs-Bellman) theory with state x and Value function V.

3 Example: mathematical fear

This example is motivated by the theory of minimax partial information control. See [4, 5, 6]. It rests upon the concept of “cost measures”, a particular class of Maslov measures introduced by Quadrat and co-workers. See [8, 9, 10].

Let $\mathcal{Q}$ be the set of *cost functions* $q : \mathbb{R}^n \to \mathbb{R}$ continuous, going to $-\infty$ as $x \to \infty$, and such that $\max_x q(x) = 0$. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be continuous. By definition, the *mathematical fear* of φ with respect to q is

$$\mathbb{F}^q \varphi := \mathbb{F}^q_x \varphi(x) := \sup_x [\varphi(x) + q(x)]\,.$$

This concept has been used to investigate partial information minimax control problems as a parallel development to stochastic control. We sketch that theory here with the notations of [6], to which we refer the reader for more details.

Let a control system in $\mathbb{R}^n$ be given by

$$\begin{aligned} \dot{x} &= f(t,x,u,w)\,, \quad x(0) = x_0\,, \\ y &= h(t,u,w)\,. \end{aligned}$$

Here, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathsf{U} \subset \mathbb{R}^m$ is the control, $w(t) \in \mathsf{W} \subset \mathbb{R}^\ell$ a disturbance and $y(t) \in \mathbb{R}^p$ a measured output. The function f is assumed to satisfy a set of regularity and growth hypotheses that insure existence and uniqueness of the *trajectory* $x(\cdot)$ for every measurable control and disturbance functions, ad h is assumed continuous. We also denote $\omega := (x_0, w(\cdot)) \in \Omega := \mathbb{R}^n \times L^1([0,T], \mathsf{W})$.

The problem at hand is to find a *non-anticipative* control strategy $u(t) = \mu(t, y(\cdot))$ that will minimize a performance index

$$H(\mu) = \mathbb{F}_\omega J(\mu(y(\cdot)), \omega) \tag{5}$$

defined via two C^1 real functions $M : \mathbb{R}^n \to \mathbb{R}$ and $L : [0,T] \times \mathbb{R}^n \times \mathsf{U} \times \mathsf{W} \to \mathbb{R}$ as

$$J(u(\cdot), \omega) = M(x(T)) + \sup_{t\in[0,T]} L(t, x(t), u(t), w(t))\,.$$

In the mathematical fear (5), the cost functions attached to x_0 and $w(\cdot)$, $Q(x_0)$ and $\int \Gamma(t, w(t))\mathrm{d}t$ respectively, are part of the data.

As a matter of fact, Γ may be taken as a function of the other variables also, so that the overall performance index considered is in fact of the form

$$H(\mu) = \sup_{\omega\in\Omega} \sup_{t\in[0,T]} \left[L(t,x(t),u(t),w(t)) + \int_0^t \Gamma(s,x(s),u(s),w(s))\,\mathrm{d}s + Q(x_0) \right].$$

But we shall stick here with the notation $\Gamma(t,w)$. See [6] for more details.

It is then showed that this problem may be solved via infinite dimensional Hamilton Jacobi Caratheodory Isaacs Bellman theory, the “state” being then the conditional state mathematical fear $Q(t,x)$ itself defined via the “worst conditional past cost function” $W(x)$ as

$$W(\xi) = \sup_{\omega} \left[\int_0^t \Gamma(s, w(s)) \, \mathrm{d}s + Q(x_0) \right]$$

where the supremum is taken among those ω that are compatible with the past controls and observations and lead to the current state $x(t) = \xi$. Then

$$Q(t,x) = W(t,x) - \max_{\xi \in \mathbb{R}^n} W(t,\xi) \,.$$

And we let $\widehat{X}$ be the set in $\mathbb{R}^n$ on which the max above is achieved.

Formally, Q satisfies the PDE given in terms of

$$\mathsf{V}(t,x|y) = [h(t,x,\cdot)]^{-1}(y)$$

and of the cost measure (denoted $\Lambda_t^\infty(y)$ in [6])

$$\Lambda(t,y) = \max_{x \in \widehat{X}} \max_{w \in \mathsf{V}(t,x|y)} \Gamma(t,w)$$

as

$$\frac{\partial Q(t,x)}{\partial t} = \max_{w \in \mathsf{V}(t,x|y)} \left[-\frac{\partial W(t,x)}{\partial x} f(t,x,u,w) + \Gamma(t,w) \right] - \Lambda(t,y) \,.$$

We let $q(t) = \{x \mapsto Q(t,x)\}$ and write the above PDE as

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \mathcal{G}(t, q(t), u(t), y(t)) \,. \tag{6}$$

With this system, we perform HJCIB theory, in terms of a performance function $U(t,q)$ satisfying formally the Hamilton Jacobi equation (we let $U_t := \mathrm{D}_1 U$)

$$\begin{aligned}
&\forall q \in \mathcal{Q}\,, \quad U(T,q) = \mathbb{F}^q M\,,\\
&\forall t \in [0,T]\,, \quad \forall q \in \mathcal{Q}\,,\\
&0 = \inf_{u \in \mathsf{U}} \max \{ U_t(t,q) + \mathbb{F}_y \mathrm{D}_2 U(t,q) \mathcal{G}(t,q,u,y), \mathbb{F}_{x,w} L(t,x,u,w) - U(t,q) \}.
\end{aligned}$$

Mathematical fears with respect to x are taken here with the cost function $q(t)$, or more explicitly $Q(t,x)$, with respect to w with the cost function $\Gamma(t,w)$, and with respect to y with the cost function $\Lambda(t,y)$.

The justification of that theory is based upon the possibility to use formula (4) for the function $U(t, q(t))$ with q satisfying the differential equation (6).

To investigate that question, we shall ignore the time dependance and investigate the differentiability of
$$U(q) = \mathbb{F}^q V \tag{7}$$
for a function $V(x)$ satisfying the condition (8) below.

We shall restrict our attention to a set up where the $\sup$ is always a $\max$. Let $V : \mathbb{R}^n \to \mathbb{R} : x \mapsto V(x)$ be a function continuously differentiable over $\mathbb{R}^n$, and let $\nabla V(x)$ be its gradient. Let us further assume that there exists a positive number α such that,

$$\forall (x, y) \in \mathbb{R}^n \times \mathbb{R}^n\,, \quad (\nabla V(x) - \nabla V(y), x - y) \le \alpha \|x - y\|^2\,. \tag{8}$$

Let $\mathcal{Q}_\alpha$ be the set of cost functions that are β-strongly concave for some $\beta > \alpha$. That is,

$$\begin{aligned}&\forall x, y \in \mathbb{R}^n\,, \quad \forall \lambda \in [0, 1]\,,\\ &q(\lambda x + (1-\lambda)y) \ge \lambda q(x) + (1-\lambda)q(y) + \beta \frac{\lambda(1-\lambda)}{2} \|x - y\|^2\,.\end{aligned}$$

This is known to be equivalent, if q is continuously differentiable, to the property

$$(\nabla q(x) - \nabla q(y), x - y) \le -\beta \|x - y\|^2\,.$$

We shall always let $\gamma = \beta - \alpha > 0$. Using (8), for any function $q \in \mathcal{Q}_\alpha$ the sum $V(\cdot) + q(\cdot)$ is γ-concave, hence it reaches its maximum over $\mathbb{R}^n$ at a unique point that we shall usually denote $\hat{x}$.

We shall embed our functions V and q in $\mathcal{C}$. Notice that $\mathcal{Q}$ is not a vector subspace, nor its subset $\mathcal{Q}_\alpha$. However, the set of p's such that for any $q \in \mathcal{Q}_\alpha$, for a small enough θ, $q + \theta p$ is β-concave with $\beta > \alpha$, is a vector space $\mathcal{P}$. (Its intersection with C^1 is the set of p's such that $(\nabla p(x) - \nabla p(y), x - y)/\|x - y\|^2$ is bounded over $\mathbb{R}^n$.) In particular, $\mathcal{P} \subset \mathcal{C}$. We shall restrict all differentials to $\mathcal{P}$.

Let $q \in \mathcal{Q}_\alpha$ and $p \in \mathcal{P}$. Let U be defined by (7). Let moreover $\hat{x}$ be the (unique) argmax in (7). We have

Proposition 4 *The function U has a Gâteaux derivative given by*

$$\mathrm{D}U(q) \cdot p = p(\hat{x}).$$

Proof It is clear that, if it exists, the Gâteaux differential is given by

$$\mathrm{D}U(q;p) = \frac{\partial}{\partial\theta}\left(\max_x[V(x)+q(x)+\theta p(x)]\right)\Big|_{\theta=0}.$$

Our hypotheses have been arranged so that for θ small enough the function $V + q + \theta p$ is γ-concave for some positive γ. Hence the max is reached at a unique $\hat{x}(\theta)$ and $\hat{x}(0) = \hat{x}$. Moreover, the derivative in θ for any fixed x is continuous in (θ and) x. Hence we are in the conditions of Danskin's theorem (see [7]). Hence $\mathrm{D}U(q;p) = p(\hat{x})$. Now, $\hat{x}$ only depends on q and not on p. Hence the operation of evaluation at $\hat{x}$ is linear, and continuous in $\mathcal{C}$. Thus there is a (Gâteaux) derivative. ∎

We shall now use the corollary 1 to show that

Theorem 3 *The function U in (7) has at any $q \in \mathcal{Q}_\alpha$ a chain derivative restricted to $\mathcal{P}$ in any direction $p \in \mathcal{P}$.*

Proof We have seen that the Gâteux differential is a derivative, i.e. continuous in p. The theorem follows from the following fact. Let $q_n \to q$ in $\mathcal{Q}_\alpha$. Let $\hat{x}_n$ be the argmax in $U(q_n) = \max_x[V(x) + q_n(x)]$. We have

Proposition 5

$$\hat{x}_n \to \hat{x} \qquad \textit{as} \quad q_n \to q\,.$$

As a matter of fact, if the proposition holds, then take any sequence $\{(q_n, p_n)\}$ in $\mathcal{Q}_\alpha \times \mathcal{P}$ converging to (q,p). We do have that $\mathrm{D}U(q_n)\cdot p_n = p_n(\hat{x}_n) \to p(\hat{x}) = \mathrm{D}U(q)\cdot p$, since $\hat{x}_n$ converging to $\hat{x}$, it remains in a compact subset containing $\hat{x}$ over which $p_n \to p$ uniformly in x.

Proof of the proposition Our proof of the proposition entails a minor restriction on the sequence q_n, which can be seen as harmless to apply the theorem[2]. Let us assume that all q_n behave for large enough $\|x\|$ as a $\bar{\gamma}$-concave function, for a uniform positive $\bar{\gamma}$. We shall use the following notation: for any positive number ε, let B_ε be the closed ball of center $\hat{x}$ and of radius ε. Let then r be a fixed positive number, and a and b be two numbers such that

$$\forall x \in B_r\,, \quad a < V(x) + q(x) < b\,.$$

Since $q_n \to q$ uniformly over B_r, it follows that the above inequalities hold with q_n instead of q for n large enough. Because we have assumed all the q_n to behave

[2]Alternatively, we may let q and the q_n's range over $\mathcal{Q}_{\bar{\alpha}}$ for some $\bar{\alpha} > \alpha$ and let $\bar{\gamma} = \bar{\alpha} - \alpha$.

as $\bar{\gamma}$-concave functions for large x, it follows that there is a large enough R such that

$$\|x-\hat{x}\| > R \Longrightarrow V(x)+q_n(x) < a\,.$$

(For instance, for a $\bar{\gamma}$-concave function, take $R = r + 2(b-a)/\bar{\gamma}$.) As a consequence, we conclude that $\hat{x}_n \in B_R$. Let $\varepsilon > 0$ (and $\varepsilon < R$). For all $x \notin B_\varepsilon$, we have $V(x)+q(x) < V(\hat{x})+q(\hat{x})-(\gamma/2)\varepsilon^2$. For n large enough, $|q_n(x)-q(x)| < (\gamma/4)\varepsilon^2$ for all $x \in B_R$. Hence, in B_R, $(V+q_n)(x) < (V+q)(x)+(\gamma/4)\varepsilon^2$. Hence, if $x \in B_R - B_\varepsilon$

$$V(x)+q_n(x) < V(x)+q(x)+\frac{\gamma}{4}\varepsilon^2 < V(\hat{x})+q(\hat{x})-\frac{\gamma}{4}\varepsilon^2 < V(\hat{x})+q_n(\hat{x})\,.$$

It follows that for n large enough, $\hat{x}_n \in B_\varepsilon$, and since ε was arbitrary, that indeed $\hat{x}_n \to \hat{x}$. ∎

4 Conclusion

Since the concept of t.v.s. is the one that seems well suited to deal with pointwise convergence, it was necessary to have a concept of derivative defined over a t.v.s. that yields the chain rule, which the classical Gâteaux derivative does not do, in order to have the relation (4) in the context of partial information minimax control, where the "state" of the system is a cost distribution satisfying a forward Hamilton-Jacobi-like evolution PDE. We have shown that the concept of "chain derivative" meets these requirements. Moreover, the theorem 2 gives a way of computing it easily by identifying it with a Gâteaux derivative when the later is suitably continuous. In the case of a mathematical fear operator, we can get that continuity with some care. Among other applications, it gives a solid footing to the sufficiency condition of [6] for the minimax control of a patially observed system. That this is not purely formal can be seen in the application of that theory made in [11].

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