A robust control approach to option pricing

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Abstract

We propose an approach of option pricing based upon game theory in a typical nonlinear robust control fashion. In this approach, one needs a description of the set of possible trajectories for the underlying stock's price process, but no probabilistic law over it.

We first quickly review the classical theory —albeit stripped of all references to probabilities—, mainly to underline the role of trajectory sets, including the bounded variation case and the "naive" hedging strategy it leads to.

Then we propose a purely (robust) control theoretic view of the problem, and solve it that way, both for continuous trading and for discrete time trading. At this time, we consider only "european" options, that is a security whose value is defined at a fixed terminal time T.

The new theory exists in both continuous and discrete time. In continuous time, we are at present restricted to a bounded variation set up, recovering the naive theory. We expect to get a more robust result once we include transaction costs, a work currently in progress.

In discrete time, the model proposed, although strongly reminiscent — if the security to be replicated is convex— of the theory of Cox, Ross, and Rubinstein, has the advantage of being realistic in its finite step version, and not only in the limit as the step goes to zero. As a matter of fact, it does not assume that the stock price evolves on a binary tree, but to the contrary, allows for a continuum of possible values at the end of each time period. It provides us with an efficient discrete time hedging strategy against that continuum of possible outcomes.

We show that in the convex case, the limit of the discrete time theory gives back either the "naive" solution or the classical Black and Scholes solution depending on whether we chose a bounded variation or a bounded quadratic variation function for the underlying stock's price history.

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1 Introduction

The crucial point in Black and Scholes's theory of option pricing [2] is that of finding a portfolio together with a self financed trading strategy that insures the same return as the option to be priced. Hence, if no riskless profitable arbitrage is to exist in that market, the price of the option should be equal to that of that portfolio.

What is requested is that the portfolio and strategy constructed replicate the option, i.e. yield the same payment to the owner *for all possible outcomes of the underlying stock's value*. As has been stressed by several authors, this statement is not in terms of probabilities, and therefore the precise (probabilist) model adopted for the stock's price should be irrelevant. As a matter of fact, it is known that if one adopts a "geometric diffusion" model

$$\frac{\mathrm{d}S}{S} = \mu\mathrm{d}t + \sigma\mathrm{d}\nu\,, \tag{1}$$

$\nu(t)$ a Weiener process, then the famous Black and Scholes equation and formula do not contain μ. In our formulation, μ just does not appear in the problem statement. We shall further argue that the volatility σ appears only as a characteristic of the set of allowable histories $S(\cdot)$, not as a probabilistic entity.

We first show an elementary theory that emphasizes this point, and let us discuss a zero volatility, but yet stochastic —in the sense that $S(\cdot)$ is a priori unknown and can thus be thought of as stochastic— model. This model leads, in continuous time, to a naive hedging strategy, which lacks robustness against transaction costs. This naive theory is usefull to set the stage for further discussion downstream, but apart from that, the fact that it fails in discrete time only serves the purpouse of stressing its limits.

The only novelty we claim in this note, if any, is to take a control theory viewpoint where the value of a portfolio is seen as a dynamical system, influenced by two exogeneous inputs : the underlying stock's price and the trading strategy of the owner. With this viewpoint, it is only natural to reinterpret the goal of replication as one of controllability in the presence of disturbances. Now, a recent trend in control theory is to deal with uncertain disturbances with little modeling, but trying to insure a desired outcome against all possible disturbance histories within a prescribed set of possible such histories. This leads to a dynamic game approach (and the theory of "capturability" [5, 1]). This is what we propose to do here.

We first investigate a continuous time theory, using the theory of differential games. If the resulting game has some interest from a game theoretic viewpoint, in its present form that theory only recovers the naive one. We expect to be able to make it into a more practical theory by introducing transaction costs into the model, a work currently in progress.

We finally investigate the discrete time case. When the terminal payment attached to the contingent claim considered is a convex function of the underlying stock's price at that time, the new approach leads to a theory strongly reminiscent of that of Cox, Ross, and Rubinstein [3, 4], but with the advantage that now the model is more realistic even with a finite time step —and not only in the limit as the time step goes to zero—, since we do not have to assume that the underlying stock's price evolves on a binary tree. We allow for a continuum of possible prices at the end of each time step, and yet find a hedging strategy. (It even gives a way of recovering at each time step part of the price paid for the portfolio to hedge against past risks that did not materialize.) When the terminal payment is not convex, our theory still yields a numerical solution to the pricing problem and a hedging strategy. The binary tree approach in that case underestimates the price of the claim.

The limit of the discrete time theory can be made, as in [3], to coincide with the classical Black and Scholes theory, or with the "naive" theory, depending on what class of stock price trajectories we choose to approach.

2 Problem formulation

The time variable, always denoted t, ranges, depending on the model, either over a continuous time interval $[0, T]$ of the real line or over the integers $\{0, 1, \ldots, T\}$, or over the discretized time interval $\{0, h, 2h, \ldots, Nh = T\}$

A given stock is assumed to have a time dependant, unpredictable, market price $S(t)$ at time t. An important part of the discussion, both in discrete and continuous time, will be the set Ω of possible time functions $S(\cdot)$ assumed.

There also exists in that economy a riskless bond, the value of one unit of which at time t is $R(t)$, characterized by $R(T) = 1$ and its rate ρ. Thus $R(t)$ is a pesent value factor. In continuous time,

$$R(t) = \mathrm{e}^{\rho(t-T)}\,,$$

and in discrete time,

$$R(t) = (1+\rho)^{t-T} = r^{t-T}\,,$$

where we have set

$$1 + \rho = r\,.$$

(In the "discretized time interval" case, we may choose either of the two models

$$R(kh) = \mathrm{e}^{\rho(k-N)h} \qquad \text{or} \qquad R(kh) = r^{(k-N)h}\,,$$

with $N = T/h$, without altering the limiting results.)

We are interested in replicating a security whose value at time T is a given function M of $S(T)$. As is well known, in the case of a call of striking price K, $M(s) = \max\{s - K, 0\}$, that we shall write $M(s) = [s - K]_+$, and in the case of a put, $M(s) = [K - s]_+$. In these cases, M is convex and has a finite growth rate at infinity, a case of interest later.

We shall consider a portfolio made of x shares of the stock and y riskless bonds. Its value at time t is thus

$$w(t) = x(t)S(t) + y(t)R(t)\,.$$

We shall consider trading strategies of the form

$$x(t) = \varphi(t, S(t))\,, \tag{2}$$

and discuss the choice of the function, or strategy, φ. In discrete time, this is a natural concept : the owner of the portfolio watches a market price $S(t)$ at time t, and essentially simultaneously buys or sells the necessary equities to hold $\varphi(t, S(t))$ shares, that he or she keeps until the next trading time $t+1$. In continuous time, deciding whether it is feasible to instantly and continuously implement such a strategy is more debatable. In line with all feedback theory, we shall accept that concept, stressing that we shall allways restrict the functions $S(\cdot)$ to a set Ω of continuous functions so that it is mathematically unambiguous.

The trading of the riskless bonds will always be decided, beyond the initial time, by the requirement that the portfolio be self financed. Hence, the amount of shares bought at time t, $\mathrm{d}x(t)$ in continuous time or $x(t) - x(t-1)$ in discrete time, at the price $S(t)$ should exactly balance the amount of trading in the bond, $\mathrm{d}y(t)$ or $y(t) - y(t-1)$ at the price $R(t)$. Therefore we should have either

$$S(t)\mathrm{d}x(t) + R(t)\mathrm{d}y(t) = 0\,, \tag{3}$$

or

$$S(t)(x(t) - x(t-1)) + R(t)(y(t) - y(t-1)) = 0\,. \tag{4}$$

As a result, inherent in our models will be that, in continuous time

$$\mathrm{d}w(t) = x(t)\mathrm{d}S(t) + y(t)\mathrm{d}R(t)\,,$$

or, using the fact that $\mathrm{d}R(t) = \rho R(t)\mathrm{d}t$ and $y(t)R(t) = w(t) - x(t)S(t)$,

$$\mathrm{d}w(t) = x(t)\mathrm{d}S(t) + \rho(w(t) - x(t)S(t))\mathrm{d}t\,. \tag{5}$$

The use we shall make of that "differential" form will be made clear later.

In discrete time, we get, using (4),

$$w(t) = S(t)x(t-1) + R(t)y(t-1)\,,$$

therefore, using the discrete time definition of $R(t)$, and shifting all time instants by one, we shall use either the form

$$w(t+1) = (S(t+1) - S(t))x(t) - \rho x(t)S(t) + (1+\rho)w(t) \tag{6}$$

or the form

$$w(t+1) = S(t+1)x(t) + r(w(t) - x(t)S(t))\,. \tag{7}$$

3 An elementary approach

3.1 Continuous time

3.1.1 Bounded variation : the naive theory

In this section, we assume that the set Ω of possible market price histories $S(\cdot)$ is that of continuous bounded variation positive functions. One possible instance would be a stochastic process driven by (1), with a *stochastic* drift μ and zero volatility σ. In spite of its zero volatility, this can be a very unpredictable stochastic process, of very high frequency, depending upon the stochastic process μ. But we shall not need that interpretation.

We want to find a function $W(t,s)$ and a trading strategy $\varphi(t,s)$ such that the use of (2) will lead to $w(t) = W(t,S(t))$ for all $t \in [0,T]$, and this for all $S(\cdot) \in \Omega$. If this is possible, and if W is of class C^1, we must have

$$\mathrm{d}W(t) = \frac{\partial W}{\partial s}(t,S(t))\mathrm{d}S(t) + \frac{\partial W}{\partial t}(t,S(t))\mathrm{d}t = \mathrm{d}w(t) \tag{8}$$

where $\mathrm{d}w(t)$ is to be taken in (5), and the differential calculus is to be taken in the sense of Stieltjes. We have a way to make this hold for every $S(\cdot) \in \Omega$ by equating the terms in $\mathrm{d}S$ through the choice

$$x(t) = \frac{\partial W}{\partial s}(t,S(t))\,, \tag{9}$$

and further equating the remaining terms in $\mathrm{d}t$ yields

$$\rho(w(t) - x(t)S(t)) = \frac{\partial W}{\partial t}(t,S(t))\,.$$

Using the previous equality again, we see that this will be satisfied if, $\forall(t,s)$,

$$\frac{\partial W}{\partial t}(t,s) - \rho W(t,s) + \rho s \frac{\partial W}{\partial s}(t,s) = 0\,. \tag{10}$$

If this partial differential equation has a solution that furthermore satisfies

$$\forall s \in \mathbb{R}^+ \quad W(T,s) = M(s)\,, \tag{11}$$

then a portfolio of total value $W(0,S(0))$ at time 0, driven by the strategy thus computed indeed replicates the security considered. An equilibrium price for the option in that model is thus $W(0,S(0))$.

The unique solution of the PDE (10)(11) is the (discounted) *parity value* :

$$W(t,s) = \mathrm{e}^{\rho(t-T)} M(\mathrm{e}^{\rho(T-t)} s)\,. \tag{12}$$

Take the case of a call say. The associated naive hedging srategy is just $x = 0$ if the call is out of the money, $S(t) < \exp(\rho(t-T)K)$, and $x = 1$ otherwise. This naïve strategy is easily seen to be indeed self financed, and replicating the call.

Its bad feature is in case the price of the underlying stock oscillates close to the discounted value of the striking price. Then the owner is perpetually in doubt as to whether the price will go up, and he must buy a share, or down and he must not. Any friction, like transaction costs, ruins that strategy. We argue that the weakness of that model, which would yield an essentially free insurance mechanism, is in the fact that we ignored transactions costs in the model, *not* in the choice of Ω, which may be more realistic than the more classical next choice.

3.1.2 Fixed quadratic relative variation

Assume now that the set Ω of allowable price processes still contains continuous positive functions, but now of unbounded total variation, and all of quadratic relative variation

$$\lim \sum_{k=1}^{N} \left(\frac{S(t_{k+1}) - S(t_k)}{S(t_k)} \right)^2 = \sigma^2 t$$

the limit being taken as the division $0 = t_0 < t_1 < \ldots < t_N = t$ has its diameter —the largest interval $t_{k+1} - t_k$— that goes to zero. (And thus N correlatively goes to infinity.) Allmost all trajectories generated by (1) have that property. But the drift has no effect on the set of possible trajectories, and hence does not appear here.

For this class of functions, we have the following lemma (see appendix) :

Lemma 1 *Let $V(t,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Let $S(\cdot) \in \Omega$ and assume that $(\partial V/\partial s)(t, S(t)) = 0$ for all $t \in [0,T]$. Then $\forall t \in [0,T]$,*

$$V(t,S(t)) = V(0,S(0)) + \int_0^t \left(\frac{\partial V}{\partial t}(\tau, S(\tau)) + \frac{\sigma^2}{2} S(\tau)^2 \frac{\partial^2 V}{\partial s^2}(\tau, S(\tau)) \right) d\tau \, .$$

We apply the lemma to $V(t,s) = W(t,s) - x(t)s - y(t)R(t)$, trying to keep $V(t,S(t))$ equal to zero. First insure that $(\partial V/\partial s)(t,S(t)) = 0$ through the choice (9). Assume that this will lead to a differentiable $x(\cdot)$, thus also $y(\cdot)$. Then (3) yields $\dot{x}S + \dot{y}R = 0$. Finally, using again $y\dot{R} = \rho(w - xS)$, we insure that $V(t,S(t))$ remains constant along any trajectory if, $\forall (t,s) \in [0,T] \times \mathbb{R}^+$,

$$\frac{\partial W}{\partial t} - \rho W + \rho s \frac{\partial W}{\partial s} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 W}{\partial s^2} = 0 \, ,$$

which, togeher with (11), is Black and Scholes'equation.

It is a simple matter to check that its famous solution indeed converges to (12) as $\sigma \to 0$, and that the corresponding hedging strategy also converges to the naive strategy.

As a final remark, let us point out that extending that elementary theory to a claim contingent on the value of *several* underlying securities is straightforward.

3.2 Discrete time

We quickly look at what that approach leads to in discrete time. Starting from (7), we see that we would like to find a function $W(t,s)$ and a strategy $\varphi(t,s)$ such that

$$\begin{aligned} &\forall S, S' \in \mathbb{R}^+, \quad \forall t \in \{1, \ldots, T\} \, , \\ &W(t+1, S') = \varphi(t,S)S' + r(W(t,S) - \varphi(t,S)S) \, . \end{aligned}$$

The right hand side of the above equality is affine in S', thus so should be the left hand side, i.e. $W(t+1,s) = A_{t+1}s + B_{t+1}$, and $A_{t+1} = \varphi(t,s)$, hence φ should be independant from s. Placing this back, for time t, in the above equation, we get

$$B_{t+1} = r[(A_t - A_{t+1})S + B_t] \, .$$

Since B_t cannot depend on S, we see that necessarily, $A_t = A_{t+1}$, i.e. A_t is a constant, say A, and $B_{t+1} = rB_t$, i.e. $B_t = r^{t-T}B$ for some constant B. As a consequence, the only possible form for the final value of the security considered is $M(s) = As + B$, totally uninteresting.

Notice that in that approach, limiting the range of variations of $S(t+1)$ given $S(t)$ would not help. It is why in their discrete time theory, Cox, Ross, and Rubinstein [3], are obliged to limit the variations of S to exactly two possible values, leading to the now classical binary tree. A model which can be considered hardly realistic as a normative one, except in the limit as the step size goes to zero.

4 The robust control approach

4.1 A controllability viewpoint

We observe that the dynamic equation governing $w(t)$ defines a dynamical system with inputs $S(\cdot)$ and $x(\cdot)$, and output $w(\cdot)$. A portfolio will be at least as good as the option, and therefore defines an upper bound to the equilibrium price, as soon as we can find a strategy where $x(t)$ depends only on *past and present information*, that leads to a final value greater or equal to that of the option for all admissible disturbances $S(\cdot)$. (This controllability property *against all admissible disturbances* is called *capturability*).

We therefore need to know which initial portfolios, if any, can thus be driven to a set of "admissible" terminal states. A good definition of the equilibrium price of the option is *the least expensive* initial portfolio that enjoys that property. As a matter of fact, we have already argued that, being controllable to the desirable terminal condition, it sets an upper bound on the equilibrium price. But in addition, if any cheaper portfolio might be faced with possible price histories $S(\cdot)$ that prevent its being driven to that desirable state, then the option is more valuable than any of those cheaper portfolios. It can therefore not be lower priced than that limiting "capturable" portfolio.

It is easily seen that if we set no restriction in the admissible time functions $S(\cdot)$, there is likely to be no capturable set. The classical way of imposing such restrictions is by representing the said function as the output of a controlled system, the simplest of all being of first order. We are thus led to the introduction of a second dynamic equation, that we may choose of the form

$$\dot{S}(t) = (\mu + \sigma\nu(t))S(t)\,, \quad |\nu(t)| \leq 1\,, \tag{13}$$

in continuous time, with $\nu(\cdot)$ measurable —this implies that we restrict $S(\cdot)$ to absolutely continuous, bounded variation, positive functions—, or

$$S(t+1) = (1 + \mu + \sigma\nu(t))S(t) = (m + \sigma\nu(t))S(t)\,, \quad |\nu(t)| \leq 1 \tag{14}$$

in discrete time. In these two models, $\nu(\cdot)$ is a "noise" signal that only serves the purpose of defining a set of admissible disturbances $S(\cdot)$, not a brownian motion. It ranges over the continuous interval $[-1,+1]$, and *not* over the finite set $\{-1,+1\}$. For that reason, we shall let, in the discrete time analysis,

$$m-\sigma=a\,,\qquad m+\sigma=b\,,$$

instead of the "d" and "u" of [3], to distinguish our model from theirs. But they shall play a very comparable role in the subsequent calculations.

We have set, or shall also use, the definitions

$$1+\mu=m\,,\quad \mu-\rho=m-r=\lambda\,.$$

The standard approach is then to consider the two equations for S and w as a dynamical system, ν as the disturbance, x as the control,

$$\mathcal{A}_T=\left\{\left(\begin{array}{c}S\\ w\end{array}\right)\in(\mathbb{R}^+)^2\;\middle|\;w\geq M(S)\right\} \tag{15}$$

as the set of admissible terminal states.

The relationship with game theory is best displayed by noticing that the prescription

$$\forall\nu(\cdot)\,,\quad\left(\begin{array}{c}S(T)\\ w(T)\end{array}\right)\in\mathcal{A}_T$$

can also be written

$$\inf_{\nu(\cdot)}[w(T)-M(S(T))]\geq 0\,, \tag{16}$$

and the existence of an admissible strategy φ that insures (16) is equivalent to

$$\max_{\varphi}\inf_{\nu(\cdot)}[w(T)-M(S(T))]\geq 0$$

at least if the maximum exists. No wonder then that the methods of robust control be the same as those of dynamical games.

4.2 Continuous time

We are therefore faced with the task of finding the initial states that can be controlled to the set (15) in the dynamical system

$$\begin{aligned}\dot S &= (\mu+\sigma\nu(t))S\,,\quad |\nu(t)|\leq 1\,,\\ \dot w &= \rho w+(\lambda+\sigma\nu(t))Sx(t)\,,\end{aligned}$$

where we choose $x(t)$ as our control.

It is known that the solution of this “qualitative game” (or “game of kind”) can be sought through the backwards construction of a barrier from the boundary of the (usable part of the) capture set. It turns out that that construction presents an unusual feature, interesting from a game theoretic viewpoint. We choose here to skip that question, and directly exhibit a solution.

We claim that a barrier is the 2D manifold parametrized by t and S as $w = W(t,S)$ where $W(\cdot,\cdot)$ is given by (12). Let us directly check this fact. At a point (t,S,w) of this manifold, we compute a normal pointing in the direction of larger w's. We use $R = \exp(\rho(t-T))$. The normal n is given by

$$n(t,S,w) = \begin{pmatrix} n_t \\ n_S \\ n_w \end{pmatrix} = \begin{pmatrix} \rho[-RM(S/R) + M'(S/R)S] \\ -M'(S/R) \\ 1 \end{pmatrix}$$

then we form

$$H = n_t + n_S\dot{S} + n_w\dot{w} = (\lambda + \sigma\nu)S(x - M'(S/R))\,.$$

Now, $\lambda < \sigma$, so that ν has the choice of the sign of $(\lambda+\sigma\nu)$, so that $\inf_\nu H$ is nonpositive. Therefore, $\max_x \inf_\nu H$ is obtained for $x = M'(S/R)$, and is indeed zero. Therefore the manifold $w = W(t,S)$ is indeed a barrier. The choice of strategy $x(t) = M'(S(t)/R(t))$ for any state lying on this manifold will prevent the state of the system from crossing it. (Actually, starting on the manifold, the state will stay on it for any $\nu(\cdot)$. This is, in disguise, the same argument as in the naive theory.)

Take the case of a call, indeed $x = M'(S/R)$ is the naive hedging strategy we have discussed. In the parlance of differential games, the locus $S(t) = KR(t)$ is a pursuer dispersal line. That is, when the state of the system is on that locus, the pursuer, here the disturbance, can behave the way it wants. The evader, here the portfolio's owner, must adapt and play $x = 0$ if S goes down, and $x = 1$ if S goes up. We have already noticed that in this dilema lies the unrealistic character of that strategy from a practical perspective.

4.3 Discrete time

4.3.1 Isaacs'equation

The system we now consider is governed by

$$\begin{aligned} S(t+1) &= (m + \sigma\nu(t))S(t)\,, \\ w(t+1) &= rw(t) + (\lambda + \sigma\nu(t))S(t)x(t)\,. \end{aligned}$$

It is convenient to make the following change of variables

$$u(t) = \frac{S(t)}{R(t)}\,, \qquad v(t) = \frac{w(t)}{R(t)}\,,$$

and to take the alternate control variables

$$\tau(t) = \frac{1}{r}(m+\sigma\nu) \in [\tilde{a} \quad \tilde{b}] = \left[\frac{a}{r} \quad \frac{b}{r}\right]\,, \qquad \psi(t) = x(t)u(t)\,.$$

Then the system is simply

$$u(t+1) = \tau(t)u(t)\,, \tag{17}$$

$$v(t+1) = v(t) + (\tau(t)-1)\psi(t)\,. \tag{18}$$

The set of admissible terminal states is still

$$\mathcal{A}_T = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in (\mathbb{R}^+)^2 \,\middle|\, v \geq M(u) \right\}\,. \tag{19}$$

Let $\mathcal{A}_t$ be the set of states at time t capturable to $\mathcal{A}_T$ at terminal time. i.e., $(\bar{u},\bar{v}) \in \mathcal{A}_t$ if and only if there exists an admissible strategy

$$\psi(t') = \varphi(t', u(t'), v(t'))\,, \quad t' \geq t\,,$$

such that the system (17)(18) initialized at $(u(t), v(t)) = (\bar{u},\bar{v})$, driven by that strategy ends in an admissible state at time T for all admissible disturbances $\tau(\cdot)$.

We notice first that if a state $(\bar{u},\bar{v})$ belongs to $\mathcal{A}_t$, then clearly so do all $(\bar{u}, v)$'s for $v \geq \bar{v}$. Let therefore

$$V_t(u) = \min\left\{ v \,\middle|\, \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{A}_t \right\}\,.$$

The function $V_t(\cdot)$ completely describes the set $\mathcal{A}_t$ as its epigraph.

We shall now proceed backwards, by dynamic programming. The sequence of sets $\mathcal{A}_t$ and the hedging strategy $\psi(t) = \varphi(t, S)$ are simultaneously defined by Isaacs'equation:

$$1\!\!1_{\mathcal{A}_t}(u,v) = \max_{\psi}\inf_{\tau} 1\!\!1_{\mathcal{A}_{t+1}}(\tau u, v + (\tau-1)\psi) \tag{20}$$

where

$$1\!\!1_{\mathcal{A}_{t+1}}(u', v') = \begin{cases} 1 & \text{if } v' \geq V_{t+1}(u')\,, \\ 0 & \text{if } v' < V_{t+1}(u')\,, \end{cases}$$

and

$$V_t(u) = \min\{v \mid 1\!\!1_{\mathcal{A}_t}(u,v) = 1\}\,. \tag{21}$$

This actually provides us with a computational procedure thats lets us compute $V_0(S)$ for every admissible $S(0) = S$, and thus a valuation formula for the contingent claim.

4.3.2 The convex case

Notice now that if $M(\cdot)$ is convex, so is the set $\mathcal{A}_T$. Furthermore, for a given state $(u(T-1), v(T-1))$ and a fixed $\psi(T-1)$, the set of possible $(u(T), v(T))$ is a line segment in (u, v) space. It is entirely contained in the convex set $\mathcal{A}_T$ if and only if its end points are. Therefore we investigate for which (u, v) there exists a fixed ψ such that

$$v + (\tilde{a} - 1)\psi \leq M(\tilde{a}u)\,, \qquad v + (\tilde{b} - 1)\psi \leq M(\tilde{b}u)\,.$$

Remember that $\tilde{a} < 1 < \tilde{b}$, so that the above pair of inequalities is satisfied provided that

$$\frac{1}{\tilde{b} - 1}[-v + M(\tilde{b}u)] \leq \psi \leq \frac{1}{1 - \tilde{a}}[v - M(\tilde{a}u)]\,.$$

There exists such a ψ if and only if

$$\frac{1}{\tilde{b} - 1}[-v + M(\tilde{b}u)] \leq \frac{1}{1 - \tilde{a}}[v - M(\tilde{a}u)]\,,$$

or equivalently, $v \geq V_{T-1}(u)$, with

$$V_{T-1}(u) = \frac{\tilde{b} - 1}{\tilde{b} - \tilde{a}} M(\tilde{a}u) + \frac{1 - \tilde{a}}{\tilde{b} - \tilde{a}} M(\tilde{b}u)\,.$$

This therefore describes the set $\mathcal{A}_{T-1}$. Now, it is a simple matter to check that, due to the convexity of M, the function V_{T-1} above is still convex, (notice for that purpose that u is the convex combination of $\tilde{a}u$ and $\tilde{b}u$ with the respective coefficients $(\tilde{b} - 1)/(\tilde{b} - \tilde{a})$ and $(1 - \tilde{a})/(\tilde{b} - \tilde{a})$) and therefore also the set $\mathcal{A}_{T-1}$.

As a consequence, the same calculation can be propagated backwards, defining a sequence of functions $V_t(\cdot)$ through

$$V_{t-1}(u) = \frac{\tilde{b} - 1}{\tilde{b} - \tilde{a}} V_t(\tilde{a}u) + \frac{1 - \tilde{a}}{\tilde{b} - \tilde{a}} V_t(\tilde{b}u)\,, \qquad V_T(u) = M(u)\,. \tag{22}$$

It is convenient to get back in the original variables, and to set

$$W(t, S) = R(t)V_t(S/R(t)) = r^{t-T} V_t(r^{T-t} S)$$

and to rewrite the above recursion as

$$W(t-1, S) = \frac{1}{r}\left[\frac{b - r}{b - a} W(t, aS) + \frac{r - a}{b - a} W(t, bS)\right]\,, \qquad W(T, S) = M(S)\,, \tag{23}$$

which is the same as in [3], equation (2) or (4). This recursion enables one to numerically compute a sequence of functions $W(t,S)$ for a given $M(\cdot)$. And it yields an equilibrium price $W(0,S(0))$ for the security considered.

Notice that, if we are to trade bonds for shares of the underlying stock a finite number of times, i.e. at discrete instants of time, this model is the relevant one. As we shall see, it gives a higher price for the option than that obtained for any of the continuous limits we shall consider. The difference is the price to be paid for less attentive behaviour. However, part of this price may be recovered through careful trading if the risk it was meant to hedge against does not materialize. Here is how.

Assume that at time $t-1$, $w = W(t-1,S)$, i.e. we are on the boundary of admissible states, the portfolio's current value is the minimum one if we are to hedge against all admissible disturbances in the future stock's price. We use the strategy —the portfolio composition— advised by the present theory (explicitly given below). Now, assume also that the price $S(t)$ that happens to materialize is not an extreme one, but interior to the admissible interval $[aS(t-1), bS(t-1)]$. Then, in general (this is not always so), we shall get that the difference $\Delta w(t) = w(t) - W(t,S(t))$ will be strictly positive. Therefore, an admissible hedging strategy is to first sell equities in the portfolio for an amount equal to that difference, and being then back to a worth $w(t^+) = W(t,S(t))$, choose a portfolio composition according to the now unique hedging strategy easily derived from the above analysis :

$$x(t)S(t) = \frac{1}{b-a}[W(t+1,bS(t)) - W(t+1,aS(t))]\,.$$

The complement $y(t)R(t) = w(t) - x(t)S(t)$ is thus :

$$y(t)R(t) = \frac{1}{b-a}[\frac{b}{r}W(t+1,aS(t)) - \frac{a}{r}W(t+1,bS(t))]\,.$$

It might be interesting to notice that the amount $\Delta w(t)$ recovered at each step by selling the excess portfolio value is directly related to the convexity of the successive $W(t,\cdot)$, hence of $M(\cdot)$, as is shown by the following formula. (Write S^- and S^+ for $\tilde{a}S(t-1)$ and $\tilde{b}S(t-1)$ respectively to make the formula more legible.)

$$\Delta w(t) = \frac{(S^+ - S(t))(S(t) - S^-)}{S^+ - S^-}\left(\frac{W(t,S^+) - W(t,S(t))}{S^+ - S(t)} - \frac{W(t,S(t)) - W(t,S^-)}{S(t) - S^-}\right).$$

An interesting question, then, would be to evaluate the expectation of the (discounted) sum of these benefits if ν is, say, evenly distributed between -1 and 1.

4.3.3 Non convex contingent claims

We relied in the above theory on the convexity of the function $M(s)$ giving the payoff at terminal time. The natural question then is what if this payoff is non convex, such as a spread : $M(s) = \min\{\max\{0, s-K\}, L\}$, or a "digital call" such as $M(s) = L(1 + \mathrm{sign}(s-K))/2$. Then, the fact that the endpoints of the segment described by (17)(18) be in $\mathcal{A}_{t+1}$ is no longer sufficient to guarantee that (u_{t+1}, v_{t+1}) be in that set for all possible $\nu(t)$. Therefore, that theory alone, i.e. the Cox, Ross and Rubinstein valuation formula, *underestimates* the value of the replicating portfolio, and of course gives no hint as to a hedging strategy.

In that case, we must revert to numerical implementation of the Isaacs equation (20)(21). For each time step t, we traverse the state space in u, and for fixed u perform a dichotomic search for the lowest admissible v. That way, if one accepts our model of $S(\cdot)$, which is at least more realistic than a binary tree, the present theory can be used to compute both a hedging strategy and a valuation. The computational load remains light for classical contingent claims based upon a single underlying good.

We give hereafter the results of computations that display the disparity between the Cox Ross and Rubinstein valuation and the new one proposed here.

4.4 Vanishing step size

4.4.1 The general framework

We wish now to investigate the case of vanishing step sizes. In [3], the solution of (23) in the case of a call is explicitly computed, and its limit as the step size goes to zero is computed, with two limit procedures : one corresponding to a geometric diffusion process for $S(\cdot)$, and the other one to a jump process. Here, in keeping with our previous theory, we shall consider the case of a continuous bounded variation function and that of a continous, fixed quadratic variation, one. However, we choose a path that lets us deal with a more general contingent claim characterized by its convex terminal value $M(S(T))$, assumed to have bounded growth rate at infinity. We shall deal elsewhere with non convex claims.

In both cases, we need to let a and b, or $\tilde{a}$ and $\tilde{b}$, depend on the step size h. We denote them a_h and b_h, or $\tilde{a}_h$ and $\tilde{b}_h$. We call $V_k^{(h)}(\cdot)$ the function obtained from the recursion (22), with $\tilde{a}_h$ and $\tilde{b}_h$ in place of $\tilde{a}$ and $\tilde{b}$, for the time $t = kh$. Therefore, the "initialization" of that recursion at terminal time is now given by

$$V^{(h)}_{\frac{T}{h}}(u) = M(u)\,. \tag{24}$$

We wish to obtain

$$V_0^0(u) = \lim_{h\to 0} V_0^{(h)}(u)\,,$$

or, for that matter,

$$V_t^0(u) = \lim_{h\to 0} V_{\frac{t}{h}}^{(h)}(u)\,,$$

for any fixed t. Our method will be to show that the recursion (22) approximates the solution of a partial differential equation, Black and Scholes' in the fixed nonzero quadratic variation case. To do so, notice that using (22) and substracting $V_k^{(h)}(u)$ from both sides, we get

$$\begin{aligned}&\tfrac{1}{h}(V_{k-1}^{(h)}(u) - V_k^{(h)}(u)) = \\ &\frac{\tilde{b}_h - 1}{\tilde{b}_h - \tilde{a}_h}\frac{1}{h}\left[V_k^{(h)}(\tilde{a}_h u) - V_k^{(h)}(u)\right] + \frac{1-\tilde{a}_h}{\tilde{b}_h - \tilde{a}_h}\frac{1}{h}\left[V_k^{(h)}(\tilde{b}_h u) - V_k^{(h)}(u)\right].\end{aligned} \tag{25}$$

We also notice the following easy facts :

Propositions

1. The application $M \mapsto V_t(\cdot)$, or equivalently $M \mapsto V_k^{(h)}(\cdot)$ defined by (22), is a contraction in the large for the distance of uniform convergence.
2. If $M \in C^1(\mathbb{R}^+)$, and $|M'(u)| \le C_1$ for all u, then all the $V_k^{(h)}(\cdot)$ have first derivatives bounded by C_1.
3. If $M \in C^2(\mathbb{R}^+)$, and $|M"(u)| \le C_2$ for all u, then all the $V_k^{(h)}(\cdot)$ have second derivatives. They are uniformly bounded by a number D provided that
$$\forall h, \quad (\tilde{a}_h + \tilde{b}_h - \tilde{a}_h\tilde{b}_h)^{T/h} C_2 \le D\,. \tag{26}$$

Let $M_\varepsilon(\cdot)$ be a convex function, of class C_2 for each fixed epsilon, approaching M uniformly as ε goes to zero as

$$\forall u \in \mathbb{R}^+\,, \quad |M_\varepsilon(u) - M(u)| \le \varepsilon\,.$$

Such an M_ε exists because we have assumed that M has a finite growth rate at infinity. We can consider the recursion (22) initialized by $V_{T/h}^{(h)} = M_\varepsilon$ instead of (24). Because of proposition (1) above, each $V_k^{(h)}(u)$ lies within ε of the corresponding one when the recursion is initialized at M. Therefore so does their limit $V_t^0(u)$. We may therefore evluate this limit for a C^2 terminal value M_ε and let that function converge to M in the resulting expression.

4.4.2 Bounded variation

For the sake of completeness, we first examine the case where the sequence $\{S(t)\}$ is requested to approach a bounded variation function. This is done by choosing

$$\tilde{a}_h = \tilde{a}^h \quad \text{and} \quad \tilde{b}_h = \tilde{b}^h \,.$$

In that way, the amplitude of admissible variations of S over a fixed time interval is preserved for all h. Notice then that

$$(\tilde{a}_h + \tilde{b}_h - \tilde{a}_h\tilde{b}_h)^{T/h} \to 1$$

as $h \to 0$ so that it is indeed bounded.

Consider equation (25) with the above $\tilde{a}_h$ and $\tilde{b}_h$. Notice further that

$$\frac{\tilde{b}^h - 1}{\tilde{b}^h - \tilde{a}^h} \to \frac{\ln \tilde{b}}{\ln \tilde{b} - \ln \tilde{a}}\,, \qquad \frac{1 - \tilde{a}^h}{\tilde{b}^h - \tilde{a}^h} \to \frac{-\ln \tilde{a}}{\ln \tilde{b} - \ln \tilde{a}}\,,$$

and, assuming that the derivatives exist (we have replaced M by M_ε as explained above)

$$\frac{1}{h}\left[V_k^{(h)}(\tilde{a}^h u) - V_k^{(h)}(u)\right] \to \frac{\mathrm{d}V_k^{(h)}}{\mathrm{d}u}(u)u\ln\tilde{a}\,,$$

$$\frac{1}{h}\left[V_k^{(h)}(\tilde{b}^h u) - V_k^{(h)}(u)\right] \to \frac{\mathrm{d}V_k^{(h)}}{\mathrm{d}u}(u)u\ln\tilde{b}\,.$$

As a consequence, $(1/h)[V_{k-1}^{(h)}(u) - V_k^{(h)}(u)] \to 0$. More precisely, using the bound on the second derivatives of the $V_k^{(h)}$,

$$\frac{1}{h}\left|V_{k-1}^{(h)}(u) - V_k^{(h)}(u)\right| \leq hD$$

so that, for any t,

$$V_{\frac{T}{h}}^{(h)}(u) - h(T-t)D \leq V_{\frac{t}{h}}^{(h)}(u) \leq V_{\frac{T}{h}}^{(h)}(u) + h(T-t)D\,,$$

Hence, in the limit $V_t^0 = M_\varepsilon$, and leting ε go to zero, $V_t^0 = M$ for all t.

We therefore recover the naive theory where the equilibrium price of any such security is constant in discounted values. The corresponding srategy is the naive strategy.

4.4.3 Fixed quadratic variation

We now want to approach the case where the admissible functions $S(\cdot)$ have fixed quadratic variation. This is done by choosing

$$\tilde{a}_h = 1 + \mu_h - \sigma\sqrt{h}\,, \quad \tilde{b}_h = 1 + \mu_h + \sigma\sqrt{h}\,,$$

where μ_h/h converges to some μ_0' as h goes to zero.

Let us consider (25) and let h go to zero. The left hand side formally converges to $-\partial V_t^0/\partial t$. Calculation of the limit of the right hand side requires that the differences be expanded to second order, since the variation in the arguments is only of the order of $\sqrt{h}$. It is however an elementary task to check that (25) formally converges to

$$-\frac{\partial V_t^0}{\partial t}(u) = \frac{\sigma^2}{2}u^2\frac{\partial^2 V_t(u)}{\partial u^2}\,, \tag{27}$$

i.e. Black and Scholes'partial differential equation in discounted variables.

To make that statement precise, we first use the previous trick of replacing M by a C^2 M_ε, and let it converge to M in the limiting V_t^0. We shall therefore assume that the $V_k^{(h)}$ are all twice differentiable in u. Notice also that we now have that

$$(\tilde{a}_h + \tilde{b}_h - \tilde{a}_h\tilde{b}_h)^{T/h} \to \mathrm{e}^{\sigma^2}$$

so that again, that function is bounded and continuous in h as h approaches 0. Thus it has a finite maximum and we are still in the conditions of proposition (3) above.

The calculation proposed above lets us conclude that

$$\left| -\frac{1}{h}[V_{k-1}^{(h)}(u) - V_k^{(h)}(u)] + \frac{\sigma^2}{2}u^2\frac{\partial^2 V_k^{(h)}}{\partial u^2} \right| \le \sqrt{h}D\,. \tag{28}$$

Let us consider a dyadic sequence of time divisions, in steps $h_n = 2^{-n}$. The first and second derivatives in u of the $V_k^{(h)}$ are all bounded, uniformly in n, k, u. By Tychonov's theorem, there is a subsequence for which these derivatives have a pointwise limit. By repeated application of the dominated convergence theorem, these limits are the first and second derivatives of a limit $V_t^0(u)$ of the $V_{t/h}^{(h)}(u)$'s. Let us further, for each fixed h, interpolate linearly the said $V_k^{(h)}(u)$'s for values of t between the discretization points, and, with a transparent abuse of notations let us call this $V_t^{(h)}(u)$. It results from (28) that $(\partial V_t^{(h)}/\partial t)(u)$ converges to $-(\sigma^2/2)(\partial^2 V_t^0/\partial u^2)(u)$, and again from the dominated convergence theorem, this limit is $(\partial V_t^0/\partial t)(u)$. Thus, $V_t^0(u)$ satisfies equation (28). The solution of that equation with boundary value M_ε at T is unique. Therefore it is the whole sequence that converges. Finally, the solution of that equation can be seen to be continuous in its boundary value, thus we have that indeed, $V_{t/h}^{(h)}$ converges to the solution of Black and Scholes'equation with boundary value M at T.

5 conclusion

We conclude that the classical Black and Scholes equation can be seen as a property of *the set of trajectories* assumed for the underlying market prices, not of a probabilistic measure it would be endowed with. A weakness of the bounded quadratic variation case (i.e. Black and Scholes's theory) is that the it *does not suffice* to assume that the quadratic variation be bounded to calculate a valuation. We need to know the exact volatility. In contrast, in the bounded variation case —the zero volatility case— the total variation does not need to be known. And the problem does not seem much less meaningfull. The weakness of that theory, in its present form, being that it yields our "naive" hedging strategy, that we have seen to be non robust to even small transaction costs. We are currently investigating its solution in the presence of these costs, using our robust control approach.

As a matter of fact, we have a coherent piece of theory based upon robust control and game theoretic tools, concerning the pricing of a contingent claim at a fixed expiration date, such as european options. In continuous time, we only have a "bounded variation" theory, easily recovering our naive theory. But as we said, we hope to get a much more realistic hedging strategy by including transaction costs.

In the case of discrete time trading, by exhibiting a hedging strategy against infinitly many possibilities, we give a normative value to the theory of Cox, Ross and Rubinstein, and improve over it in the case of non convex contingent claims. In that last case, we have shown that the previous theory underestimates the value of the contingent claim (which was known to a large extent) and we have a numerical tool to efficiently compute a valuation, if one accepts our model for market prices.

We have carried out numerical checks of that theory. If carefully done, the computation proves to be very fast, and completely corroborates the theory. We give here two graphics concerning the digital call, chosen because it is not simply constructed from calls and puts. Similar caluclations have been performed for standard calls and puts, and various spreads. In the results shown, the discount factor has been chosen unrealistically large to better separate the curves. The first graphic shows the valuation of the digital call for various maturities. The second one provides a comparison with a "Cox Ross and Rubinstein" approach.

Finally, we extended the convergence results of the Cox, Ross and Rubinstein theory for vanishingly small stepsizes in two directions : on the one hand, we allow for a wider class of contingent claims than just call or put, on the other hand, we show that the bounded variation theory can also be recovered that way. An aim of further research is to exploit recent advances in the numerical approach to dynamic games, using the tool of viscosity solutions, to investigate the limit of our valuation in non convex cases.

Finally, the methodology advocated here lends itself to deal with other types of contingent claims, american options, asiatic options, barrier options, etc. We plan to cover these in forthcoming papers.

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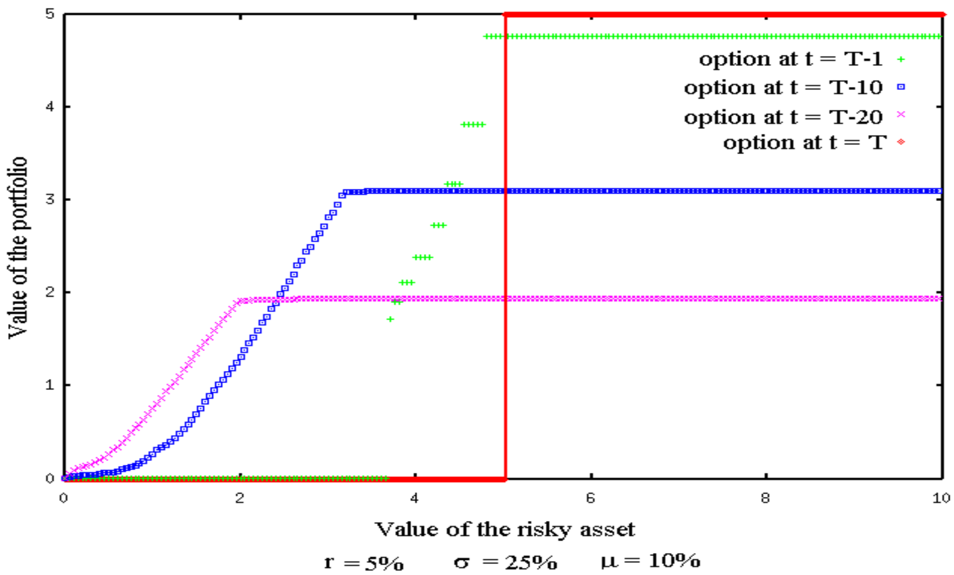

Figure 1: Pricing of a digital call

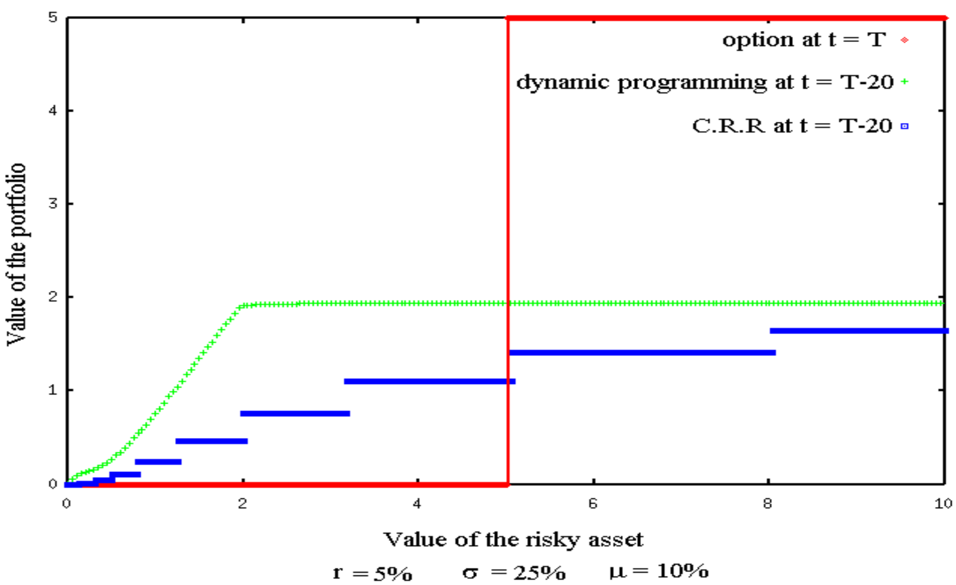

Figure 2: Comparison of two pricing methodologies

7 Appendix

We prove here a lemma of deterministic Itô calculus, a slightly modified version of which is lemma 1 in the text.

Lemma 2 *Let $\sigma(\cdot)$ be a measurable real function and $z(\cdot)$ be a continuous real function, both defined over the interval $[0, T]$, such that for any $t \in [0, T]$ and any sequence of divisions $0 = t_0 < t_1 < t_2 < \ldots < t_N = t$ with a diameter h going to 0 (and therefore $N \to \infty$), it holds that*

$$\lim \sum_{k=0}^{N-1} (z(t_{k+1}) - z(t_k))^2 = \int_0^t \sigma^2(\tau)\, d\tau\,. \tag{29}$$

Let $f(t, x)$ be a function from $[0, T] \times \mathbb{R}$ to $\mathbb{R}$ twice continuously differentiable. And assume that, for all $t \in [0, T]$,

$$\frac{\partial f}{\partial x}(t, z(t)) = 0\,.$$

Then, for all $t \in [0, T]$,

$$f(t, z(t)) = f(0, z(0)) + \int_0^t \left(\frac{\partial f}{\partial t}(\tau, z(\tau)) + \frac{\sigma(\tau)^2}{2} \frac{\partial^2 f}{\partial x^2}(\tau, z(\tau)) \right) d\tau\,.$$

Proof Consider a division $0 = t_0 < t_1 < t_2 < \ldots < t_N = t$. Using a Taylor expansion to second order with exact rest, we have

$$\begin{aligned} f(t_{k+1}, z(t_{k+1})) - f(t_k, z(t_k)) = \left(\frac{\partial f}{\partial t}(t_k, z(t_k)) \right) (t_{k+1} - t_k) + \\ \left(\frac{\partial f}{\partial x}(t_k, z(t_k)) \right) (z(t_{k+1}) - z(t_k)) + \frac{1}{2} \frac{\partial^2 f}{\partial t^2}(t_{k+1} - t_k)^2 + \\ \frac{\partial^2 f}{\partial t \partial x}(t_{k+1} - t_k)(z(t_{k+1}) - z(t_k)) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(z(t_{k+1}) - z(t_k))^2, \end{aligned}$$

where all second partial derivatives are evaluated at a point (t'_k, z'_k) on the line segment $[(t_k, z(t_k)) \quad (t_{k+1}, z(t_{k+1}))]$.

By assumption, the second term in the r.h.s. is equal to zero. We summ these expressions for $k = 0$ to $N - 1$. The l.h.s. is just $f(t, z(t)) - f(0, z(0))$. We want to investigate the limit of the four remaining sums in the r.h.s.

The first sum is elementary : since $t \mapsto (\partial f / \partial t)(t, z(t))$ is continuous, we get

Proposition 1

$$\lim \sum_{k=0}^{N-1} \left(\frac{\partial f}{\partial t}(t_k, z(t_k))\right)(t_{k+1} - t_k) = \int_0^t \frac{\partial f}{\partial t}(\tau, z(\tau))\, \mathrm{d}\tau\,.$$

Let us examine the quadratic terms. Again, it is a trivial matter to check that

Proposition 2

$$\lim \sum_{k=0}^{N-1} \left(\frac{\partial^2 f}{\partial t^2}(t'_k, z'_k)\right)(t_{k+1} - t_k)^2 = 0\,.$$

Let us show

Proposition 3

$$\lim \sum_{k=0}^{N-1} \left(\frac{\partial^2 f}{\partial t \partial x}(t'_k, z'_k)\right)(t_{k+1} - t_k)(z(t_{k+1}) - z(t_k)) = 0\,.$$

Proof This is hardly any more complicated than the previous fact, but let us be careful with terms involving $z(\cdot)$. However, we know that that function is continuous over $[0, t]$, hence uniformly so. Therefore, for any positive ε, there exists a sufficiently small h such that if the diameter of the division is less than h, then $|z(t_{k+1}) - z(t_k)| \leq \varepsilon$ for all k. Moreover, $z(t)$ remains within a compact, and thus so do all the z'_k's. Hence, f being of class C^2, its second derivative evaluated in (t'_k, z'_k) is bounded by a number C. Therefore, for a small enough diameter, the absolute value of the above sum is less than

$$\sum_{k=0}^{N-1} C\varepsilon(t_{k+1} - t_k) = C\varepsilon t\,,$$

hence the result claimed.

We now want to prove

Proposition 4

$$\lim \sum_{k=0}^{N-1} \left(\frac{\partial^2 f}{\partial x^2}(t'_k, z'_k)\right)(z(t_{k+1}) - z(t_k))^2 = \int_0^t \sigma(\tau)^2 \frac{\partial^2 f}{\partial x^2}(\tau, z(\tau))\mathrm{d}\tau\,.$$

To that aim, we show two intermediary facts :

Fact 4.1 For any continous real function $a(\cdot)$, we have that, for $t_k \leq t'_k \leq t_{k+1}$,

$$\lim \sum_{k=0}^{N-1} a(t'_k)(z(t_{k+1}) - z(t_k))^2 = \int_0^t a(\tau)\sigma(\tau)^2 \, \mathrm{d}\tau \, .$$

Proof Notice that the result trivially follows from the assumption (29) whenever the function a is piecewise constant. (In the limit as $h \to 0$, only a finite number of intervals $[z(t_k), z(t_k + 1)]$ contain a discontinuity of a, and their weight in the sum vanishes. For all the other ones, just piece together the intervals where a is constant. There the differences are multiplied by a constant number.)

Now, both the finite sums of the l.h.s. and the integral are continous with respect to $a(\cdot)$ for the uniform convergence. (Concerning the finite sums, they are linear in $a(\cdot)$. Check the continuity at zero.) And a being continuous over $[0, T]$, it is uniformly continuous and can be approximated arbitrarily well, in the distance of the uniform convergence, by a piecewise continuous function. The result follows.

Fact 4.2 For any continuous function $b(t, z)$, we have for $t_k \leq t'_k \leq t'_{k+1}$ and $z(t_k) \leq z'_k \leq z(t_{k+1})$

$$\lim \sum_{k=0}^{N-1} b(t'_k, z'_k)(z(t_{k+1}) - z(t_k))^2 = \int_0^t \sigma(\tau)^2 b(\tau, z(\tau)) \mathrm{d}\tau \, .$$

Proof Replace first z'_k by $z(t'_k)$ as the second argument of b in the l.h.s. Then just let $a(t) = b(t, z(t))$ in proposition 4.1, and he limit follows. Now, as $h \to 0$, and because b and z are continuous, $b(t'_k, z'_k) - b(t'_k, z(t'_k))$ converges to zero uniformly in k. (They both approach $b(t_k, z(t_k))$ uniformly.) The result follows.

Set $b = \partial^2 f / \partial x^2$ in the fact 4.2 above to get proposition 4.

Propositions 1,2,3, and 4 together yield the lemma.

Remark Use $b(t, S) = S^2(\partial^2 V / \partial s^2)(t, S)$ in Proposition 4.2 to derive the slightly different lemma 1 in the text.