A Survey of Linear Quadratic Robust Control

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Abstract

We review several control problems, all related to robust control in some way, that lead to a minimax linear quadratic problem. We stress the fact that although an augmented preformance index appears, containig an L^2 norm of a disturbance signal, only the *non augmented* quadratic performance index is of interest per se in each case.

Keywords : linear-quadratic, robust control, Riccati equations, H_∞-optimal control.

1 Introduction

Robust control has been one of the most active areas of control research for the last twenty years or so. By the phrase "robust control", one means those control mechanisms that explicitly take into account the fact that the system model (or the noise model) is unprecise. Of course, one has to know something about this model. A distinctive feature of most of robust control theory is that the a priori information on the unknown model errors (or signals) is non probabilistic in nature, but rather in terms of *sets of possible realizations*[1]. Typically (though not always), bounded errors of some kind. (Yet we shall show a strange parallel with a stochastic problem formulation in the so called *risk averse* control problem.) As a consequence, robust control aims at synthesizing control mechanisms that control in a satisfactory fashion (e.g. stabilize, or bound an output) a *family* of models.

If so called $\mathcal{H}_\infty$-optimal control has been the subject of the largest share of that research, it is by no means the only approach to robust control. Most prominent among other approaches are those "à la Kharitonov". Kharitonov's theorem states a sufficient condition for a particular family of polynomials to have all their roots in the left half complex plane. Applied to the characteristc polynomial of a family of linear time invariant (LTI) systems, it yields interesting robust control results. The so called "edge theorem" is an attempt at a similar result for more interesting families of polynomials. It is the starting point of some robust control results. We shall not review that line of thought here. For a nice review, see Barmish 1988.

$\mathcal{H}_\infty$-optimal control started with the work of G. Zames 1981, although some other papers such as Doyle and Stein 1981 can be seen as forerunners of the new theory. It was developped in the context of LTI systems and frequency domain representation, where it won its name. A good review of that approach can be found in Francis 1987. It was with the important paper of Doyle, Glover, Khargonekar, and Francis 1989, first given at the 1988 Conference on Decision and Control (CDC), that the first link with state space was established, the role of a game-like Riccati equation shown, and an observer-like form, reminiscent of the linear quadratic gaussian (LQG) theory, exhibited. The link with games was elucidated the next year independently by Başar 1989, Papavassilopoulos and Safonov 1989, Tadmor 1989. At that time, the available game theory did not allow one to explain the full result of Doyle, Glover, Khargonekar, and Francis 1989 and its observer-like structure. This was to be explained by our minimax certainty equivalence theorem in Bernhard 1990, 1991 and Bernhard and Rapaport 1995 and exploited to its full strength by Başar and Bernhard 1991.

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[1] One may argue a posteriori that this is because being concerned with disturbances in the coefficients of the model, a stochastic formulation would lead to a differential equation with stochastic coefficients, a technically complex object, difficult to use.

This allowed us to develop a theory entirely in state space, indeed in the realm of the classical linear quadratic theory, with the same type of tools. It also let us extend the theory to non time invariant systems, finite horizon criterions, and successfuly deal with such features as sampled data control, time lags, $\mathcal{H}_\infty$-optimal estimation, etc.

Other important works were conducted in parallel on the time domain approach, such as Limebeer, Green and Walker 1989, Stoorvogel 1992 or Kwakernaak 1991, most of them, however, restricted to LTI systems.

Consideration of L^2 norms in a linear system with state x and control u naturally leads to the consideration of a quadratic perforamce index, specifically the integral of a quadratic form in $x(t)$ and $u(t)$, that we call J. In some sense, we wish to keep it small in spite of unknown factors, disturbances and/or model errors. For technical reasons there shall appear an augmented performance index $J_\gamma = J - \gamma^2 \|w(\cdot)\|^2$ and its minimax. Our presentation is intended to stress the fact that this is for pure technical reasons, and that the real problems at hand are only concerned with J and its being kept small.

In section 2, we first show a simple noise attenuation problem, that shows up as an alternative to the gaussian noise model to deal with the control of a disturbed linear system. No uncertain model in that problem, and no set of models per se. But it shares a feature of robust control in as much as the noise description is in terms of an admissible set of time functions : L^2, the set of finite “energy” signals. (The set of finite “power” signals leads to the same theory).

In section 3, we consider a slightly generalized version of Zames’original robust control formulation. Here lies the claim of $\mathcal{H}_\infty$-optimal control to solve robust control problems. The second edition of Başar and Bernhard 1991 (1995) will serve as the basis of sections 2 and 3.

In a fourth section, we show the link with so called “risk averse” control. As a matter of fact, Whittle’s separation theorem largely predates our certainty equivalence theorem. It is a version of the latter restricted to the discrete time, time invariant, linear quadratic case (with a simple observation equation). We introduce the relationship between $\mathcal{H}_\infty$-optimal control and risk averse control via what we nickname “Whittle’s magic formula”. This in a way makes that relationship look at best accidental, at worst magic. Some deeper reason is probably at work, but not completely clear to us at this time.

A final section gives a couple of elementary examples to show how the mathematics deal with model uncertainty. We want there to emphasize that *robust* is not synonym to *cautious*. Robust control may be more or less cautious than classical LQG, depending on the structure of the model uncertainties.

To keep the exposition as short and simple as possible, we shall restrict it to continuous-time models. There exist discrete time parallels to (essentially) everything we shall discuss. In some instances the mathematics are even simlper, but the formulas are always more complicated there.

2 Robust noise attenuation

2.1 System and notations

Consider a linear system with state $x \in \mathbb{R}^n$, two inputs : the control $u \in \mathbb{R}^m$ and a disturbance signal $w \in \mathbb{R}^\ell$, and two outputs : a measurement output $y \in \mathbb{R}^p$ and a controlled output $z \in \mathbb{R}^q$:

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw\,, && (1)\\ y &= Cx \qquad\quad + Ew\,, && (2)\\ z &= Hx + Gu \qquad\quad . && (3) \end{aligned}$$

The matrices A, B, D, C, E, H, and G are of appropriate sizes. In infinite horizon problems, they will be assumed constant. In finite horizon problems, they may be time varying, say piecewise continuous, right-continuous and left-limited. But we shall always omit that possibility in our notations. Notice that the following *system matrix* can be built from them :

$$\mathcal{S} = \begin{pmatrix} A & B & D \\ C & 0 & E \\ H & G & 0 \end{pmatrix} . \tag{4}$$

The 0 matrix in the definition of y is of no consequence. As a matter of fact, y will be the measured output, i.e. the information available to the controller to choose u. If there were a term $+Fu$ in it, one, knowing $u(t)$, could instantly substract out that term and recover our y. Therefore there is no loss of generality here. This is not so for the 0 matrix in z. We shall keep it because the theory is simpler that way. But T. Başar has extended the theory to the case where this extra term is present (see the second ed. of Başar and Bernhard 1991). See also Bernhard 2000 for a discussion of that question in the framework of minimax control.

Also to keep things simple, we shall always make the following assumptions :

Assumptions A

1. the matrix G is injective, and has its mth (smallest) singular value bounded away from zero, (hence $m < q$),
2. the matrix E is surjective, and has its pth (smallest) singular value bounded away from zero, (hence $p < \ell$).

We shall also use the notations

$$\begin{pmatrix} H^tH & H^tG \\ G^tH & G^tG \end{pmatrix} =: \begin{pmatrix} Q & S \\ S^t & R \end{pmatrix}, \tag{5}$$

so that hypothesis **A1** translates into $R > 0$ and R^{-1} bounded, and

$$\begin{pmatrix} DD^t & DE^t \\ ED^t & EE^t \end{pmatrix} =: \begin{pmatrix} M & L^t \\ L & N \end{pmatrix}, \tag{6}$$

so that hypothesis **A2** translates into $N > 0$ and N^{-1} bounded.

In many applications, one has $S = 0$ (no cross terms in xu in J below) and $L = 0$ (the dynamics noise Dw and measurement noise Ew are unrelated). This simplifies somewhat the various equations below, but not by much.

We shall write $\|u\|_R^2 = (u, Ru) = u^tRu$, $\|w\|_N^2 = (w, Nw) = w^tNw$ and likewise for other quadratic forms, even when the weighting matrix shall not be positive definite.

We shall consider infinite horizon problems where $t \in (-\infty, +\infty)$, where implicitly what is meant is that the state at time $-\infty$ was zero :

$$x(t) = \exp(At)\int_{-\infty}^{t} \exp(-As)[Bu(s) + Dw(s)]\,\mathrm{d}s\,.$$

(This is well defined provided that the system be stable, or adequately stabilized by feedback.) Therefore, $x(\cdot)$ is then a function of $u(\cdot)$ and $w(\cdot)$ alone, and there is no consideration of final state either, we shall always require that the system be stabilized : $x(t) \to 0$ as $t \to \infty$. We shall then have $z(\cdot) \in L^2(-\infty, +\infty)$ and use the notation

$$J(u(\cdot), w(\cdot)) = \|z(\cdot)\|^2 = \int_{-\infty}^{\infty} [\|x(t)\|_Q^2 + 2x(t)^tSu(t) + \|u(t)\|_R^2]\,\mathrm{d}t\,, \tag{7}$$

and for any positive number γ

$$J_\gamma = J - \gamma^2\|w(\cdot)\|^2 = \int_{-\infty}^{\infty} [\|x(t)\|_Q^2 + 2x(t)^tSu(t) + \|u(t)\|_R^2 - \gamma^2\|w(t)\|^2]\,\mathrm{d}t\,. \tag{8}$$

For the finite horizon case, where $t \in [0, T]$, we shall need the notations

$$\zeta = \begin{pmatrix} z(\cdot) \\ x(T) \end{pmatrix} \in L^2([0, T] \to \mathbb{R}^n) \times \mathbb{R}_X^n\,, \tag{9}$$

for a given non negative definite matrix X, so that

$$\|\zeta\|^2 = \|x(T)\|_X^2 + \int_0^T [\|x(t)\|_Q^2 + 2x(t)^tSu(t) + \|u(t)\|_R^2]\,\mathrm{d}t\,, \tag{10}$$

and

$$\omega = \begin{pmatrix} x_0 \\ w(\cdot) \end{pmatrix} \in \mathbb{R}^n_Y \times L^2([0,T] \to \mathbb{R}^\ell) =: \Omega\,, \tag{11}$$

where $x_0 = x(0)$ and Y is a given positive definite matrix, so that

$$\|\omega\|^2 = \int_0^T \|w(t)\|^2 \,\mathrm{d}t + \|x_0\|_Y^2\,. \tag{12}$$

And we shall write

$$J(x_0, u(\cdot), w(\cdot)) = J(u(\cdot), \omega) = \|\zeta\|^2\,, \tag{13}$$

and

$$J_\gamma(u(\cdot), \omega) = J(u(\cdot), \omega) - \gamma^2 \|\omega\|^2\,, \tag{14}$$

hence

$$J_\gamma = \|x(T)\|^2 + \int_{-\infty}^{\infty} [\|x(t)\|_Q^2 + 2x(t)^t S u(t) + \|u(t)\|_R^2 - \gamma^2 \|w(t)\|^2] \,\mathrm{d}t - \gamma^2 \|x_0\|_Y^2 .$$

2.2 The problem

In an unprecise statement, the aim is to “choose $u(t)$, knowing only the past $y(s)$, $s < t$”, in such a way as to “keep $z(\cdot)$ small in spite of the unpredictable disturbances”. All we shall assume concerning these disturbances is that the time function $w(\cdot)$ is square integrable over the time interval considered, either finite or infinite.

The aim of the mathematical models is to propose mathematical metaphors of that problem, more or less well suited to various experimental or logical contexts. One very famous metaphor has been to construct a probabilistic model for the disturbances, and accordingly for the state trajectory, with the necessary aparatus to account for the causality of the admissible control laws. One then strives to minimize the *expceted value* of J. This leads to the famous LQG theory.

This is known to be a very usefull piece of theory, and a very brilliant one, but the point here is that it is only *one* possible way of making a mathematical metaphor of the basic problem. It is well suited if, on the one hand, one has reasons to believe that the disturbances qualitatively resemble a random walk, and on the other hand, the average value of J over several experiments is of interest. But assume that it be known, for instance, that the distrbance is a *bias*, a constant over time. (Still of zero expectation.) There is no way by which this can resemble a random walk, nor be represented as the output of a linear system driven by such a process, because it is not ergodic. (The time average differs from the ensemble average.) This is one situation, and others may arise, where the foregoing approach might be better suited.

We shall stick with the decision that “keeping $z(\cdot)$ small” will be juged by looking at the L^2 norm of that ouput function, either of $z(\cdot)$ in the infinite horizon case, or, to be slightly more general, ζ in the finite horizon case. Thus our aim is, as previously, to keep J as given by (7) or (13) small.

Admissible control laws will be *causal* functions of the measured output, i.e. of the form[2]

$$u(t) = \mu(t, y(s); s < t) \tag{15}$$

and such that when substituted for u in (1), (2) it yields for all $\omega \in \Omega$ a unique solution $x(\cdot)$. Let $\mathcal{M}$ be the set of all such admissible controls.

Assume for awhile that we are restricted to *linear* control laws μ. Then, once μ is substituted into the dynamics, ζ becomes a linear function of ω alone, say $\zeta = T_\mu \omega$. Hence, there is no way to avoid that $J = \|\zeta\|^2$ grow as $\|\omega\|^2$. A reasonable mathematical problem, which indeed is a valid metaphor of the original problem, is to try to keep the ratio $J/\|\omega\|^2$ as small as possible. Equivalently, since the norm of a linear operator T_μ is defined as the smallest number $\|T_\mu\|$ such that

$$\forall \omega \in \Omega\,, \quad \|\zeta\| \leq \|T_\mu\| \|\omega\|$$

the problem at hand is to *find an admissible control law μ that makes $\|T_\mu\|$ small.*

Thus, it would be nice to be able to solve the problem $\min_{\mu \in \mathcal{M}} \|T_\mu\|$. Unfortunately, this problem is not well behaved and does not admit a simple solution. In particular, the discussion of whether the min is reached or not is difficult.

It turns out that it is usefull to rephrase that problem in the following way :

> **Problem** $\mathcal{P}_\gamma$ Given a positive number γ, does there exist an admissible control law μ that will insure that $\|T_\mu\| \leq \gamma$, or equivalently (16) below, and if yes find one.

Equivalently, this reads (remember that here $\zeta = T_\mu \omega$)

$$\forall \omega \in \Omega\,, \quad \|\zeta\| \leq \gamma \|\omega\|\,. \tag{16}$$

Now, the above property is equivalent to

$$\forall \omega \in \Omega\,, \quad \|\zeta\|^2 - \gamma^2 \|\omega\|^2 = J_\gamma(\mu, \omega) \leq 0\,,$$

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[2] Roxin has proposed the following equivalent definition of causality : an application μ from $L^2([0,T] \to \mathbb{R}^p)$ to $L^2([0,T] \to \mathbb{R}^m)$ is causal if $\forall t \in [0,T]$, the equality $y(s) = y'(s)$ for allmost all $s < t$ implies $\mu(y)(t) = \mu(y')(t)$.

and thus also to

$$\sup_{\omega\in\Omega} J_\gamma(\mu,\omega) \le 0\,.$$

Finally, existence of an admissible control law that achieves (16) is equivalent to (if the min exists)

$$\min_{\mu\in\mathcal{M}} \sup_{\omega\in\Omega} J_\gamma(\mu,\omega) \le 0\,, \tag{17}$$

(and this does not depend on μ being linear).

Hence, we end up solving a differential game, or minimax control problem, for the cost function J_γ. But only because checking whether (17) holds is a means of answering the question of problem $\mathcal{P}_\gamma$, or equivalently to attempt to insure (16), and because if the answer is yes, then the minimizing μ in (17) solves it.

2.3 Solution

For the sake of completeness, we recall here the solution of that game problem.

2.3.1 Finite horizon

Let us first consider the finite horizon case. The solution of the problem involves two matrix Riccati equations, for symmetric matrices $P(t)$ and $\Sigma(t)$:

$$\dot P + PA + A^tP - (PB+S)R^{-1}(B^tP+S^t) + \gamma^{-2}PMP + Q = 0\,, \quad P(T) = X\,, \tag{18}$$

and

$$\dot\Sigma = A\Sigma + \Sigma A^t - (\Sigma C^t + L^t)N^{-1}(C\Sigma + L) + \gamma^{-2}\Sigma Q\Sigma + M\,, \quad \Sigma(0) = Z\,, \tag{19}$$

where we have set $Z := Y^{-1}$.

The main theorem of $\mathcal{H}_\infty$-optimal control is as follows. (For any square matrix K, $\rho(K)$ stands for *spectral radius* of K.)

Theorem 1 *If equations (18) and (19) have solutions $P(\cdot)$ and $\Sigma(\cdot)$ over $[0,T]$, and if furthermore these solutions satisfy the following inequality :*

$$\forall t\in[0,T] \quad \rho(\Sigma(t)P(t)) \le \gamma^2\,,$$

then the answer to problem $\mathcal{P}_\gamma$ is positive, and a control law that achieves the desired disturbance attenuation level γ is given by equations (20) and (21) hereafter.

Conversely, if one of the above conditions fails, then for any $\tilde\gamma > \gamma$ the problem $\mathcal{P}_{\tilde\gamma}$ has no solution.

The control law proposed is obtained as a "certainty equivalent" feedback on a "worst possible state" $\hat{x}(t)$:

$$u(t) = -R^{-1}(B^t P(t) + S^t(t))\hat{x}(t)\,, \tag{20}$$

where $\hat{x}$ is the solution of the following differential equation :

$$\begin{aligned} \dot{\hat{x}} &= (A - BR^{-1}(B^t P + S^t) + \gamma^{-2} MP)\hat{x} \\ &\qquad + (I - \gamma^{-2}\Sigma P)^{-1}(\Sigma C^t + L^t)N^{-1}[y - (C + \gamma^{-2} LP)\hat{x}]\,, \\ \hat{x}(0) &= 0\,. \end{aligned} \tag{21}$$

One may notice the similarity with the optimal LQG control. Indeed, the feedback law (20) has exactly the same form (though with a different $\hat{x}$ of course), and equation (21) has the same structure as a Kalman filter. The differences with the latter case are in the presence of a "worst" disturbance $w = \gamma^{-2} D^t P\hat{x}$ in the dynamics and in the corrective term (it disappears from the correcting term if $L = 0$), and in the fact that the gain matrix of the corrective term is premultiplied by the coefficient $(I - \gamma^{-2}\Sigma P)^{-1}$. Notice that the spectral radius condition of the theorem precisely guarantees the required invertibility.

As a matter of fact, one way of showing this theorem is through a certainty equivalence theorem that states that under some conditions satisfied here, a minimax control in imperfect information is obtained by substituting in the minimax state feedback (i.e. the optimal control law in the case of perfect state information) a "worst current state compatible with the past measurements", that (21) provides.

It is a worthwhile fact to state (and *not* a corollary of the above) that

Theorem 2 *If the available measurement is $x(t)$ (exact state measurement), the theorem 1 holds wihout the condition on equation (19) (existence of Σ), with the spectral radius condition restricted to initial time : $\rho(ZP(0)) \le \gamma^2$, and with $x(t)$ instead of $\hat{x}(t)$ in (20). (Hence equation (21) is not required either.)*

2.3.2 Infinite horizon

The stationary theory, which predated the finite horizon one, can be obtained as a limiting case of the above, with some care though. We assume now that $\mathcal{S}$ is constant. And we need an extra set of assumptions :

Assumptions B

1. The pair (A, D) is stabilizable,[3]

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[3] Recall that (A, D) stabilizable means that there exists a feedback matrix F such that $A - DF$ be asymptotically stable, and (A, D) controllable suffices.

2. the pair (H, A) is reconstructible.[4]

Then we have the following result :

Theorem 3 *Under assumptions* **A** *and* **B***, if the following three conditions hold :*

1. *The Riccati equation (18) integrated from* $P(0) = 0$ *has a solution that converges to some* P^* *as* $t \to -\infty$*,*
2. *the Riccati equation (19) integrated from* $\Sigma(0) = 0$ *has a solution that converges to some* Σ^* *as* $t \to \infty$*,*
3. $\rho(\Sigma^* P^*) \leq \gamma^2$*,*

then the answer to the problem $\mathcal{P}_\gamma$ *is positive, an admissible controller is given by equations (20) and (21) with* $P(t)$ *and* $\Sigma(t)$ *replaced by* P^* *and* Σ^* *respectively. In that case,* P^* *and* Σ^* *are the* least positive definite solutions *of the* algebraic Riccati equations *obtained by placing* $\dot{P} = 0$ *and* $\dot{\Sigma} = 0$ *in (18) and (19) respectively.*

Conversely, if one of the above three conditions fail, for any $\tilde{\gamma} > \gamma$*, the problem* $\mathcal{P}_{\tilde{\gamma}}$ *has no solution.*

Furthermore, if in addition to assumptions **B** *we have that* (A, B) *is stabilizable and* (C, A) *reconstructible, then there will always be a positive* γ^* *such that the conditions of the theorem be satisfied for* $\gamma > \gamma^*$ *and violated for* $\gamma < \gamma^*$*.*

A carefull analysis of the problem shows that usually, as γ is decreased from values larger than γ^*, the first condition of the theorem to be violated will be the third one. What happens for $\gamma = \gamma^*$ is more complicated (a reduced order controller may exist), but is of little practical importance.

3 Robust stabilization and control

3.1 Model uncertainty

We now turn to the original problem that brought $\mathcal{H}_\infty$-optimal control to life, a real robust control problem in that it deals with model uncertainty. To justify the description we shall use of plant uncertainty, we begin with an example.

Let a linear system be of the form

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \\ y &= Cx\,. \end{aligned}$$

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[4] (H, A) is reconstructible if (A^t, H^t) is stabilizable. Thus (H, A) observable suffices.

Assume that the matrices A, B and C are not exactly known. All we know are approximate values A_0, B_0, and C_0 and bounds on how bad these approximations may be, in terms of norms of matrices : three positive numbers α, β, and γ are given, together with the information

$$\|A - A_0\| \leq \alpha\,, \quad \|B - B_0\| \leq \beta\,, \quad \|C - C_0\| \leq \gamma\,.$$

We shall be concerned with the problem of stabilizing and controlling that system, hence the family of all models thus described.

We rewrite the system's equations as

$$\begin{array}{rcl} \dot{x} & = & A_0 x + B_0 u + [I \quad 0] w\,, \\ y & = & C_0 x + [0 \quad I] w\,, \\ z & = & \begin{pmatrix} I \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u \end{array} \tag{22}$$

with the added relation

$$w = \begin{pmatrix} \Delta A & \Delta B \\ \Delta C & 0 \end{pmatrix} z\,, \tag{23}$$

where $\Delta A := A - A_0$, and likewise for ΔB and ΔC. This is indeed the same system.

The system (22), called the *nominal* system, is of the form

$$\begin{pmatrix} y \\ z \end{pmatrix} = \mathcal{G} \begin{pmatrix} u \\ w \end{pmatrix}$$

where $\mathcal{G}$ is entirely known: there is no uncertain coefficient in it. Only an unknown disturbance input w. Furthermore, it is exactly of the form of our system of the previous section. (With $S = 0$ and $L = 0$.) All the uncertainty has been placed in a the feedback term (23). We rewrite that last term as

$$w = \Delta\mathcal{G} z$$

and the available information on the uncertainties translates into (as a matter of fact, is degraded into)

$$\|\Delta\mathcal{G}\| \leq \delta \tag{24}$$

for some number δ, function of the given uncertainty bounds.[5]

The above example is meant to substantiate the claim that the following uncertainty description is indeed very general. For the sake of convenience however, we

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[5] It is an elementary matter to check that $\delta^2 = (\alpha^2 + \beta^2 + \gamma^2 + \sqrt{(\alpha^2 + \beta^2 + \gamma^2)^2 - 4\beta^2\gamma^2})/2$.

shall rename respectively r and s the input and output w and z above, that play a special role in the uncertainty description

We consider a linear system described as a linear operator $\mathcal{G}$ acting on inputs to deliver outputs. One may think of $\mathcal{G}$ as meaning an abstract operator from L^2 spaces into L^2 spaces, or, equivalently, as meaning the transfer function as a concrete representation of a linear operator (that last interpretation is restricted to a infinite horizon time invariant problem, not the former one). We shall use three (vector) inputs now : w renamed r and u as above, and an exogeneous disturbance v. Likewise, we may use three (vector) outputs : z renamed s and y as above, and a to-be-controlled output e (an "error" signal seen as the deviation of an actual output from a desired one, that should be kept small). By linearity, the system may be written as

$$\begin{aligned} s &= \mathcal{G}_{sr}r + \mathcal{G}_{sv}v + \mathcal{G}_{su}u\,, && (25)\\ e &= \mathcal{G}_{er}r + \mathcal{G}_{ev}v + \mathcal{G}_{eu}u\,, && (26)\\ y &= \mathcal{G}_{yr}r + \mathcal{G}_{yv}v + \mathcal{G}_{yu}u\,. && (27) \end{aligned}$$

The uncertainty in the system resides in the fact that we know that

$$r = \Delta\mathcal{G}s \tag{28}$$

for some linear operator $\Delta\mathcal{G}$ of which we only know a norm bound δ as in (24).

The output y is the measurement available to choose our control u, whose aim is to stabilize this family of models —this is the topic of the next subsection— and if possible, in doing so to attenuate as much as possible the effect of the exogeneous disturbance v in the controlled output e, this is dealt with in a later subsection.

3.2 Robust stabilization

Because we want to deal with stability, we restrict our attention here to linear time invariant systems and infinite horizon controls.

Assume we are constrained to linear operators (this restriction may be waived, but we shall not consider that question here) of the form (15), that we rewrite

$$u = \mu y\,. \tag{29}$$

to stress the linearity. Then, substituting into (27) one may, formally solve for y, and thus u, in terms of v and r, and substitute this form of u in (25), leading to a linear expression of the form

$$s = T_\mu r + S_\mu v\,. \tag{30}$$

This is indeed exactly the same argument as in section 2.2. Thus the *controlled* system is now given by (30) (28).

How to choose μ to insure stability of this controlled system for any $\Delta\mathcal{G}$ within the norm bound ? The fundamental remark is as follows : under suitable assumptions, that are satisfied by a standard canonical state variable model, the system is stable if and only if the above equations have a solution in L^2 for any v in L^2. (Sufficiency stems from the fact that if all inputs are in L^2 so is $\dot{x}$, thus x is in H^1 and hence $x(t) \to 0$ as $t \to \infty$. Necessity requires some observability.)

Now, substitute (28) into (30). It comes

$$s = T_\mu \Delta\mathcal{G} s + S_\mu v\,. \tag{31}$$

This is a fixed point equation for s. By Banach's theorem, a sufficient condition for the existence of a (unique) solution is that $\|T_\mu \Delta\mathcal{G}\| < 1$. Notice that we have

$$\|T_\mu \Delta\mathcal{G}\| \leq \|T_\mu\|\|\Delta\mathcal{G}\| \leq \|T_\mu\|\delta\,,$$

so that a sufficient condition of stability of all our models is that $\|T_\mu\| < \delta^{-1}$.

The so called "small gain theorem", states that if S_μ is onto, this condition is also necessary to insure existence of a solution to (30)(28) for all $\Delta\mathcal{G}$ of norm no more than δ.[6]

Hence, the search for a controller μ that stabilizes all the models in the family may in practice be replaced by the requirement that $\|T_\mu\| \leq \gamma$ for a well chosen γ. Of course, the remarkable fact is that this is the problem considered for another reason in the previous section.

3.3 Robust stabilizing control

We now want to simultaneously stabilize and control our family of models. Consider now the combined input

$$w = \begin{pmatrix} r \\ v \end{pmatrix}$$

and the combined output

$$z = \begin{pmatrix} \delta s \\ \beta e \end{pmatrix}$$

for some positive β (and the same δ as above). Assume a control law μ is chosen, and let $\widetilde{T}_\mu$ be the ensuing linear operator from w to z. Choose $\gamma < 1$ (but very close

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[6] Assume $\|T_\mu\| = \gamma \geq \delta^{-1}$. There exists an $\hat{s}$ of norm 1 such that $T_\mu T_\mu^* \hat{s} = \gamma^2 \hat{s}$. Choose $\Delta\mathcal{G}$ defined by $\Delta\mathcal{G} u = \gamma^{-2} T_\mu^* \hat{s}(\hat{s}, u)$, and pick v such that $S_\mu v = \hat{s}$. Here $\|\Delta\mathcal{G}\| = \gamma^{-1} \leq \delta$. And it is readily seen that if s were the solution of (31), one would have $s = [(\hat{s}, s) + 1]\hat{s}$, and taking the scalar product with $\hat{s}$, $(\hat{s}, s) = (\hat{s}, s) + 1$, a contradiction.

to one). If it is possible to choose μ such that $\|\widetilde{T}_\mu\| \leq \gamma$, then this in particular implies that, on the one hand, the operator from r to δs has norm less than one, hence the operator T_μ from r to s has norm less than δ^{-1}, insuring robust stability, and on the other hand that the operator from v to βe also has norm less than one, insuring a disturbance rejection ratio of at least β^{-1}. Thus, the larger the β for which this is possible, the better the control law.

Again, we are back to a problem of the form treated to begin with in section 2.

One should notice however that up to this point, the control problem addressed by this approach is not completely satifactory, because the system norm we have strived to control to insure noise attenuation is that of the *nominal* system. It would be interesting to be able to say something of the operator from v to e in the *perturbed* system, where indeed $r = \delta\mathcal{G}s$. It is a late and surprising theorem (Chilali 1996) that indeed, in that case we have *also* insured that the perturbed system admits the same norm bound. Thus this does provide simultaneous robust stabilization and control.

This is still an elementary stage of the theory however. Two important extensions have been developed. On the one hand, it is possible to exploit a more refined knowledge on the disturbance system than just a norm bound, typically in terms of a frequency dependant bound. This is done using shaping filters, very much like what is done with the classical LQG theory to deal with colored noise. On the other hand, we have stressed in the example that by reducing our knowledge about the disturbance to a single operator norm, we degrade our information. Thus, means have been developed to distinguish several channels in both r and s, with a diagonal structure on $\Delta\mathcal{G}$, and separate norm bounds on each block of that structure. This is the aim of "μ-synthesis", after the name of the "structured singular value" oftentimes called μ.

4 Risk averse control

We now outline a seamingly completely different problem that leads to the consideration of the same minimax problem as in (17).

Consider a linear model as in (1)(2) (3), but for the time being, and following Whittle 1981, in discrete time :

$$\begin{aligned}
x(t+1) &= Ax(t) + Bu(t) + Dw(t)\,, && (32)\\
y(t) &= Cx(t) \qquad\qquad + Ew(t)\,, && (33)\\
z(t) &= Hx(t) + Gu(t) \qquad\qquad . && (34)
\end{aligned}$$

This will make things simpler on technical grounds, but the theory has since been extended to the continuous time problem, though with much technical difficulties, by Bensoussan and Van Schuppen (1985). We also restrict our attention to a finite horizon problem. Thus $w(\cdot)$ is now a finite sequence, thus a finite dimensional variable, that we shall still write w when no confusion is possible. Its l^2 norm is exactly the euclidean norm of the composite vector of dimension $T\ell$ made of all the $w(t)$'s.

As in classical LQG theory, we modelize the disturbances $w(\cdot)$ as a normalized white noise, i.e. a sequence of independant normal gaussian random variables.

We want to modelize a risk averse controller. One way of doing so is to assume that the controller seeks to minimize the expected value of the exponential of the classical quadratic performance index. Because the exponential function is convex, this penalizes upwards deviations from the mean more than it saves on downwards deviations, making it important to reduce the variance of the quadratic performance index.

More precisely, we take as the performance index

$$G_\gamma(x_0,u) = \mathbb{E}\exp\left(\frac{1}{2\gamma^2}J(x_0,u,w)\right) . \tag{35}$$

(For obvious reasons, it is customary to consider more precisely $\widetilde{G}_\gamma := 2\gamma^2 \ln G_\gamma$, but it is clearly equivalent to minimize $\widetilde{G}_\gamma$ or G_γ.)

Expanding the expectation operator, this leads to

$$G_\gamma(x_0,u) = (2\pi)^{-\frac{T\ell}{2}}\int \exp\left(\frac{1}{2\gamma^2}[J(u,w)-\gamma^2\|w\|^2]\right)\,\mathrm{d}w\,,$$

The exponent involves the familiar $J_\gamma = J - \gamma^2\|w\|^2$. It is a non homogeneous quadratic form in u and w, and can be written as

$$J_\gamma = (u,\mathcal{R}u) + 2(w,\mathcal{S}u) - \gamma^2(w,\mathcal{T}w) + 2(a,u) + 2(b,w) + c$$

with $\mathcal{T} = I - \frac{1}{2\gamma^2}J_{ww}$, and for some linear operators $\mathcal{R}$, $\mathcal{S}$, some time functions a and b and a number c. We have used a minus sign in front of the quadratic term in w to stress the fact that the expectation is defined (finite) if and only if the operator $\mathcal{T}$ is positive definite. Otherwise the integral in w diverges.

It is a classical fact that one may "complete the square", i.e. re-write the above quadratic form in terms of the linear operators $\mathcal{N} = \gamma^{-2}\mathcal{T}^{-1}\mathcal{S}$ and $v = \gamma^{-2}\mathcal{T}^{-1}b$ as

$$J_\gamma = -\gamma^2(w-\mathcal{N}u-v,\mathcal{T}(w-\mathcal{N}u-v)) + K_\gamma(u)\,. \tag{36}$$

The remainder K_γ is easily computed. The important fact is that it does not depend on w. Because we need $\mathcal{T}$ to be positive definite, the form (36) immediately shows that

$$K_\gamma(u) = \max_w J_\gamma(u,w)\,. \tag{37}$$

But also, we have

$$G_\gamma = \exp\left(\frac{1}{2\gamma^2}K_\gamma(u)\right)(2\pi)^{-\frac{T\ell}{2}}\int\exp\left(-\frac{1}{2}\|w-\mathcal{N}u-v\|^2_{\mathcal{T}}\right)\,\mathrm{d}w\,,$$

and a simple change of variable shows that the last integral does not depend on u, yielding

$$G_\gamma = \frac{1}{\sqrt{\det\mathcal{T}}}\exp\left(\frac{1}{2\gamma^2}K_\gamma(u)\right)\,.$$

Therefore, the problem of minimizing G_γ is equivalent to minimizing $K_\gamma(u)$, of which we have seen that it is the max over w of J_γ. Hence we are indeed back to problem (17).

The above assumes an open loop control u (a prior commitment), but we have a similar situation if we want to accept a control law of the form (15). Let us again restrict the control law to be linear. It amounts to an affine map $u = \mathcal{F}w + f$ in an admissible family of such maps (in particular, the matrix of $\mathcal{F}$ will be triangular to insure causality). Substituting this in J_γ, we again obtain a non homogeneous quadratic form in w. The same technique of completing the square will lead to the same conclusion, that G_γ is proportional to the exponential of $(1/2\gamma^2)\max_w J_\gamma(\mu(y), w)$.

The above result is what we like to call "Whittle's magic formula". P. Whittle was able to go further, proving, in a slightly simpler case, a separation theorem which implies our certainty equivalence theorem in that case. This approach was used by James Barras and Elliott 1993 and 1994 to derive a solution to the partial information minimax control problem when the certainty equivalence theorem does not hold. Their derivation was contemporaneous to, and independant from, our own derivation of the same result using tools introduced in Başar and Bernhard 1991, and since generalized in the framework of the $(\max, +)$ algebra (see Bernhard 2000).

5 "Robust" is not necessarily "cautious"

Among the missconceptions concerning robust control, one is that it is by nature cautious, because it does not rely on an uncertain model. This is not necessarily so, and the following examples are meant to illustrate that point. We shall all along assume perfect state information, and concentrate on the state feedback gain.

5.1 Disturbance attenuation

Let us consider the simple system where all variables are scalar :

$$\dot{x} = -x + u + w\,,$$

$$z \;=\; \begin{pmatrix} x \\ u \end{pmatrix}.$$

Consider as a reference LQG control theory, with $w(\cdot)$ taken as a normalized "white noise". The corresponding algebraic Riccati equation is

$$-2P - P^2 + 1 = 0\,.$$

The positive root is $P = \sqrt{2} - 1 \simeq .414$. This is also the optimal feedback gain $F = P$ as well as the expectation of the (limit of the) integrand in the quadratic performance index, i.e.

$$\lim_{t\to\infty} \mathbb{E}\|z(t)\|^2 = \mathbb{E} \lim_{T\to\infty} \frac{1}{T}\int_0^T \|z(t)\|^2 \,\mathrm{d}t = \sqrt{2} - 1 = 0.414\,.$$

Using essentially the same theory (or that of $\mathcal{H}_\infty$norms), one can also check that with that control, the noise attenuation from w to z is $\gamma = \sqrt{2-\sqrt{2}} \simeq .765$.

Let us now use the theory of "robust" noise attenuation. The Riccati equation is

$$-2P - (1-\gamma^{-2})P^2 + 1 = 0\,.$$

It has a positive real root down to $\gamma = 1/\sqrt{2} = 0.707$, for which the positive real root is unique and is $P = 1$. Since again, the feedback gain is $F = P$, we see that it is *larger* than in the previous case. Of course, the noise attenuation, as measured by L^2 norms, is better (it is optimized here): .707 versus .765, while correlatively, if we assume that w is a normalized "white noise" as previously, this leads to a worse output covariance (it was minimized in the previous case), specifically $\mathbb{E}x^2 = 1/4$, and thus (since $u = -x$) $\mathbb{E}\|z\|^2 = .5$ versus .414.

To summarize, the "robust noise attenuation" control leads to more control effort, for a better L^2-norm noise attenuation, at the expense of a larger control power that degrades the output covariance in the case where the disturbance is (looks like) a normalized white noise.

5.2 Robust stabilization

Let us now examine how the theory of robust stabilization (still with perfect state measurement) works on such simple examples. We consider two situations depending on whether the plant uncertainty resides with the free dynamics or the control channel efficiency.

5.2.1 Uncertain free dynamics

Let us consider the system

$$\dot{x} = -x + \delta a\, x + u\,, \quad |\delta a| < \alpha\,,$$

to be stabilized not knowing the exact value of δa.

Application of the above theory leads to the consideration of the system

$$\begin{aligned} \dot{x} &= -x + u + r\,, \\ s &= \begin{pmatrix} \alpha x \\ \varepsilon u \end{pmatrix} \end{aligned}$$

with the uncertain feedback

$$r = \Delta G s\,, \qquad \Delta G = [\frac{\delta a}{\alpha} \quad 0]\,.$$

The bound on the uncertainty is now

$$\|\Delta G\| < 1\,. \tag{38}$$

We were obliged to introduce the extra εu component in s to insure that the equivalent of the G matrix of equation (3) be injective. We shall use it as a tuning parameter of the design method.

The Riccati equation is now

$$-2P - (\varepsilon^{-2} - 1)P^2 + \alpha^2 = 0\,.$$

Its smaller positive root is, for small ε's

$$P = \frac{1}{\varepsilon^{-2} - 1}\left(\sqrt{1 + \alpha^2(\varepsilon^{-2} - 1)} - 1\right).$$

The corresponding feedback gain is $F = \varepsilon^{-2}P$. We see that for small ε's, it is close to α/ε. Hence, the theory says : if you want to stabilize the above uncertain system, just use a large nagative feedback gain. The larger the uncertainty (α) the larger the feedback gain. But we may make it arbitrarily large, since we did not attempt to simultaneously control an output containing u.

If we are interested in limiting the feedback gain, we may look at the same design procedure for large ε's. We see that there exists a positive root to the Riccati equation if and only if $1-(1-\varepsilon^{-2})\alpha^2 > 0$. Thus if $\alpha < 1$, we may take ε arbitrarily large, and correlatively F arbitrarily small. We do not have to control at all, the system is spontaneously stable whatever δa within its bounds. If $\alpha > 1$, however, the limiting ε is $\alpha/\sqrt{\alpha^2-1}$, and the corresponding feedback gain is $F = \alpha^2 - 1$.

As a matter of fact, this leads to the closed loop system

$$\dot{x} = -(\alpha^2 - \delta a)x$$

which is stable for every $\delta a < \alpha^2$, and a fortiori for the bound (38).

That we do not find $F = \alpha - 1$ is only a reflexion on the fact that our design procedure is conservative. As a matter of fact, with (38), we have allowed any $\Delta G = [p \quad q]$ with $\sqrt{p^2+q^2} < 1$. Thus we have controlled the family of systems

$$\dot{x} = -(1 - p\alpha)x + (1 + q\varepsilon)u\,, \qquad p^2 + q^2 < 1\,.$$

It is a simple exercise to place $u = -Fx$ in that system, and investigate for which ε there is an F that insures $-1 + p\alpha - (1 + q\varepsilon)F < 0$. One indeed finds that ε should not be larger than $\alpha/\sqrt{\alpha^2-1}$, and that at this limiting value, the only satisfactory F is $\alpha^2 - 1$. (Avoiding that degree of conservatism is the aim of "μ-synthesis".)

5.2.2 Uncertain control channel efficiency

We use a similar approach to stabilize the unstable system

$$\dot{x} = x + (1 + \delta b)u\,, \qquad |\delta b| < \beta\,.$$

(Applying the design procedure to a stable system would lead to no control.) We proceed in the same fashion, using the system

$$\begin{aligned} \dot{x} &= x + u + r\,, \\ s &= \begin{pmatrix} \varepsilon x \\ \beta u \end{pmatrix}, \end{aligned}$$

with

$$r = \Delta G s, \qquad \Delta G = [0 \quad \frac{\delta b}{\beta}]\,.$$

and again the bound (38). The term εx in s is now needed to insure the observability condition of the theory.

The Riccati equation is now

$$(1 - \beta^{-2})P^2 + 2P + \varepsilon^2 = 0\,.$$

For $\beta < 1$ it always has a positive root

$$P = \frac{1}{\beta^{-2} - 1}\left(1 + \sqrt{1 + \varepsilon^2(\beta^{-2} - 1)}\right)$$

leading to the feedback gain $F = P/\beta^2$, the limit of which as $\varepsilon \to 0$ is now $F = 2/(1-\beta^2)$. The worst closed loop system is then, for $\delta b = -\beta$,

$$\dot{x} = -\frac{1-\beta}{1+\beta}x$$

which is indeed stable since here $\beta < 1$.

Again for β close to 1, we need a large feedback gain to compensate for the fact that the control channel may be very inefficient.

For $\beta > 1$, the problem clearly has no solution : our system is unstable, and we do not know the sign of the coefficient of u in the dynamics. The Riccati equation always has two negative roots.

5.2.3 Mixed case

We consider the uncertain system

$$\dot{x} = -(1-\delta a)x + (1+\delta b)u\,, \qquad (\delta a)^2 + (\delta b)^2 < \rho^2\,.$$

We may expect that for $\rho \leq \sqrt{2}$ that family of systems can be stabilized, because if $1 - \delta a < 0$, making the free system unstable, then the sign of $1 + \delta b$ is known to be positive, so that it is possible to control the system.

As a matter of fact, the Riccati equation associated to that problem is

$$(1-\rho^{-2})P^2 - 2P + \rho^2 = 0\,,$$

which has a positive root provided that $\rho \leq \sqrt{2}$. For $\rho > 1$, that root leads to a feedback gain $F = (1 - \sqrt{2-\rho^2})/(\rho^2 - 1)$, equal to 1 if $\rho = \sqrt{2}$.

We do not have large gains any more, this is indeed a cautious control, because one has to balance the risk of not compensating an unstable free dynamics and that of exerting a “positive feedback”.

Notice that if the a priori bound on the uncertainty is of the form $|\delta a| < \alpha$ and $|\delta b| < \beta$, within the current simple $\mathcal{H}_\infty$-optimal control theory, we cannot do better than the above, with $\rho^2 = \alpha^2 + \beta^2$.

5.3 Robust stabilizing control

At last, we consider the simultaneous stabilization and control of our simple system.

5.3.1 Uncertain free dynamics

We want to control z in the disturbed uncertain system

$$\begin{aligned} \dot{x} &= -x + (\delta a)x + u + w\,, \qquad |\delta a| < \alpha\,, \\ z &= \begin{pmatrix} x \\ u \end{pmatrix}. \end{aligned}$$

We may seek to minimize the system norm from $(r \quad w)$ to z, and check whether this norm is less than $1/\alpha$, which, in view of the small gain theorem, is sufficient to insure stability. One finds that the limiting γ is 1. Thus this procedure suceeds only if $\alpha < 1$.

For $\alpha > 1$, we can apply the standard procedure advocated above : introduce both $s = \alpha x$ and $z' = (1/\gamma)z$, i.e. an output in $\mathbb{R}^3$. Then seek for which γ we can insure that the operator normm from $(r \quad w)$ to $(s \quad z')$ be less than one, guaranteeing both robust stability and a disturbance attenuation of γ from w to z. We propose a slightly different approach, which turns out to give much better results. (This also serves the purpouse of showing that this whole theory is to be applied with cleverness.)

We write the system

$$\begin{aligned} \dot{x} &= -x + u + r + w\,, \\ s &= \begin{pmatrix} \alpha x \\ \frac{1}{\gamma} u \end{pmatrix}, \end{aligned}$$

again $r = \Delta G s$ with $\|\Delta G\| < 1$. We attempt to insure an operator norm from $(r \quad w)$ to s less than 1. This insures that the system is stable, and, if $\gamma > 1/\alpha$, (as will occur for $\alpha > 1)$ a fortiori an operator norm from w to z less than γ. (Since $\|z\| < \gamma\|s\|$.)

The Riccati equation associated to that new problem is

$$(2 - \gamma^2)P^2 - 2P + \alpha^2 = 0\,,$$

wich has a positive root provided that $\gamma^2 \geq 2 - \alpha^{-2}$. (Notice that this gives back $\gamma \geq 1$ if $\alpha = 1$, we recover the same limiting case as above.) For the limiting γ, we have $P = \alpha^2$ and $F = \gamma^2 P = 2\alpha^2 - 1$. Thus, this solution rules out “large gains”. In that respect it embodies a degree of caution. But as compared to the smallest gain obtained in the robust stabilization section, with no regard for an output, we use a larger feedback gain : $2\alpha^2 - 1$ versus $\alpha^2 - 1$. In that respect, robust control is not cautious control.

5.3.2 Uncertain control channel

A symmetric situation results for the uncertain control channel case :

$$\dot{x} = -x + (1 + \delta b)u + w\,, \qquad |\delta b| < \beta\,,$$

with the same controlled output, leading when $\beta > 1/\sqrt{2}$ to the limit $\gamma^2 \leq 2 - \beta^{-2}$ and a feedback gain $F = 2\beta^2 - 1$.

5.3.3 Mixed case

As a last example, we consider the system with uncertainties on both the free dynamics and the control channel, that we want to simultaneously satbilize and control :

$$\begin{aligned} \dot{x} &= -(1 - \delta a)x + (1 + \delta b)u + w\,, \qquad |\delta a| < \alpha\,, \quad |\delta b| < \beta\,, \\ z &= \begin{pmatrix} x \\ u \end{pmatrix} . \end{aligned}$$

Again, the standard approach proposed in the general theory would have us introduce a two dimensional output s, in addition to the controlled output z, and thus apply $\mathcal{H}_\infty$-optimal control theory with a four dimensional output. We avoid that higher dimension via another trick. We need to have a way to tune the relative weight of the two objectives : stabilization and control of z, in such a way as to achieve the best possible disturbance attenuation in z without sacrificing stability. A way to achieve that goal is to introduce an output s modeled as previously, embodying both x and u, and introduce a scaling weight $1/\gamma$ on the disturbance's input channel, i.e. let $v = (1/\sqrt{2}\gamma)w$. (The $\sqrt{2}$ is there for normalisation purposes.) Then we apply the $\mathcal{H}_\infty$-control theory to the system from $(r \quad v)$ to s.

We shall therefore work with the system

$$\begin{aligned} \dot{x} &= -x + u + r + \frac{1}{\gamma\sqrt{2}}v\,, \\ s &= \begin{pmatrix} \alpha x \\ \beta u \end{pmatrix} , \end{aligned}$$

with the uncertainty model

$$r = [\frac{\delta a}{\alpha} \quad \frac{\delta b}{\beta}]s = \Delta G s\,,$$

and the bound $\|\Delta G\| < \sqrt{2}$. Stability is insured if the norm of the operator from $(r \quad v)$ to s is less than $1/\sqrt{2}$. In that case, the disturbance attenuation factor, in L^2

norm, is better than $\gamma/\min\{\alpha,\beta\}$. (Of course, this is an efficient design procedure only if α and β are of the same order of magnitude.)

The Riccati equation of that problem is

$$(2+\gamma^{-2}-\beta^{-2})P^2-2P+\alpha^2=0\,.$$

It has a positive solution provided that

$$\alpha^{-2}+\beta^{-2}\geq 2 \tag{39}$$

and

$$\gamma^{-2}\leq\alpha^{-2}+\beta^{-2}-2$$

and the feddback gain that yields the limiting attenuation factor is

$$F=\frac{\alpha^2}{\beta^2}\,. \tag{40}$$

That this gain is indeed stabilizing can be checked directly : it leads to

$$\dot{x}=-\alpha^2(\alpha^{-2}+\beta^{-2}-\frac{\delta a}{\alpha^2}-\frac{\delta b}{\beta^2})x+w \tag{41}$$

and it is easily seen that

$$\frac{\delta a}{\alpha^2}+\frac{\delta b}{\beta^2}<\frac{1}{\alpha}+\frac{1}{\beta}<\sqrt{2}\sqrt{\alpha^{-2}+\beta^{-2}}$$

so that the condition (39) is precisely sufficient to insure that (41) be stable.

Formula (40) has an interesting interpretation : the feedback gain should be chosen large or small depending on whether the larger uncertainty is in the free dynamics or the control channel respectively. If we trust the control channel, we may use it to correct an uncertain free dynamics. If, on the contrary, we trust more a stable dynamics than the control channel, we should exert control with care. Hence, $\mathcal{H}_\infty$-optimal control theory appears as more or less cautious depending on where the uncertainties in the system lie.

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