ESS and Replicator Dynamics

Tutorial

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RTP M3D, june 16, 2008

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Capture the interplay between individual and collective behaviour in a large population of identical agents.

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- ¿ Stock market ?

Strategies

Pure strategy

Individual behavior: choice of a *phenotye* or *trait* (evolution) or *strategy* (routing, sociology) x in a *trait* or *strategy space* X either finite or infinite.

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Mixed strategy

$p, q, \ldots \in \Delta(X)$

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- individual

- population *monomorphic* or *polymorphic*

Generating function

An individual using strategy x in a population using p gets $G(x, p)$

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Fitness function A mixed strategy q yields a *fitness*

$$F(q, p) = \int_X G(x, p) q(\mathrm{d}x) \, .$$

Generating function

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Linear case G may be given via an *pairwise encounter* function $H(x, y)$

$$G(x, p) = \int_X H(x, y) p(\mathrm{d}y).$$

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Interpret “fitness” as reproductive success. Assume most of the population behaves according to strategy p, while a small proportion ε behaves according to q. The overall population then behaves according to

$$q_\varepsilon := (1-\varepsilon)p + \varepsilon q.$$

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$$F(p,p) \geq F(q,p)\,,$$

and in the linear case

$$[F(p,p) = F(q,p)] \Rightarrow F(p,q) > F(q,q)\,.$$

A menagery of concepts

Other concepts, close to ESS, have been introduced : locally superior, evolutioarily robust (ERS), uninvadable, continuously stable, etc.

Definition A strategy p is locally superior if there exists a neighborhood $\mathcal{N}(p)$

$$\forall q \in \mathcal{N}(p), q \neq p\,, \quad F(p,q) > F(q,q)\,.$$

In the infinite cases, depends on the topology used on $\Delta(X)$.

Definition A strategy is Evolutionarily Robust if it is locally superior in the weak topology of $\Delta(X)$.

Theorem In the linear case, ERS $\Rightarrow$ ESS.
In the finite linear case, ESS $\Leftrightarrow$ ERS.

Relation with Nash

Define a non zero sum two players game on $\Delta(X) \times \Delta(X)$ by

$$J_1(p,q) = F(p,q)\,, \quad J_2(p,q) = F(q,p)\,.$$

Then the first ESS condition (linear case) is equivalent to (p,p) being a Nash point of this game. As a consequence we get

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Theorem [Von Neuman 1944, Kuhn 1950, Wardrop 1952] Let p be an ESS.

1. $\forall x \in X\,, \quad G(x,p) \le F(p,p)$,

2. let $N = \{x \in X \mid G(x,p) < F(p,p)\}$, then, $p(N) = 0$.

Finite linear case

Let $X = \{x_i, i = 1, \ldots, n\}$. $\Delta(X)$ is the simplex of $\mathbb{R}^n$.

Let $H(x_i, x_j) = a_{ij}, \quad A = (a_{ij}), \quad F(p,q) = \langle p, Aq \rangle$

$$\text{Wardrop} \Leftrightarrow \quad p = \begin{pmatrix} p_1 \\ 0 \end{pmatrix}, \quad Ap = \begin{pmatrix} v\mathbb{1} \\ r_2 \end{pmatrix}, \quad r_2 < v\mathbb{1}\,;$$

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The second condition is a second order condition. Partition A according to the partition of p. A sufficient condition is that the restriction of the quadratic form $\langle q_1, A_{11} q_1 \rangle$ to the orthogonal subspace of $\mathbb{1}$ be negative definite.

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A necessary condition is that the same restricted quadratic form for the submatrix A_{22} of A corresponding to the strictly positive elements of p_1 be nonpositive definite.

A second order criterion

Given a 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

define its *symmetric difference* $\sigma(M) = a - b - c + d$.

Given an $n \times n$ matrix M, define the $n-1 \times n-1$ matrix $\sigma(M)$ by replacing each block of four adjacent numbers in M by their symmetric difference.

The sufficient condition is that $\sigma(A_{11}) + \sigma(A_{11})^t < 0$.
The necessary condition is that $\sigma(A_{22}) + \sigma(A_{22})^t \leq 0$.

Example

A rare example of a non trivial ESS in a continuous setting.

$X = [0,1]$, $H(x,y) = \max\{x-y, \lambda(y-x)\}$, $\lambda \in (0,1)$.

Theorem p is a global ERS:

$$p = \frac{\lambda}{1+\lambda}\delta_0 + \frac{1}{1+\lambda}\delta_1 \,.$$

For a given $q \in \Delta(X)$, define x_0 [...] such that $q([0,x_0]) = \lambda/(1+\lambda)$, and $a = \int_0^{x_0} xq(\mathrm{d}x)$, x_1 and b symmetrically in the neighborhood of 1. We show

$$F(p,q) - F(q,q) \geq (1+\lambda)\left(\frac{a^2}{x_0} + \frac{b^2}{1-x_1}\right) \,.$$

Hawk and Doves

Sharing a prey

$Opponent$ Him	D	H
D	$1/2$	0
H	1	$-\theta$

Hawk and Doves

Sharing a prey

$Population$	p	$1-p$
$Opponent$	D	H
Him		
D	$1/2$	0
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Hawk and Doves

Sharing a prey

$Population$ / $Opponent$ / Him	p / D	$1-p$ / H
D	$1/2$	0
H	1	$-\theta$

Wardrop condition:

$$0.5p = p - \theta(1-p) \quad \Rightarrow \quad p = \frac{\theta}{\theta + 0.5}$$

Replicator Dynamics

If $G(x, q)$ measures the excess of births over deaths, like begets like, and q is interpreted as the distribution of strategies among the population, it follows that

$$\dot{q}(t, x) = q(t, x)[G(x, q(t)) - F(q(t), q(t))] .$$

This is called the *replicator dynamics*.

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This is called the *replicator dynamics*. "Evolutionary game ?"

Theorem In the finite linear case, every locally stable point of (1) is a Nash point. Every ESS is a locally asymptotically stable point of (1) and its attraction basin contains a neighborhood of the relative interior of the lowest dimensional face of $\Delta(X)$ it lies on.

Proof

Use the Lyapunov function

$$V(q) = \sum_i p_i \ln \left(\frac{p_i}{q_i} \right) .$$

$\ln x \leq 1 + x \Rightarrow V(q) > 0$ if $q \neq p$ and

$$\dot{V}(q) = F(q, q) - F(p, q) .$$

Stability in the infinite case

with A. J. Shaiju

We are investigating the stability of a differential equation over a space of measures. The Lyapunov function used to prove the above theorem is not continuous in the weak topology in the infinite case. We do not know whether an ERS (even global) is necessarily a stable point of (1).

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Theorem If $H(x, y) = H(y, x)$, an ERS is locally asymptotically stable.

Proof: use the Lyapunov function $F(p, p) - F(q, q)$.

Two by two case

Players have only two pure strategies (two phenotypes in the population). The game is described by a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let $\sigma = a - b - c + d$, (assume here $\sigma \neq 0$.)

Candidate Nash (via Wardrop condition) $p^\star = \frac{d-c}{\sigma}$. Is an ESS iff $\sigma < 0$.

Example : for Hawk and Dove, $\sigma = -0.5 - \theta < 0$.

2×2 **ESS**

In this most simple context,

Definition p is an ESS if

1. (p, p) is Nash, $\Longleftrightarrow$

$$\forall q \in [0,1]\,, \quad (\, p-q \quad q-p\,) A \begin{pmatrix} p \\ 1-p \end{pmatrix} \geq 0\,,$$

and

2. If $p \notin \{0,1\}$, then necessarily $p = p^\star$ and the above is 0,

$$\forall q \neq p^\star\,, \quad (\, p^\star - q \quad q - p^\star\,) A \begin{pmatrix} q \\ 1-q \end{pmatrix} > 0\,.$$

2×2 **ESS continued**

Corollary If $a \neq c$ and $b \neq d$, there always exists an ESS

1. If $\sigma = 0$ or $\sigma \neq 0$, but $p^\star \notin (0, 1)$, the unique ESS is pure,

2. If $\sigma \neq 0$ and $p^\star \in (0, 1)$,

   (a) If $\sigma < 0$, $p^\star$ is the only ESS,

   (b) If $\sigma > 0$ there are two pure ESS and no mixed ESS.

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Proof

$$(\, p^\star - q \quad q - p^\star \,) A \begin{pmatrix} p^\star \\ 1 - p^\star \end{pmatrix} = -\sigma (q - p^\star)^2 \, .$$

Replicator equation

If “fitness” measures the reproductive efficiency, i.e. the rate of the excess of births over deaths (possibly negative), p evolves accordingly to the simple replicator equation which becomes here

If $\sigma \neq 0$,

$$\dot{p} = \sigma p(1 - p)(p - p^\star)$$

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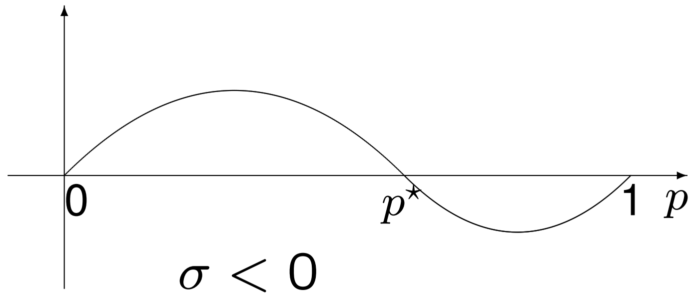
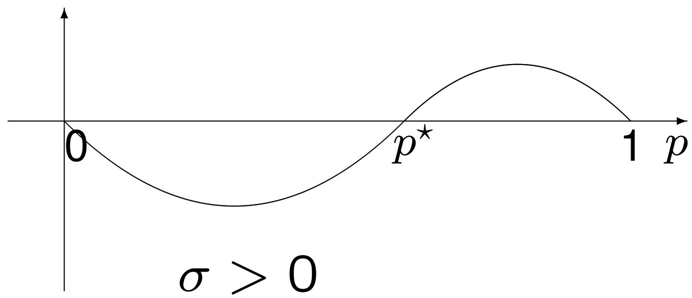
If $\sigma = 0$,

$$\dot{p} = (b-d)p(1-p)$$

Elementary replicator equation analysis



$\sigma < 0$



$\sigma > 0$

Population game

Two populations, with two behaviors each, interacting with each other. States p_1 and p_2, proportions of individuals using the first behavior.

Fitness accrued by an individual of population i using its first behavior $a_i p_j + b_i(1 - p_j)$, and with its second behavior $c_i p_j + d_i(1 - p_j)$.

Replicator equations become

$$\dot{p}_i = \sigma_i p_i (1 - p_i)(p_j - p_j^\star)\,.$$

Phase portrait

1. Unique (pure) Nash equilibria are stable,

2. If $(p_1^\star, p_2^\star) \in (0,1) \times (0,1)$,

   (a) If $\sigma_1\sigma_2 < 0$, the unique Nash equilibrium $(p_1^\star, p_2^\star)$ is a center, and all trajectories are periodic.

   (b) If $\sigma_1\sigma_2 > 0$, the Nash equilibrium $(p_1^\star, p_2^\star)$ is a saddle, the two pure Nash are stable.

Proof The following function is a first integral:

$$V(p_1, p_2) = \sum_{i=1}^{2} (-1)^i \sigma_i \left[p_i^\star \ln \frac{p_i^\star}{p_i} + (1 - p_i^\star) \ln \frac{1 - p_i^\star}{1 - p_i} \right]$$

Lynxes and wolves

$L\backslash W$	$cow.$	$agr.$
$cow.$	$1-\lambda$ λ	1 0
$agr.$	0 $1-\mu$	$-\theta$ $1-\nu$

$$\lambda+\mu>1>\nu$$

$$\sigma_1=\lambda+\mu-\nu\,,\quad p_2^\star=(1-\nu)/(\lambda+\mu-\nu)\,,$$
$$\sigma_2=-\lambda-\theta\,,\qquad p_1^\star=\theta/(\lambda+\theta)\,.$$

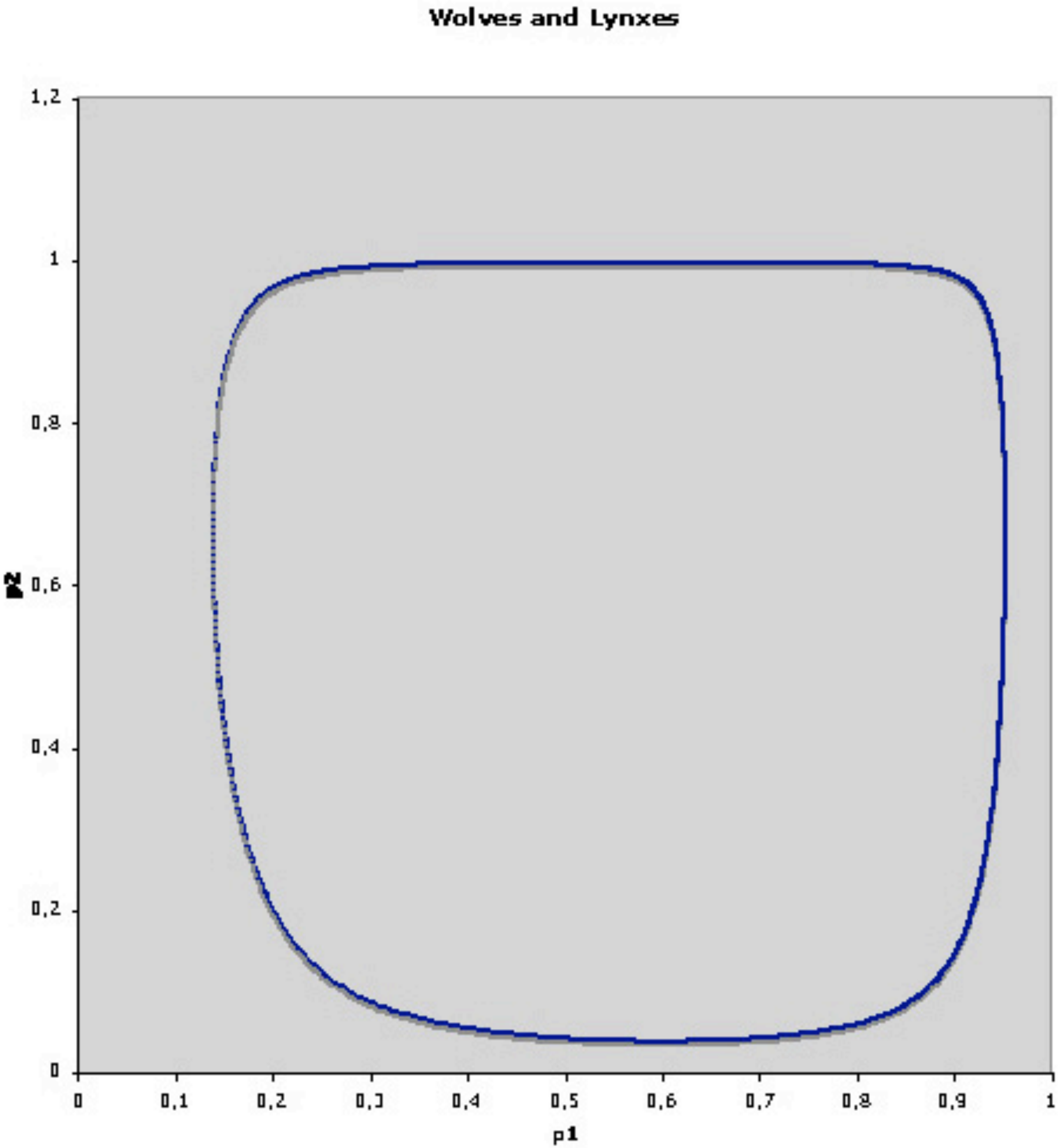
Lynxes and wolves

$L\backslash W$	$cow.$	$agr.$
$cow.$	λ , $1-\lambda$	0 , 1
$agr.$	$1-\mu$, 0	$1-\nu$, $-\theta$

$$\lambda + \mu > 1 > \nu$$

$$\sigma_1 = \lambda + \mu - \nu\,, \quad p_2^\star = (1-\nu)/(\lambda+\mu-\nu)\,,$$
$$\sigma_2 = -\lambda - \theta\,, \qquad p_1^\star = \theta/(\lambda+\theta)\,.$$

Draw case $\lambda = \nu = 0{,}5,\quad \mu = 0{,}75,\quad \theta = 1{,}5.$

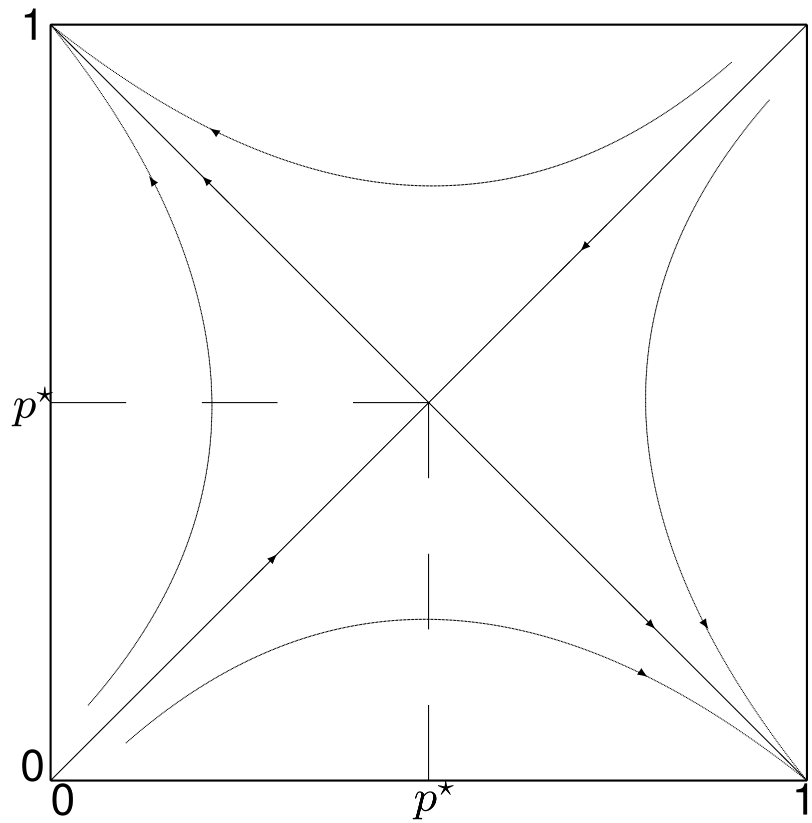

Wolves and Lynxes
1,2
1
0,8
0,6
0,4
0,2
0
p2
0
0,1
0,2
0,3
0,4
0,5
0,6
0,7
0,8
0,9
1
p1

Two identical populations

Two identical but distinct populations. Potentially, $p_1 \neq p_2$.
But $A_1 = A_2$, hence $\sigma_1 = \sigma_2$. Assume further that $\sigma_i < 0$. The single population replicator equation is stable at $p^\star$, but $\sigma_1 \sigma_2 > 0$ hence $(p^\star, p^\star)$ is *not* stable.

The diagonal $p_1 = p_2 = p$ represents the solution of the one-population game. Highly unstable in the two-population game.

Two identical populations



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