The Mathematics of Routing in Massively Dense Ad-Hoc Networks

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Abstract. Computing optimal routes in massively dense adhoc networks becomes intractable as the number of nodes becomes very large. One recent approach to solve this problem is to use a fluid type approximation in which the whole network is replaced by a continuum plain. Various paradigms from physics have been used recently in order to solve the continuum model. We propose in this paper an alternative modeling and solution approach similar to a model by Beckmann [3] developed more than fifty years ago from the area of road traffic.

1 Introduction

An important approach to routing in ad-hoc network has been to to design traffic dependent adaptive protocols that send packets along paths that have smallest delays. This metrics goes back to an early paper by Gupta and Kumar [8] who show that by doing so, resequensing delays (that are undesirable in real time traffic and that are very harmful in data transfers using the TCP protocol) are minimized. A recent line of research has been to study the such protocols in massively dense static ad-hoc networks that are characterized by the property that each node has many other nodes in its transmission range. We are interested here in the recent fluid limit approach in which the nodes are modeled as a continuum, and where the discrete graph describing the links and their costs is replaced by a cost density (which depends on the traffic intensity) over the plain. The rational of using such fluid limit approximations is that whereas the complexity of finding optimal routes grows with the number n of nodes, the fluid limit does not depend on n and hence the complexity of finding optimal routes in the fluid approximation does not grow with n.

Various approaches inspired by physics have been proposed starting with the pioneering work of Jacquet (see [10]) who used ideas from geometrical optics[1]. Approaches based on electrostatics have been designed in [20,21,18,17,9] (see the survey [19] and references therein).

The physics-inspired paradigms allow one to minimize various metrics related to the routing. In contrast, Hyytia and Virtamo propose in [15] an approach based on load balancing arguing that if shortest path (or cost minimization) arguments were used, then some parts of the network would carry more traffic than others and may use more

[1] We note that this approach is restricted to costs that do not depend on the congestion.

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energy than others. This would result in a shorter lifetime of the network since some parts would be out of energy earlier than others and earlier than any part in a load balanced network.

The development of the original theory of routing in massively dense ad-hoc networks has emerged in a complete independent way of the related theory developed within the community of road traffic engineers, introduced in 1952 by Wardrop [22] and by Beckmann [3][2], and which is still an active research area among that community, see [5,6,13,14,24] and references therein.

This community further developed numerical approaches to solve the continuous approximation model through discretization[3].

Inspired by Dafermos [5] who considered routing over two possible directions (North to South and West to East), we have studied in [1] routing in static ad-hoc networks (e.g. sensor networks) where the limitation to two directions can be justified by the use of directional antennas. In the present work, we study the case where any general direction can be chosen at any point.

Two types of objectives are sought in the research on routing in the road traffic context. The first is to maximize the global utility for the whole society, and the second is to find a routing configuration (called "traffic assignment") such that each transmission uses only paths with minimum costs. Configurations satisfying this property are known as "Wardrop Equilibrium", and they coincide with the solution concept used by Gupta and Kumar [8]. We study the two types of objectives in this paper in the context of massively dense ad-hoc networks. For the first objective (which corresponds to a co-operation between nodes) we use and strengthen results of Beckmann by using tools from optimization and control theory that have not been available at the middle of the last century. We further study the Wardrop equilibrium and establish conditions under which it coincides with the global optimization.

The paper is structured as follows. After describing the model in the next section, we provide in Section 3 the mathematical foundations for globally optimizing the fluid model. The mathematical foundation for describing and solving the non-cooperative case (i.e. the Wardrop equilibrium) are introduced in Section 4. This is followed by Section 5 with two examples for congestion cost. We end with a concluding section that summarizes our contributions.

2 The Problem

2.1 Routing in a Dense Network

We consider a routing problem in a dense ad-hoc network. A domain Ω of the plane (x, y) is densely covered by potential routers. Messages have to flow from a region $\mathcal{S}$ of the boundary Γ of Ω to a disjoint region $\mathcal{R}$ of Γ. The intensity $\sigma(x, y)$ of message

[2] See also [3, p 644, footnote 3] for the abundant literature of the early 50's.

[3] Although it may seem that one is back to the starting point with yet another discrete problem to solve, the new discrete problem is simpler, each node in it has only a small number of neighbors, and the number of nodes in the new discrete model is independent of the number of nods in the original system.

generation on $\mathcal{S}$ given, while the intensity $\rho(x, y)$ of signal sink on $\mathcal{R}$ is not. It is only assumed that these are consistent: the total flow of messages emitted and received are equal. On the rest $\mathcal{T}$ of the boundary of Ω, no message should enter nor leave Ω.

The congestion cost per packet transmitted (say in terms of delays, or energy use) at each point in Ω is a function $c(x, y, \varphi)$ of the point and of the intensity φ of the flow of messages through that point.

We wish to investigate the optimal routing policy and its relationship with a Wardrop kind of optimality.

2.2 A Mathematical Model

Formal Equations. We shall use the notation $\mathbf{x} = (x, y)$ to denote a point of $\mathbb{R}^2$. Let Ω be an open domain of $\mathbb{R}^2$ with a smooth boundary Γ, Ω being at every point of Γ on a single side of Γ, so that an exterior normal to Ω, say $n(\mathbf{x})$ is well defined and smooth on Γ.

We model the flow of messages as a vector field $f : \Omega \to \mathbb{R}^2$, and we let $\varphi(\mathbf{x}) = \|f(\mathbf{x})\|$ be its intensity. The flux of messages through $\mathcal{S}$ is given as a $\mathcal{C}^1$ function $\sigma(\cdot) : \mathcal{S} \to \mathbb{R}_+$. The consistency assumption now reads

$$\int_{\mathcal{R}} \rho(\mathbf{x})\,\mathrm{d}s = \int_{\mathcal{S}} \sigma(\mathbf{x})\,\mathrm{d}s\,. \tag{1}$$

Let $\mathcal{Q} = \mathcal{S} \cup \mathcal{T}$ and extend the function σ to the whole of $\mathcal{Q}$ by $\sigma(\mathbf{x}) = 0$ on $\mathcal{T}$. We model the conditions on the boundary as

$$\forall \mathbf{x} \in \mathcal{Q}\,, \quad \langle n(\mathbf{x}), f(\mathbf{x})\rangle = -\sigma(\mathbf{x}) \tag{2}$$

There is no source nor sink of messages in Ω, which we model as a constraint

$$\forall \mathbf{x} \in \Omega\,, \quad \mathrm{div} f(\mathbf{x}) = 0\,. \tag{3}$$

It follows that

$$\int_{\Gamma} \langle n(\mathbf{x}), f(\mathbf{x})\rangle \mathrm{d}s = 0\,,$$

which suffices to insure the consistency condition (1).

The congestion cost per packet c is supposed to be a strictly positive $\mathcal{C}^1$ function $c(\mathbf{x}, \varphi) : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$, increasing and convex in φ for each $\mathbf{x}$. The total cost of congestion will be taken as

$$G(f(\cdot)) = \int_{\Omega} c(\mathbf{x}, \|f(\mathbf{x})\|)\|f(\mathbf{x})\|\,\mathrm{d}\mathbf{x}\,. \tag{4}$$

The path followed by a packet is specifed by its direction of travel $e_\theta = (\cos\theta,\ \sin\theta)$ along its path, according to $\dot{\mathbf{x}} = e_\theta$. The cost incurred by one packet traveling from $\mathbf{x}_0 \in \mathcal{S}$ at time t_0 to $\mathbf{x}_1 \in \mathcal{R}$ reached at time t_1 is

$$J(e_\theta(\cdot)) = \int_{\mathbf{x}_0}^{\mathbf{x}_1} c(\mathbf{x}, \|f(\mathbf{x})\|)\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_{t_0}^{t_1} c(\mathbf{x}(t), \|f(\mathbf{x}(t))\|)\,\mathrm{d}t\,. \tag{5}$$

Notice that this "time" t may be a fictitious time, related to physical time, say τ, by $\mathrm{d}\tau = c\,\mathrm{d}t$ for instance. Then c is the inverse of a speed of travel, a delay due to congestion, and J is the time taken by the message to go from source to destination.

Regularity and Function Spaces. We shall seek $f(\cdot)$ in a space we call V. We next discuss the choice of function spaces. A non mathematical oriented reader may skip the description of the funcntion spaces we introduce.

We may choose $V = (H^1(\Omega))^2$, but this will require $\sigma(\cdot)$ to be slightly more regular than necessary, viz. $H^{1/2}(\Gamma)$. To keep with the classical hypothesis in fluid dynamics, we may choose $V = (H_{\mathrm{div}}(\Omega))^2$, the space of L^2 functions whose divergence is in L^2. Then we my choose $\sigma(\cdot)$ in $L^2(\Gamma)$.

The above Sobolev spaces have been introduced by the modern theory of PDEs [7]. An extensive extensive Theory of PDEs and their numerical approximations is now available in these spaces.

This choice of spaces allows one to have complete spaces for functions and for their derivatives along with a scalar product of L^2. The completeness is needed to have existence of minima. The scalar product allows to have duality. The completeness together with the duality allows KKT Theorem to hold, which we make use of in this paper.

Let V_0 be the closure in V of the set of C^∞ functions with compact support in Ω. Let $V_\mathcal{R}$ and $V_\mathcal{Q}$ be the closures in V of the set of $\mathcal{C}^\infty$ functions that are null in a neighborhood of $\mathcal{R}$ and $\mathcal{Q}$ respectively. They are vector spaces, supersets of V_0. Let also $\tilde{f}(\cdot) : \Omega \to \mathbb{R}^2$ be a vector field in V satisfying the constraint (2) (for instance a smooth extension of $\sigma(\mathbf{x})n(\mathbf{x})$). Let $\mathcal{V}$ be the affine space $\tilde{f} + V_\mathcal{Q}$.

We shall also need the space $H^1_\mathcal{R}$ of functions of $H^1(\Omega)$ whose trace on $\mathcal{R}$ is zero.

Finally, we let $\Omega_0 = \{\mathbf{x} \mid f^\star(\mathbf{x}) = 0\}$, or more precisely, since $f^\star$ is not necessarily continuous, the largest open subset of Ω over which $\int_{\Omega_0} \|f^\star(\mathbf{x})\|^2 \,\mathrm{d}\mathbf{x} = 0$.

2.3 The Case of Elastic Traffic

Let's assume that we do not have to ship the whole demand $\sigma(x)$ to the destination. We shall send less if there is congestion. The standard way to model that is first to define a utility $u(s)$ for having s units of information shipped; we take $s(x) \le \sigma(x)$. The new objective is to minimize the sum of $C(f) - U(s)$ where $U(s)$ is the integral of $u(s(x))$ over x.

One way to solve the problem is to define a new sink S. Then add an alternative route from each source to S; the cost to ship f units from a source x to S is $-u(\sigma(x) - f)$. Thus instead of directly adding utilities to the optimization problem, they appear through costs of new routes that are added. The elastic routing problem is thus transformed into an equivalent routing problem with fixed demand. This transformation is standard, see [12,16], and we shall not pursue it here.

3 Global Optimum

3.1 The Completely Differentiable Case

We seek here the vector field $f^\star \in (L^2(\Omega))^2$ satisfying the constraints (2) and (3) and minimizing $G(f)$.

Let $C(\mathbf{x}, \varphi) = c(\mathbf{x}, \varphi)\varphi$. It is convex in φ and coercive (i.e. goes to infinity with φ). As a consequence, $f(\cdot) \mapsto G(f(\cdot))$ is continuous, convex and coercive. Moreover, the constraints are linear. Therefore an optimum exists, and we may apply the theorem "KKT" (Karush, Kuhn and Tucker).

We dualize only the constraint (3) and look for $f^\star$ in $\mathcal{V}$. Let therefore $p(\cdot) \in L^2(\Omega)$ be the dual variable, we let

$$\mathcal{L}(f,p) = \int_\Omega \Big(C(\mathbf{x}, \|f(\mathbf{x})\|) + p(\mathbf{x}) \mathrm{div} f(\mathbf{x}) \Big) \mathrm{d}\mathbf{x} \,.$$

Using Green's formula, we may also write

$$\mathcal{L}(f,p) = \int_\Omega \Big(C(\mathbf{x}, \|f(\mathbf{x})\|) - \langle \nabla p(\mathbf{x}), f(\mathbf{x}) \rangle \Big) \mathrm{d}\mathbf{x} + \int_\Gamma p(\mathbf{x}) \langle n(\mathbf{x}), f(\mathbf{x}) \rangle \mathrm{d}s \,.$$

The optimal vector field $f^\star$ should minimize $\mathcal{L}$ over $\mathcal{V}$, for some p. Therefore, 0 must belong to the subdifferential with respect to f of the restriction of $\mathcal{L}$ to $\mathcal{V}$.

Wherever $f^\star \neq 0$, $\mathcal{L}$ is actually differentiable, so that the subdifferential contains only the derivative. Actually, we only need the restriction of the derivative to $V_\mathcal{Q}$. which

$$\mathrm{D}\mathcal{L}\cdot g = \int_\Omega \Big(\mathrm{D}_2 C(\mathbf{x}, \|f^\star(\mathbf{x})\|) \frac{\langle f^\star(\mathbf{x}), g(\mathbf{x}) \rangle}{\|f^\star(\mathbf{x})\|} - \langle \nabla p(\mathbf{x}), g(\mathbf{x}) \rangle \Big) \mathrm{d}\mathbf{x} + \int_\mathcal{R} p(\mathbf{x}) \langle n(\mathbf{x}), g(\mathbf{x}) \rangle \mathrm{d}s \,,$$

should be zero for every $g \in V_\mathcal{Q}$. Pick first g in V_0. The last integral vanishes. It follows that necessarily

$$\forall \mathbf{x} : f^\star(\mathbf{x}) \neq 0 \,, \quad \mathrm{D}_2 C(\mathbf{x}, \|f^\star(\mathbf{x})\|) \frac{f^\star(\mathbf{x})}{\|f^\star(\mathbf{x})\|} = \nabla p(\mathbf{x}) \,. \tag{6}$$

It follows from this equation that $p(\cdot) \in H^1(\Omega)$, and also that the first integral in the r.h.s. must be zero for every g in $V_\mathcal{Q}$. Picking now $g \in V_\mathcal{Q}$. It follows that

$$p(\cdot) \in H^1_\mathcal{R} \tag{7}$$

Wherever $\|f^\star(\mathbf{x})\| = 0$, a discussion arises. If $D_2 C(\mathbf{x}, \varphi)/\varphi$ remains bounded as $\varphi \to 0$, there is nothing to add to equations (6) and (7) above. (We shall see the typical example $C(\mathbf{x}, \varphi) = (1/2) c(\mathbf{x}) \varphi^2$ below.) Otherwise the situation is more complicated.

3.2 Lack of Differentiability

We investigate now the case where $\mathrm{D}_2 C(\mathbf{x}, \varphi)/\varphi \to \infty$ as $\varphi \to 0$. This typically arises, e.g. if $\mathrm{D}_2 C(\mathbf{x}, 0) \neq 0$. We shall see the typical example $C(\mathbf{x}, \varphi) = c(\mathbf{x}) \varphi$ below.

Then $f \mapsto C(\mathbf{x}, \|f\|)$ is not differentiable (with respect to f) at 0. Its subdifferential is the set

$$\partial_f C(\mathbf{x}, 0) = \{ q \in \mathbb{R}^2 \mid \forall g \in \mathbb{R}^2 \,,\ C(\mathbf{x}, \|g\|) - C(\mathbf{x}, 0) \geq \langle q, g \rangle \} \,.$$

Since C is assumed differentiable and convex in its second argument, this is equivalent to

$$\partial_f C(\mathbf{x}, 0) = \{ q \mid \forall g \in \mathbb{R}^2 \,,\ \mathrm{D}_2 C(\mathbf{x}, 0) \|g\| \geq \langle q, g \rangle \} \,,$$

which in turn is equivalent to $\|q\| \le |D_2C(\mathbf{x},0)|$. Now, since C is assumed increasing in φ, $D_2C \ge 0$. Placing this back into the subdifferential of $\mathcal{L}$, we get, for $\mathbf{x} \in \Omega_0$,

$$\exists q(\mathbf{x}) \text{ such that } \|q(\mathbf{x})\| \le D_2C(\mathbf{x},0) \text{ and } \forall g \in V_{\mathcal{Q}}\,,\ \int_{\Omega_0} (q(\mathbf{x}) - \nabla p(\mathbf{x}))g(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0\,.$$

Combining both cases, we conclude that, for a function $f^\star(\cdot) \in V$ with null set Ω_0 to be optimal, there must exist a $p(\cdot) \in H^1_{\mathcal{R}}$ such that

$$\begin{array}{l} \forall \mathbf{x} \in \Omega\,, \qquad \|\nabla p(\mathbf{x})\| \le D_2C(\mathbf{x},0)\,,\\ \forall \mathbf{x} \in \Omega - \Omega_0\,,\ \nabla p(\mathbf{x}) = D_2C(\mathbf{x}, \|f(\mathbf{x})^\star\|)\frac{1}{\|f^\star(\mathbf{x})\|} f^\star(\mathbf{x})\,. \end{array} \tag{8}$$

We may notice that the first condition above also yields

$$\forall \mathbf{x} : f^\star(\mathbf{x}) \neq 0\,, \quad \|\nabla p(\mathbf{x})\| = D_2C(\mathbf{x}, \|f^\star(\mathbf{x})\|)\,,$$

Overall, the problem of determining the optimum $f^\star$ is equivalent (if that system has a single solution) to determining simultaneously $f^\star$ and p satisfying (2),(3) and (8).

This system certainly has at least one solution, since our problem is convex coercive with affine constraints, and thus has a minimum. Uniqueness on the other hand, is by no means simple. It may be noticed that one might look for the two scalar functions φ and p, satisfying

$$\begin{array}{l} \forall \mathbf{x} : \varphi(\mathbf{x}) \neq 0\,, \|\nabla p(\mathbf{x})\| = D_2C(\mathbf{x}, \varphi(\mathbf{x}))\,,\\ \forall \mathbf{x} : \varphi(\mathbf{x}) = 0\,, \|\nabla p(\mathbf{x})\| \le D_2C(\mathbf{x}, 0)\,,\\ \forall \mathbf{x} \in \mathcal{R}\,, \qquad p(\mathbf{x}) = 0\,, \end{array}$$

and impose furthermore the constraints (2) and (3) on

$$f^\star(x) = \frac{\varphi(\mathbf{x})}{D_2C(\mathbf{x}, \varphi(\mathbf{x}))}\nabla p(\mathbf{x})\,.$$

We shall investigate a typical case hereafter.

4 Wardrop Equilibrium

Assume the message flow obeys the above necessary conditions. We want to investigate whether it is optimal for a single message to follow the route prescribed by $f^\star$, i.e. an integral line of that field, assuming that its lone deviation from that scheme would have no effect on the overall congestion map. (This is the so called "atomicity" assumption.)

We investigate the optimization of the criterion (5) via its Hamilton-Jacobi-Bellman equation. Let $V(\mathbf{x})$ be the return function, it must be a viscosity solution of

$$\begin{array}{l} \forall \mathbf{x} \in \Omega\,, \min_\theta \langle e_\theta, \nabla V(\mathbf{x})\rangle + c(\mathbf{x}, \|f^\star(\mathbf{x})\|) = 0\,,\\ \forall \mathbf{x} \in \mathcal{R}\,, V(\mathbf{x}) = 0\,. \end{array}$$

hence

$$\begin{array}{l} \forall \mathbf{x} \in \Omega\,, -\|\nabla V(\mathbf{x})\| + c(\mathbf{x}, \|f^\star(\mathbf{x})\|) = 0\,,\\ \forall \mathbf{x} \in \mathcal{R}\,, V(\mathbf{x}) = 0\,. \end{array} \tag{9}$$

And the optimal direction of travel is opposite to $\nabla V(\mathbf{x})$, i.e. $e_\theta = -\nabla V(\mathbf{x})/\|\nabla V(\mathbf{x})\|$.

Clearly, this is the same system of equations as previously, upon replacing $p(\mathbf{x})$ by $-V(\mathbf{x})$, and $\mathrm{D}_2 C(\mathbf{x},\varphi)$ by $c(\mathbf{x},\varphi)$. We thus conclude that the Wardrop equilibrium can be obtained by solving the globally optimal problem in which the cost density is replaced by $\int_0^\varphi c(\mathbf{x},\varphi)d\varphi$. This is the continuous version of the potential function approach of Beckmann et al. [4]. This transformation has been frequently used in the road traffic context but only for one particular cost structure [23,24,25,26] the equivalence was shown to hold in [23,25].

Monomial cost. In the case where $c(\mathbf{x},\varphi) = c(\mathbf{x})\varphi^\alpha$, then $C(\mathbf{x},\varphi) = \alpha c(\mathbf{x},\varphi)$, and therefore the two systems of equations coincide, or more precisely, they coincide in the domain $\{\mathbf{x} \mid f^\star(\mathbf{x}) \neq 0\}$. We shall show that for a given $\varphi(\cdot)$, p is uniquely defined. We therefore have the following property :

Proposition 1. *For a monomial cost, any global equilibrium where $\Omega_0 = \emptyset$ is a Wardrop equilibrium.*

5 Two Examples

5.1 Linear Congestion Cost

We investigate here the simple typical case, where the cost of congestion is linear : $c(\mathbf{x},\varphi) = \frac{1}{2}c(\mathbf{x})\varphi$, so that

$$C(\mathbf{x},\varphi) = \frac{1}{2}c(\mathbf{x})\varphi^2\,.$$

Then, $\mathcal{L}$ is differentiable everywhere, and the necessary condition of optimality is just that there should exist $p : \Omega \to \mathbb{R}^2$ such that $\nabla p(\mathbf{x}) = c(\mathbf{x})f^\star(\mathbf{x})$. Placing this into (3) and (2), we see that we end up with a simple elliptic equation with mixed Dirichlet - (non-homogeneous) Neuman boundary conditions :

$$\left.\begin{array}{ll} \forall \mathbf{x} \in \Omega\,, & \mathrm{div}\big(\frac{1}{c(\mathbf{x})}\nabla p(\mathbf{x})\big) = 0\,,\\ \forall \mathbf{x} \in \mathcal{Q}\,, & \dfrac{\partial p}{\partial n}(\mathbf{x}) = c(\mathbf{x})\sigma(\mathbf{x})\,,\\ \forall \mathbf{x} \in \mathcal{R}\,, & p(\mathbf{x}) = 0\,, \end{array}\right\} \tag{10}$$

for which we easily get existence and uniqueness of the solution.

A more or less explicit solution can then be given in terms of the Green function $\mathcal{G}(\mathbf{x},\xi)$ of the domain

$$f^\star(\mathbf{x}) = \int_{\mathcal{Q}} \frac{1}{c(\mathbf{x})}\nabla_1 \mathcal{G}(\mathbf{x},\xi)\sigma(\xi)\mathrm{d}s(\xi)\,.$$

If he Green function is not available, according to a classical approach, we may derive a finite element method from the variational form : Find $p \in H^1_{\mathcal{R}}$ such that, for any $q \in H^1_{\mathcal{R}}$,

$$\int_\Omega \frac{1}{c(\mathbf{x})}\langle\nabla p(\mathbf{x}), \nabla q(\mathbf{x})\rangle \mathrm{d}x - \int_{\mathcal{Q}} \sigma(\mathbf{x})q(\mathbf{x})\,\mathrm{d}s = 0\,.$$

This can be read as $\mathrm{D}K(p) = 0$ where $K : H^1_{\mathcal{R}} \to \mathbb{R}$ is given by

$$K(p) = \frac{1}{2}\int_{\Omega} \frac{1}{c(\mathbf{x})}\|\nabla p(\mathbf{x})\|^2 - \int_{\mathcal{Q}} \sigma(\mathbf{x})p(\mathbf{x})\,\mathrm{d}s\,.$$

Thanks to Poincaré's inequality, it is convex coercive. We therefore obtain:

Proposition 2. *Equations (10) have a unique solution* $p \in H^1_{\mathcal{R}}$.

5.2 Uncongested Network

An Algorithm. We consider now a situation where the network operates far from congestion. The "cost" $c(\mathbf{x})$ may be regarded as a delay, then the cost of any trajectory is just the time it takes, or an energy expenditure. In any case, it is related to the state of the infrastructure, not to its load. Then, c is independent of $\|f(\mathbf{x})\|$, and we get $C(\mathbf{x},\varphi) = c(\mathbf{x})\varphi$. Then, (8) simplifies into

$$\begin{aligned} &\forall \mathbf{x}\in\Omega\,, \quad \|\nabla p(\mathbf{x})\| \le c(\mathbf{x})\,,\\ &\forall \mathbf{x} : f^\star(\mathbf{x}) \ne 0\,, \quad \nabla p(\mathbf{x}) = c(\mathbf{x})\frac{f^\star(\mathbf{x})}{\|f^\star(\mathbf{x})\|}\,. \end{aligned}$$

Let

$$\varphi(\mathbf{x}) = \|f^\star(\mathbf{x})\|\,, \quad \psi(\mathbf{x}) = \frac{\varphi(\mathbf{x})}{c(\mathbf{x})}\,.$$

The above system yields

$$\forall \mathbf{x}\in\Omega\,, \quad \psi(\mathbf{x})\ge 0\,, \quad \|\nabla p(\mathbf{x})\| \le c(\mathbf{x})\,, \quad \psi(\mathbf{x})[\|\nabla p(\mathbf{x})\| - c(\mathbf{x})] = 0\,, \tag{11}$$

and also $f^\star(\mathbf{x}) = \psi(\mathbf{x})\nabla p(\mathbf{x})$, which placed in (3) and (2) yields

$$\begin{aligned} &\forall \mathbf{x}\in\Omega\,, \quad \psi(\mathbf{x})\Delta p(\mathbf{x}) + \langle\nabla\psi(\mathbf{x}),\nabla p(\mathbf{x})\rangle = 0\,,\\ &\forall \mathbf{x}\in\Gamma\,, \quad \psi(\mathbf{x})\langle n(\mathbf{x}),\nabla p(\mathbf{x})\rangle = \sigma(\mathbf{x})\,. \end{aligned} \tag{12}$$

We do not have a satisfactory theory of this equation. If, as we noticed, existence is guaranteed, we do not know whether that solution is unique. It should be noticed that the uniqueness proof given for a very similar equation in [3] does nor carry over here, because it relies critically on the strict convexity of the cost in $\|f\|$.

As an attempt, we provide here an iterative algorithm which, if it converges, converges toward a solution of the system. It provides us with a uniqueness result under a strong hypothesis. We suspect that a more general result is true, and also that the algorithm converges even without that hypothesis.

We seek ψ in $H^1(\Omega)$, and p in $H^1_{\mathcal{R}}$.

Using the classical variational trick, we may reformulate system (12) as $\forall q \in V_{\mathcal{R}}$,

$$\int_{\Omega}[\psi(\mathbf{x})\Delta p(\mathbf{x}) + \langle\nabla\psi(\mathbf{x}),\nabla p(\mathbf{x})\rangle]q(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{\mathcal{Q}}[\psi(\mathbf{x})\langle n(\mathbf{x}),\nabla p(\mathbf{x})\rangle - \sigma(\mathbf{x})]q(\mathbf{x})\mathrm{d}s = 0\,.$$

Using Green's formula for $q \in H^1(\Omega)$:

$$\int_\Omega [\psi(\mathbf{x})\Delta p(\mathbf{x}) + \langle \nabla\psi(\mathbf{x}), \nabla p(\mathbf{x})\rangle] q(\mathbf{x})\,\mathrm{d}\mathbf{x} = \\ -\int_\Omega \psi(\mathbf{x})\langle \nabla p(\mathbf{x}), \nabla q(\mathbf{x})\rangle\,\mathrm{d}\mathbf{x} + \int_\Gamma \psi(\mathbf{x})\langle n(\mathbf{x}), \nabla p(\mathbf{x})\rangle q(\mathbf{x})\,\mathrm{d}s\,,$$

system (12) can therefore be stated as:

$$\forall q \in V_{\mathcal{R}}\,, \quad \int_\Omega \psi(\mathbf{x})\langle \nabla p(\mathbf{x}), \nabla q(\mathbf{x})\rangle\,\mathrm{d}\mathbf{x} - \int_{\mathcal{Q}} \sigma(\mathbf{x})q(\mathbf{x})\,\mathrm{d}s = 0\,. \tag{13}$$

This equality may also be interpreted as $\mathrm{D}_1 J(p,\psi)q = 0$ where $J : V_{\mathcal{R}} \to \mathbb{R}$ is defined by

$$J(p,\psi) = \frac{1}{2}\int_\Omega \psi(\mathbf{x})\|\nabla p(\mathbf{x})\|^2\,\mathrm{d}\mathbf{x} - \int_{\mathcal{Q}} \sigma(\mathbf{x})p(\mathbf{x})\,\mathrm{d}s\,.$$

Poincaré's inequality states that there exists $C > 0$ such that,

$$\forall p \in V_{\mathcal{R}}, \|p\|^2 \le C\|\nabla p\|^2\,. \tag{14}$$

Thus the functional J above is coercive and has a single minimum.

One may guess the following algorithm: fix $\psi^0(\mathbf{x})$ (say $= 1$). Given ψ^n, minimize J with respect to p, say solving the finite element equations corresponding to (13). Call p^n the solution, and do

$$\psi^{n+1}(\mathbf{x}) = \max\{0, \psi^n(\mathbf{x}) + \theta(\|\nabla p^n(\mathbf{x})\|^2 - c(\mathbf{x})^2)\} \tag{15}$$

for some positive θ. We shall prove the following theorem :

Proposition 3. *If there exists a solutin of equations (11)(12) such that $\|f^\star\|$ is essentially bounded away from 0 in Ω, it is unique and for θ small enough algorithm (15) converges toward that solution.*

Analysis of the Algorithm. Let $\psi^\star$, $p^\star$ be a soltion of our system of equations. Notice first that indeed, for any $\theta > 0$,

$$\forall \mathbf{x} \in \Omega\,, \quad \psi^\star(\mathbf{x}) = \max\{0, \psi^\star(\mathbf{x}) + \theta(\|\nabla p^\star(\mathbf{x})\|^2 - c(\mathbf{x})^2)\} \tag{16}$$

And any limit of he above algorithm has to satisfy this equation, which says that $\|\nabla p(\mathbf{x})\| = c(\mathbf{x})$ for every $\mathbf{x}$ where $\psi(\mathbf{x}) \ne 0$. Together with the condition that p minimizes J for ψ, this is exactly the conditions (11) and (12).

Substract (16) from (15). It results that

$$|\psi^{n+1}(\mathbf{x}) - \psi^\star(\mathbf{x})| \le |\psi^n(\mathbf{x}) - \psi^\star(\mathbf{x}) + \theta(\|\nabla p^n(\mathbf{x})\|^2 - \|\nabla p^\star(\mathbf{x})\|^2)|\,.$$

Take the square, and integrate over Ω :

$$\begin{aligned} \int_\Omega |\psi^{n+1}(\mathbf{x}) - \psi^\star(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} &\le \int_\Omega |\psi^n(\mathbf{x}) - \psi^\star(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \\ &+2\theta \int_\Omega (\psi^n(\mathbf{x}) - \psi^\star(\mathbf{x}))(\|\nabla p^n(\mathbf{x})\|^2 - \|\nabla p^\star(\mathbf{x}))\|^2)\,\mathrm{d}\mathbf{x} \\ &+\theta^2 \int_\Omega (\|\nabla p^n(\mathbf{x})\|^2 - \|\nabla p^\star(\mathbf{x})\|^2)^2\,\mathrm{d}\mathbf{x}\,. \end{aligned} \tag{17}$$

Using Cauchy-Schwarz inequality, the last term is bounded from above by

$$\int_{\Omega}(\|\nabla p^n(\mathbf{x})\|^2 - \|\nabla p^\star(\mathbf{x})\|^2)^2 \,\mathrm{d}\mathbf{x} \leq$$

$$\int_{\Omega}\|\nabla(p^n(\mathbf{x}) - p^\star(\mathbf{x}))\|^2 \,\mathrm{d}\mathbf{x} \int_{\Omega}\|\nabla(p^n(\mathbf{x}) + p^\star(\mathbf{x}))\|^2 \,\mathrm{d}\mathbf{x}\,.$$

Hence, assuming $\int_{\Omega}\|\nabla p^n(\mathbf{x})\|^2 \,\mathrm{d}\mathbf{x}$ remains bounded, there exists $a > 0$ such that

$$\int_{\Omega}(\|\nabla p^n(\mathbf{x})\|^2 - \|\nabla p^\star(\mathbf{x})\|^2)^2 \,\mathrm{d}\mathbf{x} \leq a\int_{\Omega}\|\nabla(p^n(\mathbf{x}) - p^\star(\mathbf{x}))\|^2 \,\mathrm{d}\mathbf{x}\,. \tag{18}$$

Concerning the second term of the r.h.s. of (17), write

$$\|\nabla p^\star\|^2 = \|\nabla p^n + \nabla(p^\star - p^n)\|^2 = \|\nabla p^n\|^2 + 2\langle\nabla p^n, \nabla(p^\star - p^n)\rangle + \|\nabla(p^\star - p^n)\|^2\,.$$

Thus (using short notations for convenience)

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}\psi^n\|\nabla p^\star\|^2 \,\mathrm{d}\mathbf{x} - \int_{\mathcal{Q}}\sigma p^\star \,\mathrm{d}s \\
&= \frac{1}{2}\int_{\Omega}\psi^n\|\nabla p^n\|^2 \,\mathrm{d}\mathbf{x} - \int_{\mathcal{Q}}\sigma p^n \,\mathrm{d}s + \frac{1}{2}\int_{\Omega}\psi^n\|\nabla(p^\star - p^n)\|^2 \,\mathrm{d}\mathbf{x} \\
&\qquad + \int_{\Omega}\psi^n\langle\nabla p^n, \nabla(p^\star - p^n)\rangle \,\mathrm{d}\mathbf{x} - \int_{\mathcal{Q}}\sigma(p^\star - p^n) \,\mathrm{d}s\,.
\end{aligned}$$

By the definition of p^n as solving equation (13), the second line above is zero, leaving the first line alone. In a symmetric fashion, we also get

$$\frac{1}{2}\int_{\Omega}\psi^\star\|\nabla p^n\|^2 - \int_{\mathcal{Q}}\sigma p^n = \frac{1}{2}\int_{\Omega}\psi^\star\|\nabla p^\star\|^2 - \int_{\mathcal{Q}}\sigma p^\star + \frac{1}{2}\int_{\Omega}\psi^\star\|\nabla(p^n - p^\star)\|^2\,.$$

Summing the last two equalities (and multiplying by two), we obtain

$$\int_{\Omega}(\psi^n - \psi^\star)(\|\nabla p^n\|^2 - \|\nabla p^\star\|^2) = -\int_{\Omega}(\psi^n + \psi^\star)\|\nabla(p^n - p^\star)\|^2\,.$$

Placing this and (18) in (17), we may summarize the above calculations as

$$\begin{aligned}
&\int_{\Omega}|\psi^{n+1}(\mathbf{x}) - \psi^\star(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \leq \int_{\Omega}|\psi^n(\mathbf{x}) - \psi^\star(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \\
&-2\theta\int_{\Omega}(\psi^n(\mathbf{x}) + \psi^\star(\mathbf{x}))\|\nabla(p^n(\mathbf{x}) - p^\star(\mathbf{x}))\|^2 \,\mathrm{d}\mathbf{x} \\
&+a\theta^2\int_{\Omega}\|\nabla(p^n(\mathbf{x}) - p^\star(\mathbf{x}))\|^2 \,\mathrm{d}\mathbf{x}\,.
\end{aligned} \tag{19}$$

Assume that, for almost all $\mathbf{x} \in \Omega$, $\psi^\star(\mathbf{x}) \geq b > 0$. It follows that

$$\int_{\Omega}(\psi^n(\mathbf{x}) + \psi^\star(\mathbf{x}))\|\nabla(p^n(\mathbf{x}) - p^\star(\mathbf{x}))\|^2 \,\mathrm{d}\mathbf{x} \geq b\int_{\Omega}\|\nabla(p^n(\mathbf{x}) - p^\star(\mathbf{x}))\|^2 \,\mathrm{d}\mathbf{x}\,,$$

and therefore that for any $\theta \leq b/a$,

$$\int_{\Omega} |\psi^{n+1}(\mathbf{x}) - \psi^{\star}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} \leq \int_{\Omega} |\psi^{n}(\mathbf{x}) - \psi^{\star}(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} - b\theta \int_{\Omega} \|\nabla(p^n(\mathbf{x}) - p^{\star}(\mathbf{x}))\|^2 \, \mathrm{d}\mathbf{x} \, .$$

Summing these inequalities, it follows that the series of the L^2 norms $\|\nabla p^n \nabla p^{\star}\|^2$ converges, and according to Poincarés inequality again, $p^n \to p^{\star}$ in $H^1(\Omega)$. The field of optimal directions converges as well, and assuming it is regular enough for the integral curves to be unique, the optimal field converges as well.

The algorithm is independent from the choice of $p^{\star}$ and $\psi^{\star}$ who are therefore uniquely defined.

6 Concluding Comments

We present a brief comparison of our treatment with [3], called hereafter **M.B.**. In **M.B.**, one introduces both the density $u(\mathbf{x})$ of commodity to be moved, and the speed $v(\mathbf{x})$ of this motion, which is a data. And the cost of transportation is assumed to be a function of u alone. The decision variable in **M.B.** is the vector field φ of transportation where the direction of φ is that of the transportation, and $\|\varphi\|$ its density u. Hence **M.B.**'s $v\varphi$ is our f. And his equation (11) is our equation (6).

In **M.B.** there is an area source or sink of matter to be transported. It did not seem necessary in our context, but technically, it would be trivially done just adding a nonzero r.h.s. to equation (3) and its various forms, the first equation of (10) and of (12).

Now, since the early 50's, the theory of PDE's has been considerably developed, using the tools of Sobolev spaces and the variational theory of J-L. Lions, P. Lax, and others. Thus our derivation is not formal any more, and we are able to give existence and uniqueness theorems impossible to derive in 1952. Notice that our example with no congestion, where our uniqueness theorem is not very satisfactory, does not satisfy the hypotheses of the uniqueness theorem of **M.B.**, because that paper requires that the cost function be strictly convex.

Finally, we solve for the concept of Wardrop equilibrium, and we are therefore able to compare the global optimum to the Wardrop equilibrium, which was not available to Beckmann in 1952.

By casting the routing problem in dense Ad-hoc networks in the context of the road traffic framework of Beckmann, we are able to formulate and solve various optimization problems and study various cost functions, which was not the case with the physics-inspired paradigms that had been used before to study massively dense ad-hoc networks.

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