On the standard problem of H_∞-optimal control for infinite dimensional systems

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1 Introduction

We investigate in this paper the so called "standard problem" of H_∞ optimal control in an infinite dimensional setup, general enough to account for distributed parameter systems for instance.

Our objective is to extend to infinite dimensional systems the methods and results of [4], thus recovering in a simple way the results of [1] in the case of perfect information (state feedback control) and of [2] in the partial information case (output feedback control). The motivation for doing so is, on the one hand that the methods seem much simpler, and more importantly on the other hand that one can then extend them to other problems, such as the sampled data problem which is of paramount importance to control theoretists, and various information structures such as considered in [4].

Several difficulties arise in trying to carry out this program, mainly for the infinite horizon problem that we tackle here. One is connected with the conjugate point theory as developped in [6] and [4]. It is completely overcomed here, through a technique trivially extendable to finite time problems, and that yields a stronger result.

However, in the partial information case, while the certainty equivalence theorem of [4] can easily be extended, several technical difficulties arise related to the asymptotic behaviour of the Riccati equation. We propose here a solution based upon the use of duality, more in the line of [7], which seems significantly simpler than [2].

All the necessary tools in infinite dimensional system theory needed in the sequel can be found in [5].

2 General Presentation of the Problem

2.1 Notation and Assumptions

Let X be a real separable Hilbert space, and A be the infinitesimal generator of a C_0-semigroup e^{At} in X. We denote by $D(A)$ the domain of the operator A. We say that the semigroup is exponentially stable if one has

$$||e^{At}|| \leq Me^{-\alpha t}, \alpha > 0.$$

A linear unbounded operator A in X is said *exponentially stable* if it is the infinitesimal generator of a C_0-semigroup e^{At} in X which is exponentially stable. We recall the important result of DATKO [3], namely that A is exponentially stable iff for any h in X the solution of

$$x' = Ax, x(0) = h$$

belongs to $L^2(0, \infty; X)$. Let also U and W be real separable hilbert spaces, and B and D linear bounded operators from respectively U and W to X. Consider the dynamic system governed by the equation

$$(2.\ 1)\qquad \begin{aligned} x' &= Ax + Bv + Dw \\ x(0) &= 0. \end{aligned}$$

In equation (2. 1) w stands for a disturbance and v stands for a control. We now make precise what is meant by *robust control*. Let Z be an addditional Hilbert space and $H \in \mathcal{L}(X;Z)$. Suppose the controller is interested in the cost function

$$K_0(v,w) = \int_0^\infty (|Hx|^2 + |v|^2)\, dt$$

(which may take the value $+\infty$) that he wishes to keep small in spite of the disturbances that are not known in advance. Clearly, if w is, in some sense, "large", it will be able to force a large value of K_0. A reasonable goal, in view of the linearity of the system, is to try to insure

$$K_0(v,w) \le \gamma^2 \int_0^\infty |w|^2\, dt$$

for some given (positive) number γ, provided of course that the disturbance be square integrable, which will always be assumed in this paper. We therefore introduce the ratio

$$\rho(v,w) = \frac{K_0(v,w)}{\displaystyle\int_0^\infty |w|^2\, dt},$$

that the controller shall try to hold below a fixed value regardless of w.

The next question to address is that of the *admissible control strategies*. In this section, we assume that the controller has access to instantaneous perfect measurements of the state of the system 2. 1. We therefore want to allow controls of the form $v(t) = \mu(x(t))$ for a large class of μ's. We shall let $\mathcal{M}$ be the class of all applications from X into V that are such that the differential equation

$$x' = Ax + B\mu(x) + Dw, \qquad x(0) = h$$

has a solution over $(0,\infty)$ for all $h \in X$ and for all square integrable $w(t)$, and such that the control $v(t) = \mu(x(t))$ generated be square integrable. As a matter of fact, we could replace state feedbacks by a larger class of *causal* functions of state. The sequel is unchanged by such a generalization.

We shall use the unambiguous notation $\rho(\mu,w)$ to mean the value taken by $\rho(v,w)$ when $v = \mu(x)$. We can then define the property of interest in this section:

Definition 2.1 *We say that the γ^2 robustness property (with full observation) holds for the equation (2. 1) and the cost function $K_0(v,w)$ if one has*

$$\inf_{\mu\in\mathcal{M}} \sup_w \rho(\mu,w) < \gamma^2.$$

2.2 Review of some results

Consider the pair of operators A, B, H as above.

Definition 2.2 *We say that the pair A, B is H stabilizable if for any $h \in X$ there exists $v \in L^2(0,\infty;U)$ such that the solution x of the equation*

$$(2.\ 2)\qquad \begin{aligned} x' &= Ax + Bv \\ x(0) &= h \end{aligned}$$

satisfies $Hx \in L^2(0,\infty;Z)$. We say that the pair A, B is stabilizable if it is I stabilizable.

It is a classical result that the pair A,B is stabilizable iff there exists an operator $F \in \mathcal{L}(X,U)$ such that $A+BF$ is exponentially stable. We now give another

Definition 2.3 *We say that the pair A, H is detectable if the pair A^*, H^* is stabilizable.*

It follows from the characterization of stabilizability that the pair A,H is detectable iff there exists an operator $G \in \mathcal{L}(Z,X)$ such that $A+GH$ is exponentially stable. We now state an important result in Control Theory, whose proof for infinite dimensional systems can be found in [5].

Theorem 2.1 *We assume that A, B is H stabilizable and that A, H is detectable. Then there exists one and only one operator $\Gamma \in \mathcal{L}(X,X)$ with $\Gamma = \Gamma^* \geq 0$ satisfying $A - BB^*\Gamma$ is exponentially stable and*

(2. 3) $$\Gamma A + A^*\Gamma - \Gamma BB^*\Gamma + H^*H = 0.$$

The interpretation of the operator Γ is important. Consider the functional

$$K_h(v) = \int_0^\infty (|Hx|^2 + |v|^2)\, dt$$

where x is the solution of (2. 2). Then one has

$$(\Gamma h, h) = \min K_h(v).$$

The minimum in $K_h(v)$ is attained for $u = -B^*\Gamma y$ where y is the solution of

$$y' = (A - BB^*\Gamma)y \quad y(0) = h.$$

Remark 2.1 *In equation 2. 3, one must interpret the operator*

$$\Gamma A + A^*\Gamma$$

according to the general theory of algebraic Riccati equations (see [5] for details). It is sufficient to notice that for any pair h, k in $D(A)$ the bilinear form

$$< (\Gamma A + A^*\Gamma)h, k >= (Ah, \Gamma k) + (\Gamma h, Ak)$$

makes sense.

Note also that if $h \in D(A)$ then $\Gamma h \in D(A^*)$.

3 γ^2 robustness property with full observation

3.1 Setting of the Result

We state the result due to [1].

Theorem 3.1 *We assume that A, B is stabilizable and that A, H is detectable. Then the γ^2 robustness property with full observation holds for the equation (2. 1) and the cost function $K_0(v,w)$ iff there exists a $P \in \mathcal{L}(X,X)$ with $P = P^* \geq 0$ satisfying*

(3. 1) $$PA + A^*P - P(BB^* - \frac{1}{\gamma^2}DD^*)P + H^*H = 0$$

and $A - (BB^* - \frac{1}{\gamma^2}DD^*)P$ is exponentially stable.

The operator P has an interesting interpretation. Indeed let be

$$K_h(v,w)=\int_0^\infty (|Hx|^2+|v|^2)\,dt$$

where x is the solution of

(3. 2)
$$\begin{aligned} x' &= Ax+Bv+Dw \\ x(0) &= h. \end{aligned}$$

Let also

$$J_h(v,w)=K_h(v,w)-\gamma^2\int_0^\infty |w|^2\,dt.$$

Then one has

(3. 3)
$$(Ph,h)=\max_w \min_v J_h(v,w).$$

It is important to check the following result

Lemma 3.1 *If P satisfies the properties stated in Theorem 3.1 then one has also $A-BB^*P$ is exponentially stable.*

Proof Considering the solution x of (3. 2), computing the derivative of $(Px(t),x(t))$ and integrating between 0 and T yields

(3. 4)
$$\begin{aligned} &(Px(T),x(T))-(Ph,h)+\int_0^T |Hx|^2\,dt+\int_0^T |v|^2\,dt \\ &-\gamma^2\int_0^T |w|^2\,dt=\int_0^T |v+B^*Px|^2\,dt-\gamma^2\int_0^T |w-\frac{1}{\gamma^2}D^*Px|^2\,dt \end{aligned}$$

Choose $v=-B^*Px$ and $w=0$, then we see that x is the solution of

$$x'=(A-BB^*P)x \quad ,x(0)=h$$

and from the equation (3. 4) it follows that

$$Hx\in L^2(0,\infty;Z), B^*Px\in L^2(0,\infty;U), D^*Px\in L^2(0,\infty;W).$$

Since we already know that $A-(BB^*-\frac{1}{\gamma^2}DD^*)P$ is exponentially stable, we can assert that $x\in L^2(0,\infty;X)$, hence the desired result. ♠

3.2 Proof of Main result

Sufficiency.

The control strategy that achieves the required result will be $\mu(x)=-B^*Px$. We consider the solution of

$$x'=(A-BB^*P)x+Dw \quad ,x(0)=0.$$

Since $A-BB^*P$ is exponentially stable and $w\in L^2(0,\infty;W)$ the solution x belongs to $L^2(0,\infty;X)$, as well as the control v generated. Thus this μ is admissible. We apply the relation (3. 4) with $v=-B^*Px$ and $h=0$, letting T tend to ∞. This yields

$$\int_0^\infty |Hx|^2\,dt+\int_0^\infty |B^*Px|^2\,dt=\gamma^2\int_0^\infty |w|^2\,dt-\gamma^2\int_0^\infty |w-\frac{1}{\gamma^2}D^*Px|^2\,dt.$$

Therefore

$$\rho(-B^*Px,w)=\gamma^2-\gamma^2\frac{\displaystyle\int_0^\infty |w-\frac{1}{\gamma^2}D^*Px|^2\,dt}{\displaystyle\int_0^\infty |w|^2\,dt}.$$

It follows that

$$\sup_w \rho(-B^*Px, w) \le \gamma^2 - \gamma^2 \inf_w \frac{\int_0^\infty |w - \frac{1}{\gamma^2}D^*Px|^2\, dt}{\int_0^\infty |w|^2\, dt}.$$

Now x can be viewed as the solution of

$$x' = \left(A - (BB^* - \frac{1}{\gamma^2}DD^*)P\right)x + D(w - \frac{1}{\gamma^2}D^*Px) \ \ , x(0) = 0.$$

Since $A - (BB^* - \frac{1}{\gamma^2}DD^*)P$ is exponentially stable we have

$$\int_0^\infty |x|^2\, dt \le c_0 \int_0^\infty |w - \frac{1}{\gamma^2}D^*Px|^2\, dt.$$

Hence immediately

$$\int_0^\infty |w|^2\, dt \le c_1 \int_0^\infty |w - \frac{1}{\gamma^2}D^*Px|^2\, dt.$$

Therefore we can assert that

$$\inf_w \frac{\int_0^\infty |w - \frac{1}{\gamma^2}D^*Px|^2\, dt}{\int_0^\infty |w|^2\, dt} > 0. \tag{3. 5}$$

We deduce

$$\sup_w \rho(-B^*Px, w) \le \gamma^2.$$

Necessity.
We shall prove a result stronger than that of the theorem (3.1). Namely the following

Proposition 3.1 *If*

$$\sup_w \inf_v \rho(v, w) < \gamma^2,$$

then the conditions of theorem (3.1) hold.

This is indeed stronger than the theorem, since clearly, one has

$$\inf_v \rho(v, w) \le \rho(\mu, w)$$

for any admissible μ, and therefore

$$\sup_w \inf_v \rho(v, w) \le \sup_w \rho(\mu, w), \qquad \forall \mu \in \mathcal{M}$$

and thus

$$\sup_w \inf_v \rho(v, w) \le \inf_{\mu \in \mathcal{M}} \sup_w \rho(\mu, w).$$

We now prove the proposition. We consider a control problem where the system is defined by (3. 2) in which w is given, the control being v. We minimize the cost $K_h(v, w)$. Note that the assumptions of Theorem 2.1 are satisfied, thus the equation

$$\Gamma A + A^*\Gamma - \Gamma BB^*\Gamma + H^*H = 0. \tag{3. 6}$$

has a unique solution $\Gamma = \Gamma^* \ge 0$, such that the operator $A - BB^*\Gamma$ is exponentially stable. Consider next the linear equation

$$r' + (A^* - \Gamma BB^*)r + \Gamma Dw = 0 \tag{3. 7}$$

where the solution $r(.)$ belongs to $L^2(0,\infty;X)$. Note that in (3. 7) the initial condition is given at ∞ and not at 0. The equation (3. 7) has a unique solution. The optimal feedback is then descibed by

$$\hat{u} = -B^*(\Gamma\hat{x} + r)$$

where $\hat{x}$ the optimal state is the solution of

$$\frac{d}{dt}\hat{x} = (A - BB^*\Gamma)\hat{x} - BB^*r + Dw \; ; \hat{x}(0) = h.$$

We can also express the value of $J_h(\hat{u}, w)$ as follows

$$\begin{aligned} J_h(\hat{u}, w) = &-\int_0^\infty |B^*r|^2\, dt \\ &-\gamma^2\int_0^\infty |w|^2\, dt + 2\int_0^\infty (r, Dw)\, dt + 2(h, r(0)) + (\Gamma h, h). \end{aligned} \tag{3. 8}$$

The calculations leading to the expression (3. 8) are standard and not detailed here. We now look at the problem of maximizing the expression $J_h(\hat{u}, w)$ for $w \in L^2(0,\infty;W)$. Note that although we have an LQ problem, the concavity is not a priori verified. This is where we use the assumption of γ^2 robustness property. Set $\Phi_h(w) = J_h(\hat{u}, w)$, then we note the relation

$$\Phi_h(w) = \Phi_0(w) + 2(h, r(0)) + (\Gamma h, h).$$

By the assumption of γ^2 robustness there exists a positive $\delta < \gamma$ such that $\sup_w inf_v \rho(v, w) \leq \gamma^2 - \delta^2$. It follows that

$$J_0(\hat{u}, w) \leq -\delta^2\int_0^\infty |w|^2\, dt$$

for any w. Therefore, we can assert that

$$\Phi_0(w) \leq -\delta^2\int_0^\infty |w|^2\, dt. \tag{3. 9}$$

Note also the relation

$$\Phi_h(\theta w_1 + (1-\theta)w_2) = \theta\Phi_h(w_1) + (1-\theta)\Phi_h(w_2) - 2\theta(1-\theta)\Phi_0(w_1 - w_2).$$

This relation shows immediately that $\Phi_h(w)$ is strictly concave. From (3. 9) it is a coercive functional. Therefore we can apply to the control problem (3. 7) and cost function $\Phi_h(w)$ the standard theory of existence, uniqueness of an optimal control. Moreover, the theory of necessary and sufficient conditions of optimality hold. Calling $\hat{w}$ and $\hat{r}$ the optimal control and state, we have the relations

$$\begin{aligned} &\frac{d}{dt}\hat{r} + (A^* - \Gamma BB^*)\hat{r} + \Gamma D\hat{w} = 0, \\ &\frac{d}{dt}p = (A - BB^*\Gamma)p - BB^*\hat{r} + D\hat{w}, \end{aligned} \tag{3. 10}$$

$$\begin{aligned} &p(0) = h, \\ &\gamma^2\hat{w} - D^*\hat{r} - D^*\Gamma p = 0. \end{aligned} \tag{3. 11}$$

The classical decoupling argument applies to the system (3. 10) and (3. 11), therefore there exists $\Sigma = \Sigma^*$, such that $\hat{r} = \Sigma p$. The operator Σ is a solution of the Riccati equation

$$\begin{aligned} \Sigma\left(A - (BB^* - \frac{1}{\gamma^2}DD^*)\Gamma\right) + \left(A^* - \Gamma(BB^* - \frac{1}{\gamma^2}DD^*)\right)\Sigma \\ -\Sigma(BB^* - \frac{1}{\gamma^2}DD^*)\Sigma + \frac{\Gamma DD^*\Gamma}{\gamma^2} = 0. \end{aligned}$$

Set $P = \Gamma + \Sigma$, then P is self adjoint and is a solution of (3. 1), as easily seen. Note that p is the solution of

$$\begin{aligned}\frac{d}{dt}p &= \left(A - (BB^* - \frac{1}{\gamma^2}DD^*)P\right)p \\ p(0) &= h.\end{aligned}$$

Since $p \in L^2(0,\infty;X)$, we deduce that $A - (BB^* - \frac{1}{\gamma^2}DD^*)P$ is exponentially stable. Moreover

$$\max \Phi_h(w) = \Phi_h(\hat{w}) = (Ph, h).$$

Therefore, we have

$$\max_w \min_v J_h(v,w) = (Ph,h).$$

It follows among other things that

$$(Ph,h) \geq \min_v J_h(v,0) \geq 0.$$

The proof of the necessity part has been completed. This concludes the end of the proof. ♠

It should be pointed out that we have avoided the classical conjugate point argument, such as in [4] chapt. 8 for instance, or the arguments of [8] in the stationary case. It is completely clear that the present proof extends to the finite horizon non stationary case.

4 γ^2 robustness property with partial observation

4.1 Presentation of the problem

Consider again the system

(4. 1)
$$\begin{aligned}x' &= Ax + Bv + Dw \\ x(0) &= 0.\end{aligned}$$

and the cost function

$$K_0(v,w) = \int_0^\infty (|Hx|^2 + |v|^2)\,dt.$$

In the present situation, the controller has access only to a partial observation described as follows

(4. 2)
$$y = Cx + \eta$$

where $C \in \mathcal{L}(X;Y)$, Y being a new Hilbert space, called the space of observations. Next η is another disturbance, modelling the measurement error. The controller can use only a causal functional on the observation y. A natural class of controllers is the following one

$$v = Lp,$$

$$p' = (A+M)p + Ny, \qquad p(0) = 0.$$

In other words the controller is characterized by three maps $L \in \mathcal{L}(X;U)$, $M \in \mathcal{L}(X;X)$, $N \in \mathcal{L}(Y;X)$. We call it a *feedback controller*. We have not combined $A + M$ into a single operator since A is unbounded, whereas M is bounded. If we use such a controller in the equation 4. 1, then we get a coupled system as follows

(4. 3)
$$\begin{aligned}x' &= Ax + BLp + Dw, \\ p' &= (A+M)p + NCx + N\eta, \\ x(0) &= 0, \\ p(0) &= 0.\end{aligned}$$

The coupled system introduces an operator

$$\mathcal{A} = \begin{pmatrix} A & BL \\ NC & A+M \end{pmatrix}.$$

The operator $\mathcal{A}$ is the operator related to the feedback controller L, M, N. We shall consider only controllers whose corresponding operator is exponentially stable. For such a controller the cost

$$K_0(Lp, w) = \int_0^\infty (|Hx|^2 + |Lp|^2)\, dt$$

is finite. We can then define the ratio corresponding to the feedback controller L, M, N

$$\rho(L, M, N) = \sup_{w,\eta} \frac{K_0(Lp, w)}{\int_0^\infty (|w|^2 + |\eta|^2)\, dt}.$$

We state the

Definition 4.1 *We say that the γ^2 robustness property (with partial observation) holds for the equation (4. 1), the observation (4. 2) and the cost function $K_0(v, w)$ if there exists a feedback controller L, M, N such that the corresponding operator $\mathcal{A}$ is exponentially stable and if one has*

$$\rho(L, M, N) < \gamma^2.$$

Our objective is naturally to give a necessary and sufficient condition of γ^2 robustness property (with partial observation) which leads to a computable feedback.We shall see that this property is equivalent to 3 systems and corresponding costs enjoying the γ^2 robustness property (with full observation). From section 3 corresponding Riccati equations can be introduced. In fact, the solution of one of them is expressible in terms of the others. Moreover one of the system and cost is (4. 1) with cost $K_0(v, w)$. Therefore γ^2 robustness property(with partial observation) implies γ^2 robustness property (with full observation).

4.2 Statement of the main result

We state the result due to [2].

Theorem 4.1 *We assume that A, D is stabilizable and that A, H is detectable. Then the γ^2 robustness property with partial observation holds for the equation (4. 1), the observation (4. 2) and the cost function $K_0(v, w)$ iff there exist solutions of the Riccati equations*

(4. 4)
$$\begin{array}{lcl} PA + A^*P - P(BB^* - \frac{1}{\gamma^2}DD^*)P + H^*H & = & 0 \\ P \mathit{symmetric}, \geq 0 & & \\ A - (BB^* - \frac{1}{\gamma^2}DD^*)P \mathit{is\ exponentially\ stable} & & \end{array}$$

(4. 5)
$$\begin{array}{lcl} \Sigma A^* + A\Sigma - \Sigma(C^*C - \frac{1}{\gamma^2}H^*H)\Sigma + DD^* & = & 0 \\ \Sigma \mathit{symmetric}, \geq 0 & & \\ A^* - (C^*C - \frac{1}{\gamma^2}H^*H)\Sigma \mathit{is\ exponentially\ stable} & & \end{array}$$

(4. 6)
$$I - \frac{1}{\gamma^2}P\Sigma \mathit{is\ invertible}\,; \Sigma\left(I - \frac{1}{\gamma^2}P\Sigma\right)^{-1} \geq 0$$

If the conditions (4. 4), (4. 5), (4. 6) hold, then the feedback controller

$$L = -B^*P\,, M = -(BB^* - \frac{1}{\gamma^2}DD^*)P - \Pi C^*C\,, N = \Pi C^*$$

where

$$\Pi = \Sigma\left(I - \frac{1}{\gamma^2}P\Sigma\right)^{-1}$$

satisfies the conditions of the definition 4.1.

Remark 4.1 *The Riccati equation (4. 4) is the same as the one characterizing the γ^2 robustness property with full observation. Note that the pair A, B is stabilizable, as a consequence of the fact that $\mathcal{A}$ is exponentially stable. From formula (3. 3) the solution is unique. A similar property holds for (4. 5) since it characterizes the γ^2 robustness property with full observation of a different system and cost. This will be made precise in the proof.*

4.3 Duality Considerations

We shall prove here some norm equalities which will be instrumental in the proof of Theorem 4.1 and are interesting in themselves. Consider Hilbert spaces Ξ, Φ, Ψ, identified with their duals. Consider operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, which are linear, $\mathcal{A}$ is unbounded in Ξ and is the infinitesimal generator of a C_0 semigroup in Ξ. Next

$$\mathcal{B} \in \mathcal{L}(\Phi; \Xi) \ \ , \mathcal{C} \in \mathcal{L}(\Xi; \Psi).$$

To the triple $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ we associate a linear system

(4. 7)
$$\xi' = \mathcal{A}\xi + \mathcal{B}\phi \ \ ; \xi(0) = 0$$

and a corresponding observation $\mathcal{C}\xi$. We define next a "dual" triple $\mathcal{A}^*$, $\mathcal{C}^*$, $\mathcal{B}^*$ to which corresponds the system

(4. 8)
$$\zeta' = \mathcal{A}^*\zeta + \mathcal{C}^*\psi; \ \ \zeta(0) = 0,$$

and the corresponding observation $\mathcal{B}^*\zeta$. In equations (4. 7) and (4. 8) the quantities ϕ and ψ are inputs (or controls). We now state the following

Proposition 4.1 *Assume that $\mathcal{A}$ is exponentially stable. Then one has the relation*

$$\sup_{\phi} \frac{\int_0^\infty |\mathcal{C}\xi|^2 \, dt}{\int_0^\infty |\phi|^2 \, dt} = \sup_{\psi} \frac{\int_0^\infty |\mathcal{B}^*\zeta|^2 \, dt}{\int_0^\infty |\psi|^2 \, dt}.$$

We begin with some preliminary results.

Lemma 4.1 *Let $\phi \in L^2(-\infty; +\infty; \Phi)$, then there exists one and only one function*

$$\xi \in L^2(-\infty; +\infty; \Xi) \cap C^0(-\infty; +\infty; \Xi)$$

solution of the equation

(4. 9)
$$\xi' = \mathcal{A}\xi + \mathcal{B}\phi.$$

Proof

The solution of (4. 9) is interpreted as

(4. 10)
$$\xi(t) = \int_{-\infty}^{t} e^{\mathcal{A}(t-\tau)} \mathcal{B}\phi(\tau) \, d\tau$$

for any t. Using the estimate $||e^{\mathcal{A}t}|| \leq Me^{-\alpha t}$, for some convenient constants M and $\alpha > 0$, since $\mathcal{A}$ is exponentially stable, we deduce the inequality

$$|\xi(t)| \leq M||\mathcal{B}||\beta(t)$$

where

$$\beta(t) = \int_{-\infty}^{t} e^{-\alpha(t-\tau)} |\phi(\tau)| \, d\tau.$$

From Schwartz' inequality, one checks that

$$|\beta(t)| \leq \frac{1}{\sqrt{2\alpha}} \left(\int_{-\infty}^{+\infty} |\phi(s)|^2 \, ds \right)^{1/2}$$

and using the differential equation satisfied by β and

$$0 \leq \frac{1}{2}|\beta(\tau)|^2 = \int_{-\infty}^{\tau} \beta\beta' \, dt$$

one gets the other estimation

$$\int_{-\infty}^{+\infty} |\beta(t)|^2 \, dt \leq \frac{1}{\alpha^2} \int_{-\infty}^{+\infty} |\phi(t)|^2 \, dt.$$

Therefore the expression (4. 10) is a solution. The solution is unique. Indeed, it is sufficient to prove that when $\phi = 0$, then the solution is necessarily 0. But if $\phi = 0$, we can write

$$\xi(t) = e^{\mathcal{A}(t-s)}\xi(s)$$

hence

$$|\xi(t)| \leq e^{-\alpha(t-s)}|\xi(s)|.$$

For fixed t let $s \to -\infty$, using the fact that $|\xi(s)|$ remains bounded, we conclude that $|\xi(t)| = 0$. This proves the uniqueness of the solution. ♠

Let

$$\alpha = \sup_{\phi} \frac{\int_{0}^{\infty} |\mathcal{C}\xi|^2 \, dt}{\int_{0}^{\infty} |\phi|^2 \, dt}$$

where we refer to the system (4. 7). Similarly, set

$$\beta = \sup_{\phi} \frac{\int_{-\infty}^{\infty} |\mathcal{C}\xi|^2 \, dt}{\int_{-\infty}^{\infty} |\phi|^2 \, dt}$$

referring now to the system (4. 9). Then we have

Lemma 4.2 *The two numbers α and β are equal.*

Proof

It is first clear that

$$\alpha \leq \beta.$$

This is due to the fact that, when in (4. 9) we restrict ourselves to inputs ϕ which are equal to 0 for $t < 0$, then (4. 9) reduces immediately to (4. 7). It will be convenient to introduce the linear map Θ from $L^2(-\infty; +\infty; \Phi)$ to $L^2(-\infty; +\infty; \Psi)$ defined by

$$\Theta(\phi) = \mathcal{C}\xi$$

where ξ is the solution of (4. 9) corresponding to the input ϕ. Clearly

$$\beta = ||\Theta||^2.$$

Now to any $\phi \in L^2(-\infty; +\infty; \Phi)$ we associate ϕ_T defined by

$$\phi_T(t) = \begin{cases} \phi(t) & \text{if } t > -T \\ 0 & \text{otherwise} \end{cases}$$

Let us denote by ξ_T the solution of (4. 9) corresponding to the input ϕ_T. Setting

$$\tilde{\xi_T}(t) = \xi_T(-T+t) \ \ ; \tilde{\phi_T}(t) = \phi_T(-T+t)$$

we see immediately that $\tilde{\xi_T}$ is the solution of (4. 7) corresponding to the input $\tilde{\phi_T}$. From the definition of α we can write

$$\int_0^\infty |\mathcal{C}\tilde{\xi_T}(t)|^2 \, dt \leq \alpha \int_0^\infty |\tilde{\phi_T}(t)|^2 \, dt.$$

But

$$\int_0^\infty |\mathcal{C}\tilde{\xi_T}(t)|^2 \, dt = \int_{-\infty}^\infty |\mathcal{C}\xi_T(t)|^2 \, dt$$

and

$$\begin{aligned}\int_0^\infty |\tilde{\phi_T}(t)|^2 \, dt &= \int_{-\infty}^\infty |\phi_T(t)|^2 \, dt \\ &\leq \int_{-\infty}^\infty |\phi(t)|^2 \, dt\end{aligned}$$

Therefore we have proved that

$$|\Theta(\phi_T)|^2 \leq \alpha \int_{-\infty}^\infty |\phi(t)|^2 \, dt.$$

Letting $T \to \infty$ and noting that ϕ_T tends to ϕ in $L^2(-\infty;+\infty;\Phi)$, we obtain

$$|\Theta(\phi)|^2 \leq \alpha \int_{-\infty}^\infty |\phi(t)|^2 \, dt.$$

Since ϕ is arbitrary, it follows from the definition of β that

$$\beta \leq \alpha.$$

The proof has been completed. ♠
We can now proceed with the
Proof of Proposition 4.1 In a way similar to (4. 9) we consider the dual system on $-\infty, +\infty$

(4. 11) $$\zeta' = \mathcal{A}^*\zeta + \mathcal{C}^*\psi$$

where $\psi \in L^2(-\infty;+\infty;\Psi)$ and the solution ζ belongs to $L^2(-\infty;+\infty;\Xi) \cap C^0(-\infty;+\infty;\Xi)$.Define a linear map from $L^2(-\infty;+\infty;\Psi)$ to $L^2(-\infty;+\infty;\Phi)$ by setting

$$\Upsilon(\psi) = \mathcal{B}^*\zeta.$$

In view of Lemma 4.1 the desired result will be demonstrated if we prove

(4. 12) $$||\Theta|| = ||\Upsilon||$$

To any ψ associate $\bar{\psi}$ by setting $\bar{\psi}(t) = \psi(-t)$. The key point is to verify that

(4. 13) $$\Upsilon(\psi)(t) = \Theta^*(\bar{\psi})(-t).$$

This property implies the result (4. 12). Now (4. 13) is easily deduced from the explicit formula (4. 10) and the corresponding one for ζ the solution of (4. 11). Details are left to the reader. The proof has been completed. ♠

5 Proof of Theorem 4.1

5.1 Necessary Conditions

Proof of (4. 4)
In fact, the assumptions of Theorem 3.1 are satisfied, since the property of γ^2 robustness with full observation holds and that the pair A, B is stabilizable(see Remark 4.1). Therefore There exists a unique P solution of (4. 4).
Proof of (4. 5)
We shall use the duality considerations of paragraph 4.3, see Proposition 4.1. Let $\Xi = X \times X$,$\Phi = W \times Y$, $\Psi = Z \times U$.Let next

$$\mathcal{A} = \begin{pmatrix} A & BL \\ NC & A+M \end{pmatrix}$$

and

$$\mathcal{B} = \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} H & 0 \\ 0 & L \end{pmatrix}.$$

Setting $\xi = (x,p)$,$\phi = (w,\eta)$,then we see immediately that

$$\rho(L,M,N) = \alpha$$

where the number α has been defined in the proof of Lemma 4.2. We can then make use of Proposition 4.1. The dual system of (4. 3) is (apply (4. 8) and set $\zeta = m, q$, $\psi = \lambda, \mu$)

$$\begin{array}{lcl} m' &=& A^*m + C^*N^*q + H^*\lambda \\ q' &=& L^*B^*m + (A^* + M^*)q + L^*\mu \\ m(0) &=& 0 \\ q(0) &=& 0. \end{array} \tag{5. 1}$$

Using then Proposition 4.1 we can assert that

$$\rho(L,M,N) = \sup_{\lambda,\mu} \frac{\displaystyle\int_0^\infty (|D^*m|^2 + |N^*q|^2)\,dt}{\displaystyle\int_0^\infty (|\lambda|^2 + |\mu|^2)\,dt}.$$

Therefore from the assumption, we have the property

$$\sup_{\lambda,\mu} \frac{\displaystyle\int_0^\infty (|D^*m|^2 + |N^*q|^2)\,dt}{\displaystyle\int_0^\infty (|\lambda|^2 + |\mu|^2)\,dt} < \gamma^2.$$

In particular, restricting to $\mu = 0$, we have

$$\begin{array}{lcl} m' &=& A^*m + C^*N^*q + H^*\lambda \\ q' &=& L^*B^*m + (A^* + M^*)q \\ m(0) &=& 0 \\ q(0) &=& 0. \end{array} \tag{5. 2}$$

and

$$\sup_{\lambda} \frac{\displaystyle\int_0^\infty (|D^*m|^2 + |N^*q|^2)\,dt}{\displaystyle\int_0^\infty |\lambda|^2\,dt} < \gamma^2. \tag{5. 3}$$

Consider the dynamic system

$$(5.\ 4)\qquad \begin{aligned} m' &= A^*m + C^*\upsilon + H^*\lambda \\ m(0) &= 0 \end{aligned}$$

where υ is the control and λ is the perturbation. We first note that the pair A^*, C^* is stabilizable, as a consequence of the fact that the operator $\mathcal{A}$,hence its dual $\mathcal{A}^*$ is exponentially stable. Consider also the cost function

$$\mathcal{K}_0(\upsilon, \lambda) = \int_0^\infty (|D^*m|^2 + |\upsilon|^2)\, dt.$$

¿From the assumptions A^*, D^* is detectable. Moreover, from (5. 3) we can assert that the γ^2 robustness property holds for the system (5. 4) and the cost function $\mathcal{K}_0(\upsilon, \lambda)$. Therefore, relying on Theorem 3.1we obtain the existence and uniqueness of the solution Σ of (4. 5).

Proof of (4. 6)

The proof will be decomposed in several steps. We begin with

-a:The matrix operator

$$\mathcal{A}_P = \begin{pmatrix} A + \frac{1}{\gamma^{*2}} DD^*P & BL \\ NC & A + M \end{pmatrix}$$

is exponentially stable. For that purpose consider the dynamic system

$$(5.\ 5)\qquad \begin{aligned} \bar{x}' &= \left(A + \frac{1}{\gamma^{*2}} DD^*P\right)\bar{x} + BL\bar{p} \\ \bar{p}' &= (A+M)\bar{p} + NC\bar{x} \\ \bar{x}(0) &= h \\ \bar{p}(0) &= k. \end{aligned}$$

Then we must prove that

$$(5.\ 6)\qquad \bar{x} \in L^2(-\infty; +\infty; X)\ \ ; \bar{p} \in L^2(-\infty; +\infty; X)$$

Consider then the system

$$(5.\ 7)\qquad \begin{aligned} x' &= Ax + BLp + Dw \\ p' &= (A+M)p + NCx \\ x(0) &= h \\ p(0) &= k. \end{aligned}$$

where $w \in L^2(-\infty; +\infty; W)$ and the system corresponding to $w = 0$

$$(5.\ 8)\qquad \begin{aligned} x_1' &= Ax_1 + BLp_1 \\ p_1' &= (A+M)p_1 + NCx_1 \\ x_1(0) &= h \\ p_1(0) &= k. \end{aligned}$$

Considering the differences $x - x_1$ and $p - p_1$ we can make use of the assumption

$$\rho(L, M, N) < \gamma^2$$

to assert that there exists a number $\delta < \gamma$ such that

$$(5.\ 9)\qquad \begin{aligned} &\int_0^\infty (|H(x - x_1)|^2 + |L(p - p_1)|^2)\, dt - \gamma^2 \int_0^\infty |w|^2\, dt \leq \\ &\qquad -\delta^2 \int_0^\infty |w|^2\, dt \end{aligned}$$

Furthermore, since $\mathcal{A}$ is exponentially stable

$$\int_0^\infty (|x_1|^2 + |p_1|^2)\, dt \leq C(|h|^2 + |k|^2)$$

where C is a constant independant of h, k. Combining this estimate and (5. 9), it is easy to deduce the following

$$\int_0^\infty (|Hx|^2 + |Lp|^2)\,dt - \gamma^2 \int_0^\infty |w|^2\,dt \leq$$

$$-{\delta_0}^2 \int_0^\infty |w|^2\,dt + C_0(|h|^2 + |k|^2) \tag{5. 10}$$

where $\delta_0 < \delta$ and C_0 is an appropriate constant. Let us set

$$\mathcal{J}_{h,k}(w) = \int_0^\infty (|Hx|^2 + |Lp|^2)\,dt - \gamma^2 \int_0^\infty |w|^2\,dt.$$

Note that in the functional $\mathcal{J}_{h,k}(w)$ the triple L, M, N is fixed and the control is w. The estimate (5. 10) shows easily that the functional $\mathcal{J}_{h,k}(w)$ is strictly concave and tends to $-\infty$ as $w \to \infty$. Therefore for any pair h, k there exists an optimal $\hat{w}_{h,k}$ which maximizes $\mathcal{J}_{h,k}(w)$ with respect to w.We denote by $\hat{x}_{h,k}, \hat{p}_{h,k}$ the corresponding optimal state. Let T be arbitrary and set $\bar{x}_T = \bar{x}(T), \bar{p}_T = \bar{p}(T)$ where $\bar{x}, \bar{p}$ is the solution of (5. 5). We now define

$$\tilde{x}_T(t) = \left| \begin{array}{ll} \bar{x}(t) & \text{if } t \leq T \\ \hat{x}_{\bar{x}_T,\bar{p}_T}(t-T) & \text{if } t > T \end{array} \right. \qquad \tilde{p}_T(t) = \left| \begin{array}{ll} \bar{p}(t) & \text{if } t \leq T \\ \hat{p}_{\bar{x}_T,\bar{p}_T}(t-T) & \text{if } t > T \end{array} \right.$$

and

$$\tilde{w}_T(t) = \left| \begin{array}{ll} \frac{1}{\gamma^2} D^* P \bar{x}(t) & \text{if } t < T \\ \hat{w}_{\bar{x}_T,\bar{p}_T}(t-T) & \text{if } t > T \end{array} \right.$$

By construction $\tilde{x}_T, \tilde{p}_T$ is the solution of (5. 7) corresponding to $\tilde{w}_T$.Therefore from (5. 10) we can write the estimate

$$\int_0^\infty (|H\tilde{x}_T|^2 + |L\tilde{p}_T|^2)\,dt - \gamma^2 \int_0^\infty |\tilde{w}_T|^2\,dt \leq$$

$$-{\delta_0}^2 \int_0^\infty |\tilde{w}_T|^2\,dt + C_0(|h|^2 + |k|^2) \tag{5. 11}$$

Now we have

$$\int_T^\infty \left(|H\tilde{x}_T|^2 + |L\tilde{p}_T|^2 - \gamma^2 |\tilde{w}_T|^2\right)\,dt = \int_0^\infty \left(|H\hat{x}_{\bar{x}_T,\bar{p}_T}|^2 + |L\hat{p}_{\bar{x}_T,\bar{p}_T}|^2 - \gamma^2 |\hat{w}_{\bar{x}_T,\bar{p}_T}|^2\right)\,dt$$

which is by construction

$$\max_w \left[\int_0^\infty \left(|Hx_{\bar{x}_T,\bar{p}_T}|^2 + |Lp_{\bar{x}_T,\bar{p}_T}|^2 - \gamma^2 |w|^2\right)\,dt\right]$$

where we have denoted by $x_{\bar{x}_T,\bar{p}_T}$, $p_{\bar{x}_T,\bar{p}_T}$ the solution of (5. 7) corresponding to initial conditions $h = \bar{x}_T, k = \bar{p}_T$.Clearly this quantity is larger or equal to

$$\max_w \min_v J_{\bar{x}_T}(v, w) = (P\bar{x}_T, \bar{x}_T).$$

Therefore we have proved that

$$\int_T^\infty \left(|H\tilde{x}_T|^2 + |L\tilde{p}_T|^2 - \gamma^2 |\tilde{w}_T|^2\right)\,dt \geq (P\bar{x}_T, \bar{x}_T)$$

But from the Riccati equation (4. 4) and the first equation (5. 5) we have

$$(P\bar{x}_T, \bar{x}_T) = (Ph, h) + \int_0^T \left(|B^* P\bar{x} + L\bar{p}|^2 + \frac{1}{\gamma^2} |D^* P\bar{x}|^2 - |L\bar{p}|^2 - |H\bar{x}|^2\right)\,dt.$$

Combining the two last relations we deduce

$$\int_0^\infty \left(|H\tilde{x}_T|^2 + |L\tilde{p}_T|^2 - \gamma^2|\tilde{w}_T|^2\right) dt \geq (Ph,h) + \int_0^T |B^*P\bar{x} + L\bar{p}|^2\, dt \tag{5. 12}$$

Finally from (5. 11) and (5. 12) we get the estimate

$$(Ph,h) + \int_0^T |B^*P\bar{x} + L\bar{p}|^2\, dt \leq -{\delta_0}^2 \int_0^\infty |\tilde{w}_T|^2\, dt + C_0(|h|^2 + |k|^2)$$

Recalling the definition of $\tilde{w}_T$, we deduce in particular

$${\delta_0}^2 \int_0^T |\frac{1}{\gamma^2} D^*P\bar{x}|^2\, dt \leq C_0(|h|^2 + |k|^2).$$

Since T is arbitrary, we have proved that $D^*P\bar{x} \in L^2(-\infty;+\infty;W)$. Since $\mathcal{A}$ is exponentially stable, this suffices (see (5. 5) to prove (5. 6).
-b:Consider the system

$$\begin{array}{lcl} x' &=& \left(A + \frac{1}{\gamma^2}DD^*P\right)x + BLp + Dw \\ p' &=& (A+M)p + NCx + N\eta \\ x(0) &=& 0 \\ p(0) &=& 0. \end{array} \tag{5. 13}$$

then we shall prove

$$\sup_{w,\eta} \frac{\displaystyle\int_0^\infty |B^*Px + Lp|^2\, dt}{\displaystyle\int_0^\infty (|w|^2 + |\eta|^2)\, dt} < \gamma^2. \tag{5. 14}$$

Computing $\frac{d}{dt}(Px,x)$ and integrating between 0 and T, then letting $T \to \infty$ we obtain the relation

$$\int_0^\infty |B^*Px + Lp|^2\, dt - \gamma^2 \int_0^\infty (|w|^2 + |\eta|^2)\, dt =$$
$$\int_0^\infty (|Hx|^2 + |Lp|^2)\, dt - \gamma^2 \int_0^\infty \left(|w + \frac{1}{\gamma}^2 D^*Px|^2 + |\eta|^2\right) dt$$

Next, as above using the basic assumption $\rho(L,M,N) < \gamma^2$, we can write

$$\int_0^\infty (|Hx|^2 + |Lp|^2)\, dt \leq (\gamma^2 - \delta^2) \int_0^\infty \left(|w + \frac{1}{\gamma^2} D^*Px|^2 + |\eta|^2\right) dt.$$

Therefore combining the two above relations yields

$$\frac{\displaystyle\int_0^\infty |B^*Px + Lp|^2\, dt}{\displaystyle\int_0^\infty (|w|^2 + |\eta|^2)\, dt} \leq \gamma^2 - \delta^2 \frac{\displaystyle\int_0^\infty \left(|w + \frac{1}{\gamma^2} D^*Px|^2 + |\eta|^2\right) dt}{\displaystyle\int_0^\infty (|w|^2 + |\eta|^2)\, dt}. \tag{5. 15}$$

Now let us check that

$$\frac{\displaystyle\int_0^\infty \left(|w + \frac{1}{\gamma^2} D^*Px|^2 + |\eta|^2\right) dt}{\displaystyle\int_0^\infty (|w|^2 + |\eta|^2)\, dt} \geq c_0 \tag{5. 16}$$

where c_0 is a positive constant. Indeed consider

$$\begin{array}{lcl} x_0' &=& Ax_0 + BLp_0 + Dw_0 \\ p_0' &=& (A+M)p_0 + NCx_0 + N\eta \\ x_0(0) &=& 0 \\ p_0(0) &=& 0. \end{array} \tag{5. 17}$$

Then from the exponential stability of $\mathcal{A}$ we have among other things

$$\int_0^\infty \left(|w_0 - \frac{1}{\gamma^2} D^* P x_0|^2 + |\eta|^2\right) dt \le c_1 \int_0^\infty (|w_0|^2 + |\eta|^2)\, dt.$$

Applying this estimate with

$$w_0 = w + \frac{1}{\gamma^2} D^* P x \quad x_0 = x \quad p_0 = p$$

where w, x, p correspond to (5. 13) we deduce immediately (5. 16) with $c_0 = 1/c_1$.Note that we can take c_0 as small as we wish,yet strictly positive. In particular $\gamma^2 - \delta^2 c_0 > 0$.Using (5. 16) in (5. 15) we easily deduce the desired property (5. 14).

-c:duality considerations We again use the duality considerations of paragraph 4.3, see Proposition 4.1. Let $\Xi = X \times X$, $\Phi = W \times Y$, and this time $\Psi = U$. Let then

$$\mathcal{A} = \begin{pmatrix} A + \frac{1}{\gamma^{*2}} DD^* P & BL \\ NC & A + M \end{pmatrix}$$

and

$$\mathcal{B} = \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} B^* P & L \end{pmatrix}.$$

The dual system is (apply (4. 8) and set $\zeta = m, q$, $\psi = \mu$)

$$\begin{array}{lcl} m' & = & (A^* + \frac{1}{\gamma^{*2}} PDD^*) m + C^* N^* q + PB\mu \\ q' & = & L^* B^* m + (A^* + M^*) q + L^* \mu \\ m(0) & = & 0 \\ q(0) & = & 0. \end{array} \tag{5. 18}$$

Using Proposition 4. 1 and (5. 14) we can assert that

$$\sup_\mu \frac{\int_0^\infty (|D^* m|^2 + |N^* q|^2)\, dt}{\int_0^\infty |\mu|^2\, dt} < \gamma^2. \tag{5. 19}$$

Consider the dynamic system

$$\begin{array}{lcl} m' & = & (A^* + \frac{1}{\gamma^{*2}} PDD^*) m + C^* v + PB\mu \\ m(0) & = & 0 \end{array} \tag{5. 20}$$

where v is the control and μ is the perturbation. We observe that the pair $A^* + \frac{1}{\gamma^{*2}} PDD^*, C^*$ is stabilizable, as a consequence of the fact that the operator $\mathcal{A}_P$, hence its dual $\mathcal{A}_P^*$ is exponentially stable (see part -a of the present proof and beware of the fact $\mathcal{A}_P$ has been designated now by $\mathcal{A}$ by consistency with the generic notation used when dealing with duality considerations). Consider also the cost function

$$\mathcal{K}_0(v, \mu) = \int_0^\infty (|D^* m|^2 + |v|^2)\, dt.$$

¿From the assumption that A^*, D^* is detectable it follows that the pair $A^* + \frac{1}{\gamma^{*2}} PDD^*, D^*$ is detectable. Using (5. 19) we can assert that the γ^2 robustness property holds for the system (5. 20) and the cost function $\mathcal{K}_0(v, \mu)$. Therefore, we may rely on Theorem 3.1 to obtain the existence and uniqueness of a self adjoint operator $\Pi \in \mathcal{L}(X, X) \ge 0$, solution of the Riccati equation

$$\Pi(A^* + \frac{1}{\gamma^2} PDD^*) + (A + \frac{1}{\gamma^2} DD^* P)\Pi - \Pi(C^* C - \frac{1}{\gamma^2} PBB^* P)\Pi + DD^* = 0 \tag{5. 21}$$

and $A^* + \frac{1}{\gamma^2}PDD^* - (C^*C - \frac{1}{\gamma^2}PBB^*P)\Pi$ is exponentially stable.Note also that as in Lemma 3.1 we have also $A^* + \frac{1}{\gamma^2}PDD^* - C^*C\Pi$ is exponentially stable.
- d:algebraic manipulations We check here that

$$\Sigma = \Pi(I - \frac{1}{\gamma^2}P\Sigma) \tag{5. 22}$$

If (5. 22) is true then clearly (using also the symmetry of Σ)

$$(I - \frac{1}{\gamma^2}P\Sigma)(I + \frac{1}{\gamma^2}P\Pi) = (I + \frac{1}{\gamma^2}P\Pi)(I - \frac{1}{\gamma^2}P\Sigma) = I$$

which proves that $I - \frac{1}{\gamma^2}P\Sigma$ is invertible .Moreover

$$\Pi = \Sigma(I - \frac{1}{\gamma^2}P\Sigma)^{-1} \geq 0$$

and the proof of (4. 6) will then be complete. To prove (5. 22) we proceed by algebraic manipulations combining the Riccati equations of P, Σ, Π namely (4. 4),(4. 5) and (5. 21). To simplify notation write

$$\Lambda = -\Sigma + \Pi(I - \frac{1}{\gamma^2}P\Sigma)$$

$$A_1 = A^* - (C^*C - \frac{1}{\gamma^2}H^*H)\Sigma$$

$$A_2 = A^* + \frac{1}{\gamma^2}PDD^* - (C^*C - \frac{1}{\gamma^2}PBB^*P)\Pi$$

Then we can check after an easy calculus

$$\Lambda A_1 + A_2^*\Lambda = 0. \tag{5. 23}$$

Note that A_1, A_2 are exponentially stable.Then the relation (5. 23) implies $\Lambda = 0$. Indeed consider for $h \in D(A_1), k \in D(A_2^*)$ the solutions x_1, x_2 of

$$x_1' = A_1x_1 \quad ; x_1(0) = h$$

$$x_2' = A_2x_2 \quad ; x_2(0) = k$$

then thanks to (5. 23) we check immediately that

$$\frac{d}{dt}(\Lambda x_1, x_2) = 0$$

hence $(\Lambda x_1, x_2)$ is constant. Since it vanishes at infinity by virtue of the exponential stability, it is also 0 at $t = 0$. Hence $(\Lambda h, k) = 0$ which extends by density to all values of h, k. Therefore $\Lambda = 0$ and (5. 22) has been proven.

5.2 Sufficient Conditions

So we assume that (4. 4),(4. 5),(4. 6) hold. We set

$$\Pi = \Sigma(I - \frac{1}{\gamma^2}P\Sigma)^{-1}.$$

Note that Π is symmetric. This follows from the relation

$$\Sigma(I - \frac{1}{\gamma^2}P\Sigma) = (I - \frac{1}{\gamma^2}\Sigma P)\Sigma.$$

By assumption we know that $\Pi \geq 0$.Moreover using the notation A_1, A_2, Λ as above, we have the relation (5. 23) since $\Lambda = 0$. After using the equations of Σ and P we deduce easily that the left hand side of (5. 21) multiplied to the right by $I - \frac{1}{\gamma^2}P\Sigma$ vanishes. From the invertibility of $I - \frac{1}{\gamma^2}P\Sigma$ we deduce that Π is a solution of the left hand side of (5. 21). Details to make precise this formal calculation are left to the reader. Note also that

$$A_2 = (I - \frac{1}{\gamma^2}P\Sigma)A_1(I - \frac{1}{\gamma^2}P\Sigma)^{-1}.$$

Consider the equation

$$x' = A_2 x \quad ; x(0) = h.$$

Then setting

$$x_1 = (I - \frac{1}{\gamma^2}P\Sigma)^{-1}x$$

we have

$$x_1' = A_1 x_1 \quad ; x_1(0) = (I - \frac{1}{\gamma^2}P\Sigma)^{-1}h.$$

Since we know that A_1 is exponentially stable we deduce that $x_1 \in L^2(-\infty; +\infty; X)$ hence also $x \in L^2(-\infty; +\infty; X)$.Therefore we get that A_2 is exponentially stable. Hence the operator Π satisfies all the properties stated in the part -c of the proof of (4. 6) in the necessary conditions, paragraph 5.1.

We define next L, M, N as in the statement of Theorem 4.1, definition of the feedback controller and we shall prove that this feedback controller satisfies the conditions of γ^2 robustness property(with partial observation), as stated in Definition 4.1. We associate to the triple L, M, N the matrix operator $\mathcal{A}$ as in the proof of (4. 4) in paragraph 5.1.With the present choice of L, M, N it amounts to

$$\mathcal{A} = \begin{pmatrix} A & -BB^*P \\ \Pi C^*C & A - (BB^* - \frac{1}{\gamma^2}DD^*)P - \Pi C^*C \end{pmatrix}$$

We must prove that

(5. 24) $$\mathcal{A} \text{ is exponentially stable.}$$

We decompose the proof in several steps.

-a:The matrix operator

$$\mathcal{A}_P = \begin{pmatrix} A + \frac{1}{\gamma^* 2}DD^*P & -BB^*P \\ \Pi C^*C & A - (BB^* - \frac{1}{\gamma^2}DD^*)P - \Pi C^*C \end{pmatrix}$$

is exponentially stable. Consider indeed the dynamic system

(5. 25)
$$\begin{aligned} x' &= (A + \tfrac{1}{\gamma^* 2}DD^*P)x - BB^*Pp \\ p' &= \left(A - (BB^* - \tfrac{1}{\gamma^2}DD^*)P - \Pi C^*C\right)p + \Pi C^*Cx \\ x(0) &= h \\ p(0) &= k. \end{aligned}$$

Setting

$$\xi = x - p$$

we see that ξ is the solution of

(5. 26)
$$\begin{aligned} \xi' &= (A + \tfrac{1}{\gamma^2}DD^*P - \Pi C^*C)\xi \\ \xi(0) &= h - k. \end{aligned}$$

Since $A+\frac{1}{\gamma^2}DD^*P-\Pi C^*C$ is exponentially stable we get that $\xi \in L^2(-\infty;+\infty;X)$. Next x appears as the solution of

$$(5.\ 27)\qquad \begin{array}{rcl} x' &=& (A+\frac{1}{\gamma^2}DD^*P-BB^*P)x+BB^*P\xi \\ x(0) &=& h. \end{array}$$

Since $A+\frac{1}{\gamma^2}DD^*P-BB^*P$ is exponentially stable we deduce that $x \in L^2(-\infty;+\infty;X)$ and thus also $p \in L^2(-\infty;+\infty;X)$. This completes the proof of -a.

- b:consider the system

$$(5.\ 28)\qquad \begin{array}{rcl} x' &=& (A+\frac{1}{\gamma^{*2}}DD^*P)x-BB^*Pp+Dw \\ p' &=& \Big(A-(BB^*-\frac{1}{\gamma^2}DD^*)P-\Pi C^*C\Big)\,p+\Pi C^*Cx+\Pi C^*\eta \\ x(0) &=& h \\ p(0) &=& k \end{array}$$

then one has the estimate

$$(5.\ 29)\qquad \begin{array}{rcl} \displaystyle\int_0^\infty (|Hx|^2+|B^*Pp|^2)\,dt &-& \displaystyle\gamma^2\int_0^\infty \left(|w+\frac{1}{\gamma^*2}D^*Px|^2+|\eta|^2\right)\,dt \\ &<& \displaystyle -\delta_0{}^2\int_0^\infty (|w|^2+|\eta|^2)\,dt+C_0(|h|^2+|k|^2) \end{array}$$

where δ_0 and C_0 are appropriate positive constants. To prove the estimate (5. 29) we shall exploit the Riccati equation

(5. 21) whose solution is Π. Consider the system

$$(5.\ 30)\qquad \begin{array}{rcl} m' &=& (A^*+\frac{1}{\gamma^*2}PDD^*)m+C^*v+PB\mu \\ m(0) &=& 0 \end{array}$$

where v is the control and μ is the perturbation. The pair $A^*+\frac{1}{\gamma^*2}PDD^*, C^*$ is stabilizable. Consider the cost function

$$\mathcal{K}_0(v,\mu)=\int_0^\infty (|D^*m|^2+|v|^2)\,dt.$$

The pair $A^*+\frac{1}{\gamma^*2}PDD^*, D^*$ is detectable.The existence of Π implies that the γ^2 robustness property with full observation holds for the system (5. 30) and the cost function $\mathcal{K}_0(v,\mu)$. In fact consider

$$(5.\ 31)\qquad \begin{array}{rcl} m' &=& (A^*+\frac{1}{\gamma^*2}PDD^*-C^*C\Pi)m+PB\mu \\ m(0) &=& 0 \end{array}$$

then one has the property

$$\sup_\mu \frac{\displaystyle\int_0^\infty (|D^*m|^2+|C\Pi m|^2)\,dt}{\displaystyle\int_0^\infty |\mu|^2\,dt}<\gamma^2.$$

Using duality considerations we introduce the dual system

$$(5.\ 32)\qquad \begin{array}{rcl} \xi' &=& (A+\frac{1}{\gamma^2}DD^*P-\Pi C^*C)\xi+Dw+\Pi C^*\eta \\ \xi(0) &=& 0. \end{array}$$

Then we can assert the estimate

$$(5.\ 33)\qquad \sup_{w,\eta} \frac{\displaystyle\int_0^\infty |B^*P\xi|^2\,dt}{\displaystyle\int_0^\infty (|w|^2+|\eta|^2)\,dt}<\gamma^2.$$

In particular, we can find $\delta < \gamma$ such that

$$\frac{\displaystyle\int_0^\infty |B^*P\xi|^2\,dt}{\displaystyle\int_0^\infty (|w|^2+|\eta|^2)\,dt} \le \gamma^2 - \delta^2. \tag{5. 34}$$

Consider now the system (5. 28) for initial values $h = 0; k = 0$. We denote by x_0, p_0 the corresponding solution. We see that the solution ξ of (5. 32) is equal to $x_0 - p_0$. We shall also use the following relation

$$\int_0^\infty \left(|Hx_0|^2 + |B^*Pp_0|^2 - |B^*P\xi|^2 + \gamma^2(|w|^2 - |w + \frac{1}{\gamma^2}D^*Px_0|^2)\right) dt = 0$$

which is obtained by computing $\frac{d}{dt}(Px_0, x_0)$ and integrating between 0 and ∞. In this equality, we make use of the estimate (5. 34) to obtain

$$\int_0^\infty \left(|Hx_0|^2 + |B^*Pp_0|^2 - \gamma^2(|\eta|^2 + |w + \frac{1}{\gamma^2}D^*Px_0|^2)\right) dt \le -\delta^2 \int_0^\infty (|w|^2+|\eta|^2)\,dt. \tag{5. 35}$$

Introduce $x_1 = x - x_0, p_1 = p - p_0$ which depend only on h, k and not on w, η. We replace in (5. 29) x by $x_0 + x_1$ and p by $p_0 + p_1$. Using inequalities like $|Hx|^2 \le (1+\epsilon)|Hx_0|^2 + (1+\frac{1}{\epsilon})|Hx_1|^2$ where ϵ is arbitrarily small and making use of (5. 35) we easily deduce the desired estimate (5. 29).
- c:Proof of (5. 24) Consider the system

$$\begin{array}{rcl} \bar{x}' &=& A\bar{x} - BB^*P\bar{p} \\ \bar{p}' &=& \left(A - (BB^* - \frac{1}{\gamma^2}DD^*)P - \Pi C^*C\right)\bar{p} + \Pi C^*C\bar{x} \\ \bar{x}(0) &=& h \\ \bar{p}(0) &=& k. \end{array} \tag{5. 36}$$

then we shall prove

$$\bar{x}, \bar{p} \in L^2(-\infty; +\infty; X). \tag{5. 37}$$

Let us denote by $x_{h,k}, p_{h,k}$ the solution of

$$\begin{array}{rcl} x' &=& (A + \frac{1}{\gamma^{*2}}DD^*P)x - BB^*Pp \\ p' &=& \left(A - (BB^* - \frac{1}{\gamma^2}DD^*)P - \Pi C^*C\right)p + \Pi C^*Cx \\ x(0) &=& h \\ p(0) &=& k \end{array} \tag{5. 38}$$

and for any T define the system

$$x_T(t) = \left| \begin{array}{ll} \bar{x}(t) & \text{if } t \le T \\ x_{\bar{x}_T,\bar{p}_T}(t-T) & \text{if } t > T \end{array} \right. \qquad p_T(t) = \left| \begin{array}{ll} \bar{p}(t) & \text{if } t \le T \\ p_{\bar{x}_T,\bar{p}_T}(t-T) & \text{if } t > T \end{array} \right.$$

and

$$w_T(t) = \left| \begin{array}{ll} -\frac{1}{\gamma^2}D^*P\bar{x}(t) & \text{if } t < T \\ 0 & \text{if } t > T \end{array} \right.$$

Clearly the pair $x_T(.), p_T(.)$ is the solution of the system (5. 28) corresponding to the perturbation $w(.) = w_T(.), \eta = 0$. Hence we may apply the estimate (5. 29) to assert that

$$\begin{array}{rl} \displaystyle\int_0^\infty (|Hx_T|^2 + |B^*Pp_T|^2)\,dt & - \; \gamma^2 \displaystyle\int_0^\infty |w_T + \frac{1}{\gamma^*2}D^*Px_T|^2\,dt \\ & < \; -{\delta_0}^2 \displaystyle\int_0^\infty |w_T|^2\,dt + C_0(|h|^2+|k|^2) \end{array} \tag{5. 39}$$

Now from (5. 38) it follows the relation

$$\begin{aligned}\int_0^\infty (|Hx_{h,k}|^2 + |B^*Pp_{h,k}|^2)\,dt \quad &- \quad \gamma^2 \int_0^\infty \,|\frac{1}{\gamma^*2} D^*Px_{h,k}|^2\,dt \\ &= \quad (Ph,h) + \int_O^\infty |B^*P(x_{h,k} - p_{h,k})|^2\,dt \end{aligned} \tag{5. 40}$$

We apply (5. 40) with $h = \bar{x}_T, k = \bar{p}_T$ and use the fact

$$\begin{aligned}&\int_T^\infty \left(|Hx_T|^2 + |B^*Pp_T|^2 - \gamma^2 |w_T + \frac{1}{\gamma^*2} D^*Px_T|^2 \right)\,dt \\ &\quad = \int_0^\infty \left(|Hx_{\bar{x}_T,\bar{p}_T}|^2 + |B^*Pp_{\bar{x}_T,\bar{p}_T}|^2 - \gamma^2 |\frac{1}{\gamma^*2} D^*Px_{\bar{x}_T,\bar{p}_T}|^2 \right)\,dt.\end{aligned}$$

Hence we have

$$\begin{aligned}\int_T^\infty (|Hx_T|^2 + |B^*Pp_T|^2)\,dt \quad &- \quad \gamma^2 \int_T^\infty |w_T + \frac{1}{\gamma^2} D^*Px_T|^2\,dt \\ &= \quad (P\bar{x}_T, \bar{x}_T) + \int_O^\infty |B^*P(x_{\bar{x}_T,\bar{p}_T} - p_{\bar{x}_T,\bar{p}_T})|^2\,dt\end{aligned} \tag{5. 41}$$

Therefore we deduce from (5. 39) and (5. 41) the estimate

$$\begin{aligned}\int_0^\infty (|Hx_T|^2 + |B^*Pp_T|^2)\,dt \quad &- \quad \gamma^2 \int_0^\infty \,|w_T + \frac{1}{\gamma^*2} D^*Px_T|^2\,dt = \int_0^T (|Hx_T|^2 + |B^*Pp_T|^2)\,dt + (P\bar{x}_T, \bar{x}_T) \\ &+ \quad \int_O^\infty |B^*P(x_{\bar{x}_T,\bar{p}_T} - p_{\bar{x}_T,\bar{p}_T})|^2\,dt < -{\delta_0}^2 \int_0^\infty |w_T|^2\,dt + C_0(|h|^2 + |k|^2)\end{aligned} \tag{5. 42}$$

Among other things it follows from (5. 42) that

$$\int_0^T |D^*P\bar{x}|^2\,dt \leq C_1(|h|^2 + |k|^2).$$

Letting T tend to ∞ we get $D^*P\bar{x} \in L^2(-\infty; +\infty; W)$ .Since the pair $\bar{x}, \bar{p}$ appears as the solution of (5. 28) with values

$$w = -\frac{1}{\gamma^*2} D^*P\bar{x} \quad , \eta = 0$$

it follows that $\bar{x}, \bar{p} \in L^2(-\infty; +\infty; X)$ and (5. 37) hence (5. 24) is proven.
To complete the proof of the γ^2 robustness property(with partial observation) for the triple L, M, N consider the system

$$\begin{aligned} x' &= Ax - BB^*Pp + Dw \\ p' &= \left(A - (BB^* - \tfrac{1}{\gamma^2} DD^*)P - \Pi C^*C \right) p + \Pi C^*Cx + \Pi C^*\eta \\ x(0) &= 0 \\ p(0) &= 0 \end{aligned} \tag{5. 43}$$

we must prove

$$\sup_{w,\eta} \frac{\displaystyle\int_0^\infty (|Hx|^2 + |B^*Px|^2)\,dt}{\displaystyle\int_0^\infty (|w|^2 + |\eta|^2)\,dt} < \gamma^2. \tag{5. 44}$$

Apply the estimate (5. 29) to (5. 43) with $w = w - \frac{1}{\gamma^2} D^*Px$ and $h = 0, k = 0$ to obtain

$$\begin{aligned}&\int_0^\infty \left(|Hx|^2 + |B^*Pp|^2 - \gamma^2(|w|^2 + |\eta|^2) \right)\,dt \leq \\ &-{\delta_0}^2 \int_0^\infty \left(|w - \frac{1}{\gamma^2} D^*Px|^2 + |\eta|^2 \right)\,dt\end{aligned} \tag{5. 45}$$

hence

$$\sup_{w,\eta} \frac{\displaystyle\int_0^\infty (|Hx|^2 + |B^*Px|^2)\,dt}{\displaystyle\int_0^\infty (|w|^2 + |\eta|^2)\,dt} \leq$$

$$\gamma^2 - {\delta_0}^2 \inf_{w,\eta} \frac{\displaystyle\int_0^\infty \left(|w - \frac{1}{\gamma^2} D^*Px|^2 + |\eta|^2 \right) dt}{\displaystyle\int_0^\infty (|w|^2 + |\eta|^2)\,dt}$$

Using the exponential stability of $\mathcal{A}_P$ we can check as done previously that

$$\inf_{w,\eta} \frac{\displaystyle\int_0^\infty \left(|w - \frac{1}{\gamma^2} D^*Px|^2 + |\eta|^2 \right) dt}{\displaystyle\int_0^\infty (|w|^2 + |\eta|^2)\,dt} > 0.$$

Therefore (5. 44) has been proven.
The proof of Theorem 4.1 has been completed.

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