ASYMPTOTIC BEHAVIOR OF A MULTIPLEXER FED BY A LONG-RANGE DEPENDENT PROCESS*

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Abstract

In this paper we study the asymptotic behavior of the tail of the stationary backlog distribution in a single server queue with constant service capacity \( c \), fed by the so-called “\( M/G/\infty \) input process” or “Cox input process”. Asymptotic lower bounds are obtained for any distribution \( G \) and asymptotic upper bounds are derived when \( G \) is a subexponential distribution. We find the bounds to be tight in some instances, e.g., \( G \) corresponding to either the Pareto or lognormal distribution and \( c - \rho < 1 \), where \( \rho \) is the arrival rate to the buffer.

Keywords: Asymptotic self-similar process; Long-range dependence; Subexponential distributions; Pareto distribution; Large deviations; Queues.

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1 Introduction

The recent discovery [16, 21, 30] that traffic in networks possess long-range time dependencies that cannot be easily captured by Poisson-based models has motivated queueing theorists to propose and analyze new queueing models that capture these dependencies. One such model that has received attention is a buffer with server having rate $c$ fed by an $M/G/\infty$ input process where $G$ is heavy-tailed (e.g., [1, 12, 19, 27]). This is of interest because of its versatility, i.e., the dependencies over different time-scales can be controlled by varying the tail behavior of $G$.

In this paper we consider the model introduced by Parulekar and Makowski [27]. A discrete-time single-server queue (called the multiplexer) with infinite waiting room and with service capacity $c$ is fed by an integer-valued process $\{b_t, t \in \mathbb{N}\}$. The r.v. $b_t$ is defined as the number of busy servers at time $t \in \mathbb{N}$ in an $M/G/\infty$ queue with arrival intensity $\lambda > 0$ and i.i.d. service times $\{\sigma_n\}_n$ with common cumulative distribution function (c.d.f.) $G(x) = P(\sigma_n \leq x)$ and finite mean $\overline{\sigma}$. An appealing feature of the (stationary version of the) input process $\{b_t, t \in \mathbb{N}\}$ is that it is a long-range dependent process [2] for some well-chosen subexponential c.d.f.'s $G$ (see Section 2).

Let $Q_t$ be the queue-length at the multiplexer at time $t$. Then, $Q_t$ satisfies the Lindley’s equation $Q_{t+1} = \max(0, Q_t + b_t - c)$ for all $t \in \mathbb{N}_t$, with $Q_0 = 0$. Let $Q$ be the stationary queue-length under the stability condition $c > \rho := \lambda \overline{\sigma}$ (see Section 2). The aim of this paper is to study the behavior of $\log P(Q > x)$ and of $P(Q > x)$ for large $x$. More precisely, we show that there exist positive and finite numbers $\theta_1$ and $\theta_2$, depending on $G$, such that

$$-\theta_1 \leq \liminf_{x \to \infty} \frac{\log P(Q > x)}{-\log G_1(x)} \leq \limsup_{x \to \infty} \frac{\log P(Q > x)}{-\log G_1(x)} \leq -\theta_2.$$  \hspace{1cm} (1)

The lower bound in (1) holds for any c.d.f. $G$ whereas the upper bound holds for any subexponential c.d.f. $G$ (to be defined in Section 2). Here $G_1$ is defined as

$$G_1(x) := \frac{1}{\overline{\sigma}} \int_0^x G(u) \, du, \quad x \geq 0$$  \hspace{1cm} (2)

and $F(x) = 1 - F(x)$ for any probability distribution $F$. We also show that the bounds in (1) are tight i.e. $\theta_1 = \theta_2$ when $G$ is Pareto or lognormal (see Corollary 4.1), provided that $c - \rho < 1$. In the following the bounds in (1) will be referred to as large deviations bounds. Asymptotic upper and lower bounds for $P(Q > x)$ are also obtained.

Large deviations bounds were obtained in [29] in the case when $G$ is short-tailed. Duffield observed in [12] that the approach in [27], based on the Gärtner-Ellis theorem, cannot be used to derive large deviations lower bounds for heavy-tailed $G$. By refining Theorem 2.2 in [13] and by using results in [28] Duffield was able to obtain the following large deviations upper bound (see [12])

$$\limsup_{x \to \infty} \frac{\log P(Q > x)}{\log x} \leq 1 - (\alpha - 1)(c - \rho)$$  \hspace{1cm} (3)
in the case of the Pareto distribution \( \Theta(x) \sim c_1 x^{-\alpha} \). An asymptotic lower bound for \( P(Q > x) \) was obtained by Jelenkovic and Lazar [19] in the case when \( c - \rho < 1 \) and under a technical condition on \( G_1 \) (see comment after the proof of Proposition 3.2).

In this paper we propose an alternative to the approach based on the Gärtner-Ellis theorem that will yield asymptotic lower and upper bounds. We will observe that the large deviations bounds are tight for a number of subexponential distributions when \( c - \rho < 1 \) and that, in the case of \( G \) Pareto, the large deviations upper bound that can be derived from (1) (see Proposition 4.1) is tighter than that of Duffield when \( c - \rho \leq \alpha/(\alpha - 1) \); otherwise Duffield’s is tighter.

Other models have been proposed for modeling the effects of long-range dependence in arrival processes on buffer occupancy statistics. These include fractional brownian motion [13, 25], fractional gaussian noise [27], and a finite population of on-off sources where the on state holding times are characterized by heavy-tailed distributions [5, 7, 9, 18, 19, 22, 31] (see [6] for a survey on fluid queues with long-tailed activity periods).

The rest of the paper is structured as follows. Section 2 contains a characterization of the stationary behavior of the \( M/G/\infty \) input process and the definition and characterization of the family of subexponential distributions. Asymptotic lower and upper bounds are established in Sections 3 and 4 respectively. Concluding remarks on the superposition of independent \( M/G/\infty \) input processes are given in Section 5.

## 2 Preliminaries

The lemma below gives a useful characterization of the stationary behavior of the input process \( \{b_t, t \in \mathbb{N}\} \). We will assume that customers entering the \( M/G/\infty \) queue begin their service upon arrival (see Remark 2.1).

**Lemma 2.1** The distribution of the sequence \( \{b_{t+k}, t \in \mathbb{N}\} \) converges monotonically for \( k \to \infty \) to that of a proper stationary and ergodic sequence \( \{b^t, t \in \mathbb{N}\} \) such that

\[
y_t = \sum_{j=0}^{b^t} I(\hat{\sigma}_j > t) + \sum_{s=0}^{t-1} \sum_{s \leq T_j < s+1} I(\sigma_j > t - T_j), \quad t \in \mathbb{N}
\]

where

1. (i) \( 0 < T_1 \leq T_2 \leq \cdots \) are the successive jump times of a Poisson process with intensity \( \lambda \), independent of the service times \( \{\sigma_n, n = 1, 2, \ldots\} \);

2. (ii) \( b^0 \) is a Poisson r.v. with parameter \( \rho := \lambda \pi \);
(iii) conditioned on the event \( \{ b^0 = k \}, k \geq 1 \), the r.v.'s \( \{ \hat{\sigma}_1, \ldots, \hat{\sigma}_k \} \) are i.i.d. with common c.d.f. \( G_1 \) as defined in (2), namely,

\[
P \left( \hat{\sigma}_1 \leq x_1, \ldots, \hat{\sigma}_k \leq x_k \mid b^0 = k \right) = \prod_{j=1}^{k} G_1(x_j).
\]

Further, the r.v.'s \( \{ T_j, \sigma_j, j = 1, 2, \ldots \} \) are independent of the r.v.'s \( \{ b^0, \hat{\sigma}_j, j = 1, 2, \ldots \} \).

The proof of this lemma follows from [4, Chapter 6] and [33, pp. 160-162] (see also [27]). The interpretation of (4) is the following: given that the M/G/\( \infty \) queue is in steady-state at time \( t = 0 \), the first sum in the r.h.s. gives the number of busy servers at time \( t = 1, 2 \ldots \) among all servers busy at time \( 0 \); the second sum gives the number of servers that became busy at time \( s, 0 \leq s \leq t - 1 \), and that are still busy at time \( t \).

Assume that \( \rho < c \). Since the process \( \{ b_{i+k}, t \in N \} \) converges to the stationary and ergodic process \( \{ b^i, t \in N \} \) (see Lemma 2.1) then it is well-known (see e.g. [4, Theorem 6, p. 12]) that there exists a proper r.v. \( Q \) such that

\[
P(Q > x) = \lim_{t \to \infty} P(Q_t > x) = P \left( \sup_{t \in N} \left( \sum_{s=0}^{t-1} b^{-s} - ct \right) > x \right), \quad x \in N
\]  

where \( \{ b^i, -\infty < t < \infty \} \) is a stationary and ergodic process obtained by supplementing \( \{ b^i, t \in N \} \). We will however prefer the following representation for the stationary queue length distribution:

\[
P(Q > x) = P \left( \sup_{t \in N} \left( \sum_{s=0}^{t-1} b^s - ct \right) > x \right), \quad x \in N,
\]

which follows from (5) together with the property that the number of busy servers in a stationary M/G/\( \infty \) queue is a reversible stochastic process [20, Theorem 3.11].

The rest of this paper is devoted to the computation of asymptotic lower and upper bounds for \( P(Q > x) \). Particular attention will be devoted to the case when the c.d.f. \( G \) of the service times is subexponential. Recall that a probability distribution \( F \) on \([0, \infty)\) is subexponential, denoted as \( F \in \mathcal{S} \) (or \( \overline{F} \in \mathcal{S} \) with a slight abuse of notation) if \( \overline{F}^{*2}(x) \sim 2 \overline{F}(x) \) where \( \overline{F}^{*2} \) denotes the 2nd convolution of \( F \) with itself, namely, \( \overline{F}^{*2}(x) = \int_0^\infty F(x-u) F(du) \). As usual, the notation \( f(x) \sim g(x) \) will stand for \( \lim_{x \to \infty} f(x)/g(x) = 1 \) and \( f(x) = o(g(x)) \) will stand for \( \lim_{x \to \infty} f(x)/g(x) = 0 \). The class of subexponential distributions was introduced by Chistyakov [8] and contains Pareto, Weibull and lognormal distributions (see Section 3), among others. A probability distribution \( F \) on \([0, \infty)\) belongs to the class \( \mathcal{D} \) of dominated-variation distributions if \( \limsup_{x \to \infty} F(x)/F(2x) < \infty \) and to the class \( \mathcal{L} \) of long-tailed distributions if \( \lim_{x \to \infty} F(x-y)/F(x) = 1 \) for all \( y \in (\infty, \infty) \).
For any c.d.f. $F$ on $[0, \infty)$ with finite expectation $\mu$, (i.e. $\mu := \int_0^\infty u F(du) < \infty$), define the integrated tail distribution $F_1$ by

$$F_1(x) := \frac{1}{\mu} \int_0^x \mathcal{T}(u) \, du, \quad x \geq 0.$$  

Note that $G_1$ in (2) is the integrated tail distribution of $\sigma_n$.

The next lemma reports basic properties of subexponential probability distributions.

**Lemma 2.2** The following statements hold:

(a) $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$ [15, 17];

(b) If $F$ has finite expectation and if $F \in \mathcal{D}$ then $F_1 \in \mathcal{D} \cap \mathcal{L}$ [15];

(c) If $F \in \mathcal{S}$ and $G$ is a probability distribution on $[0, \infty)$ such that $\mathcal{T}(x) \sim c_1 \mathcal{G}(x)$ for some positive constant $c_1$, then $G \in \mathcal{S}$ [26, Lemma 2].

In particular, we see from properties (a) and (b) that if $F \in \mathcal{D} \cap \mathcal{L}$ and if $F$ has finite expectation then $F, F_1 \in \mathcal{S}$.

We conclude this section by pointing out an interesting feature (already observed in [27, p. 1455]) of the process $\{b^t, t \in \mathbb{N}\}$ defined in (4). First, it has been shown in [11, formula (5.39)] that $\text{cov}(b^t, b^{t+h}) = \rho \mathcal{G}_1(h)$ for all $t, h \in \mathbb{N}$. Therefore, the stationary process $\{b^t, t \in \mathbb{N}\}$ will be long-range dependent [2] if $\sum_{h=0}^{\infty} \mathcal{G}_1(h) = \infty$, which will occur, for instance, when $G$ is Pareto (i.e. $\mathcal{G}(x) \sim x^{-\alpha}$) with parameter $1 < \alpha < 2$.

**Remark 2.1** By taking integer-valued service times our model reduces to that in [27]. This follows from the fact that in the case of integer-valued service times the number of busy servers at time $t + 1$ is the same whether customers entering the $M/G/\infty$ queue in $(t, t + 1)$ begin their service upon arrival (as in our model) or begin their service at time $t + 1$ (as in [27]).

### 3 Lower Bounds

The following representation of $A(0, t) := \sum_{s=0}^{t-1} b^s$ will prove useful:

$$A(0, t) = \sum_{s=0}^{t-1} b^s = \sum_{s=0}^{t-1} \sum_{j=1}^{b^0} I(\bar{\sigma}_j > s) + \sum_{s=0}^{t-1} \sum_{k=0}^{s-1} \sum_{k \leq T_j < k+1} I(\sigma_j > s - T_j)$$
\[
\begin{align*}
&= \sum_{j=1}^{\theta} \sum_{s=0}^{t-1} I(\hat{\sigma}_j > s) + \sum_{k=0}^{t-2} \sum_{k=T_j}^{k+1} \sum_{s=k+1}^{t-1} I(\sigma_j > s - T_j) \\
&= \sum_{j=1}^{\theta} \min(\lceil \hat{\sigma}_j \rceil, t) + \sum_{k=0}^{t-2} \sum_{k=T_j}^{k+1} \sum_{s=k+1}^{t-1} I(\sigma_j > s - T_j)
\end{align*}
\]

where \([x]\) denotes the smallest integer larger than or equal to \(x\).

The first sum in the r.h.s. of (7) gives the total number of customers arriving to the multiplexer in \([0, t]\) generated by all servers in the infinite-server queue busy at time 0; the second sum gives the total number of customers arriving to the multiplexer in \((0, t)\) generated by all servers in the infinite-server queue that become active at time \(1, 2, \ldots, t - 1\). Set

\[
a_0(t) := \sum_{j=1}^{\theta} \min(\lceil \hat{\sigma}_j \rceil, t)
\]

\[
a_s(t) := \sum_{s-1 \leq T_j < s} \sum_{i=s}^{t-1} I(\sigma_j > i - T_j), \quad s = 1, 2, \ldots, t - 1
\]

so that

\[
A(0, t) = \sum_{s=0}^{t-1} a_s(t).
\]

Denote by \([x]\) the largest integer smaller than or equal to \(x\). The following asymptotic lower bound for \(\log P(Q > x)\) holds:

**Proposition 3.1 (Large deviations lower bound)**

*For any c.d.f. \(G_*\),*

\[
\liminf_{x \to \infty} \frac{\log P(Q > x)}{-\log G_1(x)} \geq -\inf_{\beta > 0} \left\{ \lceil c - \rho + \beta \rceil + 1 \right\} \limsup_{x \to \infty} \frac{\log G_1(x)}{-\log G_1(\beta x)}.
\]

**Proof.** Fix \(\beta > 0, \epsilon > 0\), and define \(\gamma := c - \rho + \beta + \epsilon\). Note that \(\gamma > 0\) under the stability condition \(c > \rho\).

We have

\[
\begin{align*}
&\liminf_{x \to \infty} \frac{\log P(Q > x)}{-\log G_1(x)} = \liminf_{t \to \infty} \frac{\log P(Q > \beta t)}{-\log G_1(\beta t)} \\
&\geq \liminf_{t \to \infty} \frac{\log P(A(0, t) - ct > \beta t)}{-\log G_1(\beta t)} \\
&\geq \liminf_{t \to \infty} \frac{1}{-\log G_1(\beta t)} \log P \left( a_0(t) \geq \gamma t, \sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon)t \right)
\end{align*}
\]

5
\[
\liminf_{t \to \infty} \frac{-1}{\log G_1(\beta t)} \left[ \log P(a_0(t) \geq \gamma t) + \log P \left( \sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon) t \right) \right] \quad (13)
\]
\[
\geq \liminf_{t \to \infty} \frac{\log P(a_0(t) \geq \gamma t)}{-\log G_1(\beta t)} + \liminf_{t \to \infty} \frac{-1}{\log G_1(\beta t)} \log P \left( \sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon) t \right). \quad (14)
\]

Inequality (12) follows from \( P(Q > x) \geq P(A(0, t) - ct > x) \) (see (6)); (13) is a consequence of the independence of the r.v.’s \( a_0(t) \) and \( \sum_{s=1}^{t-1} a_s(t) \) (see Lemma 2.1); (14) comes from the inequality \( \liminf_n (a_n + b_n) \geq \liminf_n a_n + \liminf_n b_n \).

Let us now focus on the first limit in the r.h.s. of (14). We have for \( t > 0 \)
\[
P(a_0(t) \geq \gamma t) = P \left( \sum_{j=1}^{b_0} \min(\lceil \tilde{\sigma}_j \rceil, t) \geq \gamma t \right)
\]
\[
\geq \sum_{k=\lceil \gamma \rceil}^{\infty} P \left( \sum_{j=1}^{k} \min(\tilde{\sigma}_j, t) \geq \gamma t | b^0 = k \right) P(b^0 = k)
\]
\[
\geq \sum_{k=\lceil \gamma \rceil}^{\infty} P \left( \tilde{\sigma}_1 > t, \ldots, \tilde{\sigma}_{\lceil \gamma \rceil} > t | b^0 = k \right) P(b^0 = k)
\]
\[
= \frac{G_1(t)^{\lceil \gamma \rceil}}{G_1(\beta t)^{\lceil \gamma \rceil}} P(b^0 \geq \lceil \gamma \rceil) \quad (15)
\]
where (16) follows from Lemma 2.1(iii).

Since \( P(b^0 \geq \lceil \gamma \rceil) > 0 \) (see Lemma 2.1(ii)) we deduce from (16) that
\[
\liminf_{t \to \infty} \frac{\log P(a_0(t) \geq \gamma t)}{-\log G_1(\beta t)} \geq -\lceil \gamma \rceil \limsup_{t \to \infty} \frac{\log G_1(t)}{\log G_1(\beta t)}.
\quad (17)
\]

Let us show that the second limit in the r.h.s. of (14) is 0. We see from the definition of \( A(0, t) \) and from (8)-(10) that
\[
\sum_{s=1}^{t-1} a_s(t) \geq \sum_{s=0}^{t-1} b^s - \sum_{j=1}^{\lceil \tilde{\sigma}_j \rceil}.
\quad (18)
\]
On the other hand, the stationarity and ergodicity of the sequence \( \{b^t, t \in \mathbb{N}\} \) together with \( \rho = E[b^0] < \infty \) (see Lemma 2.1) yields
\[
\lim_{t \to \infty} \frac{1}{t} \sum_{s=0}^{t-1} b^s = \rho \quad \text{a.s.} \quad (19)
\]
from ergodic theory (see e.g. [32, Chapter V]). We therefore deduce from (18)-(19) that
\[
\liminf_{t \to \infty} \frac{1}{t} \sum_{s=1}^{t-1} a_s(t) \geq \rho \quad \text{a.s.} \quad (20)
\]
since $\sum_{j=1}^{\theta_0} \tilde{\sigma}_j < \infty$ a.s. by Lemma 2.1.

Combining [24, Proposition I-4-3] together with (20) yields

$$1 \geq \liminf_t P \left( \sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon) t \right) \geq P \left( \liminf_t \left\{ \sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon) t \right\} \right) = 1$$

(21)

which entails that

$$\liminf_{t \to \infty} \frac{1}{-\log G_1(\beta t)} \log P \left( \sum_{s=1}^{t-1} a_s(t) > (\rho - \epsilon) t \right) = 0.$$  (22)

In summary, we have shown that (cf. (14), (17), (22))

$$\liminf_{x \to \infty} \frac{\log P(Q > x)}{-\log G_1(x)} \geq -\inf_{\beta > 0, \epsilon > 0} \left\{ [\epsilon - \rho + \beta + \epsilon] \limsup_{t \to \infty} \frac{\log G_1(t)}{\log G_1(\beta t)} \right\}$$

$$\geq -\inf_{\beta > 0} \left\{ (\epsilon - \rho + \beta + 1) \limsup_{t \to \infty} \frac{\log G_1(t)}{\log G_1(\beta t)} \right\}$$

which completes the proof. 

It is worth noting that the lower bound in (11) is never trivial as it is always larger than or equal to $-(\lfloor c - \rho \rfloor + 2)$ that is obtained for $\beta = 1$.

The next result proposes asymptotic lower bounds for $P(Q > x)$.

**Proposition 3.2 (Asymptotic lower bound)**

*For any c.d.f. $G$,*

$$\liminf_{x \to \infty} \frac{P(Q > x)}{G_1(x)^{\lfloor c - \rho \rfloor + 1}} \geq \sup_{0 < \beta \leq 1 + [c - \rho] - (c - \rho)} \liminf_{x \to \infty} \left( \frac{\overline{G_1(x)}}{\overline{G_1(\beta x)}} \right)^{\lfloor c - \rho \rfloor + 1} \left( 1 - \sum_{k=0}^{\infty} \frac{\rho^k}{k!} e^{-\rho} \right).$$

(23)

**Proof.** The proof of (23) follows the same line of arguments as that of Proposition 3.1. Define $\gamma := c - \rho + \beta + \epsilon$. Let $0 < \beta \leq 1 + [c - \rho] - (c - \rho)$ and pick $\epsilon > 0$ small enough so that $\lfloor \gamma \rfloor = [c - \rho] + 1$.

In direct analogy with the derivation of (14) and by using (16) and (21) we get

$$\liminf_{t \to \infty} \frac{P(Q > x)}{\overline{G_1(x)}^{\lfloor c - \rho \rfloor + 1}} \geq \liminf_{t \to \infty} \frac{P(a_0(t) > \gamma t)}{\overline{G_1(\beta t)}^{\lfloor c - \rho \rfloor + 1}}$$

$$\geq \liminf_{t \to \infty} \left( \frac{\overline{G_1(t)}}{\overline{G_1(\beta t)}} \right)^{\lfloor c - \rho \rfloor + 1} P(t^0 \geq [c - \rho] + 1)$$

(24)
for all $0 < \beta < 1 + [c - \rho] - (c - \rho)$, from which (23) follows.

It is worth noting that the supremum in the r.h.s. of (23) is strictly positive if and only if $G_1 \in \mathcal{D}$. Indeed, it follows from [3, Corollary 2.0.6, p. 65] that if $\lim \inf_{x \to \infty} \frac{G_1(x)}{G_1(\delta x)}$ is strictly positive for some $\delta \in (0, 1)$ then this limit is strictly positive for all $\delta \in (0, 1)$, and in particular for $\delta = 1/2$.

A sufficient condition for $G_1 \in \mathcal{D}$ is that $G \in \mathcal{D}$ (e.g. G Pareto) and G has finite expectation (see Lemma 2.2(b)).

A refined lower bound has been obtained in [23] under the additional assumption that $G_1 \in \mathcal{S}$. When $c - \rho < 1$, Jelenkovic and Lazar [19, Theorem 11] have derived a tighter lower bound with the same decay function $G_1(x)$ but with a larger coefficient. The bound in [19] holds provided that

$L := \lim_{\delta \to 1} \lim \inf_{x \to \infty} \frac{G_1(x)}{G_1(\delta x)} > 0$ (Jelenkovic and Lazar [19] actually assume that $L = 1$ but this assumption can be weakened to $L > 0$; if so, then the coefficient of their lower bound in Theorem 11 has to be multiplied by $L$). Since $G_1$ is non-increasing, it is easy to see from [3, Corollary 2.0.6] that $L > 0$ is equivalent to $G_1 \in \mathcal{D}$. Hence, both bounds in Proposition 3.2 and in [19] are non-trivial if and only if $G_1 \in \mathcal{D}$.

**Corollary 3.1** When $G_1 \in \mathcal{D}$ then

$$\lim \inf_{x \to \infty} \frac{\log P(Q > x)}{-\log G_1(x)} \geq -[c - \rho] - 1. \quad (26)$$

When Corollary 3.1 applies, the lower bound in the r.h.s. of (26) is easier to compute than the lower bound in Proposition 3.1 but may not be as tight (for G Pareto both bounds in (11) and in (26) are the same as reported below).

We conclude this section by addressing the cases when G is (i) geometric, (ii) Pareto, (iii) Weibull, and (iv) lognormal.

(i) **G is geometric.** We have $P(\sigma_n = r) = (1 - q)r^{r-1}$ for $r = 1, 2, \ldots$ with $q \in (0, 1)$. Hence, $G_1(r) = q^r$ for $r = 0, 1, \ldots$. Proposition 3.2 yields a trivial lower bound ($= 0$). From Proposition 3.1 we find

$$\lim \inf_{x \to \infty} \frac{1}{x} \log P(Q > x) \geq \log q \inf_{\beta > 0} \frac{|c - \rho + \beta| + 1}{\beta} = \log q. \quad (27)$$

The r.h.s. of (27) follows from the inequalities

$$\frac{c - \rho + \beta + 1}{\beta} \geq \frac{|c - \rho + \beta| + 1}{\beta} \geq 1$$

together with $\lim_{\beta \to \infty} (c - \rho + \beta + 1)/\beta = 1$. 
(ii) **G is Pareto.** We have $\overline{G}(x) \sim c_1 x^{-\alpha}$ for some $\alpha > 1$, $c_1 > 0$. Hence,

$$\overline{G}_1(x) \sim c_2 x^{-\alpha + 1}$$

with $c_2 = c_1 / (\sigma(\alpha - 1))$. From Proposition 3.2 we get

$$\liminf_{x \to \infty} \frac{P(Q > x)}{x^{-(\alpha + 1)}} \geq c_2^\zeta (\zeta - (c - \rho))^{(\alpha - 1)\zeta} \left( 1 - \frac{1}{\sum_{k=0}^{\zeta-1} \rho^k} e^{-\rho} \right).$$

where we set $\zeta := [c - \rho] + 1$. In particular, (29) (or Proposition 3.1/Corollary 3.1) yields

$$\liminf_{x \to \infty} \frac{1}{\log x} \log P(Q > x) \geq (-\alpha + 1) \zeta.$$  

(iii) **G is Weibull.** We have $\overline{G}(x) = e^{-c_1 x^\nu}$ for some $0 < \nu < 1$ and $c_1 > 0$. Simple algebra yield

$$\overline{G}_1(x) \sim c_2 e^{-c_1 x^\nu} x^{1-\nu}$$

with $c_2 = 1/(c_1 \nu \sigma)$ and $\sigma = \Gamma(1/\nu) / (\nu c_1^{1/\nu})$ where $\Gamma(s) := \int_0^\infty x^{s-1} \exp(-x) dx$ for $s > 0$. Proposition 3.2 yields a trivial lower bound (i.e. 0). By Proposition 3.1 we get (Corollary 3.1 does not apply since $G_1 \not\in D$)

$$\liminf_{x \to \infty} \frac{1}{x^\nu} \log P(Q > x) \geq - \inf_{\beta > 0} \frac{|c - \rho + \beta| + 1}{\beta^\nu}$$

$$= \begin{cases} \min \left\{ \frac{|c - \rho| + |a|}{(|a| - q)^\nu} ; \frac{|c - \rho| + |[a]|}{([a] - q)^\nu} \right\}, & \text{if } a \geq 1 \\ \frac{|c - \rho| + 1}{(1-q)^\nu}, & \text{if } a < 1 \end{cases}$$

with $a := (\nu |c - \rho| + q)/(1 - \nu)$ and $q := c - \rho - |c - \rho|$. Indeed,

$$\inf_{\beta > 0} \frac{|c - \rho + \beta| + 1}{\beta^\nu} = \min_{i=1,2,...} \frac{|c - \rho| + i}{(i-q)^\nu}.$$  

with the mapping $g(x) := ((|c - \rho| + x)/(x - q)^\nu$ being strictly decreasing in $(0,a)$ and strictly increasing in $(a,\infty)$, so that the minimum in (33) is reached when $\beta = [a]$ or when $\beta = [a]$ if $a \geq 1$ and when $\beta = 1$ if $a < 1$.

(iv) **G is lognormal.** The c.d.f. $G$ of a r.v. $\sigma$ is lognormal if $\sigma \overset{\text{iid}}{=} \exp(Y)$ where $Y$ is a Gaussian r.v. with mean $\mu$ and variance $\delta^2$. Then, $\overline{G}(x) \sim (2\pi)^{-1/2} (\delta / (\log x - \mu)) e^{-(\log x - \mu)^2/(2 \delta^2)}$. From this we get

$$\overline{G}_1(x) \sim \frac{\delta^3 x e^{-(\log x - \mu)^2/(2 \delta^2)}}{\sigma \sqrt{2\pi} (\log x - \mu)^2}$$

with $\sigma = \exp(\mu + \delta^2/2)$. Proposition 3.2 yields a trivial lower bound (i.e. 0). From Proposition 3.1 (Corollary 3.1 does not apply since $G_1 \not\in D$) we have

$$\liminf_{x \to \infty} \frac{1}{\log x^2} \log P(Q > x) \geq - \frac{|c - \rho| + 1}{2 \delta^2}.$$
4 Upper Bounds

We begin this section by stating two lemmas that will be used in the derivation of asymptotic upper bounds in the case when \( G \) and \( G_1 \) are subexponential probability distributions.

**Lemma 4.1 (Cline [10])** Let \( F, F^1, \ldots, F^k \) be probability distributions such that \( F^j(x) \sim c_j F(x) \), \( c_j > 0 \), for all \( j = 1, 2, \ldots, k \). If \( F \in \mathcal{S} \) then \( F^1 \cdots F^k(x) \sim \sum_{j=1}^{k} c_j F(x) \).

**Lemma 4.2 (Pakes [26])** Consider a GI/GI/1 queue with i.i.d. service times \( \{\sigma_n\} \) with common c.d.f. \( F \) and i.i.d. interarrival times \( \{\tau_n\} \). Assume that \( E[\sigma_n] < E[\tau_n] \).

If \( F, F_1 \in \mathcal{S} \), then

\[
P(W > x) \sim \frac{E[\sigma_n]}{E[\tau_n] - E[\sigma_n]} F_1(x)
\]

where \( W := \sup_{n \in \mathbb{N}} \left( \sum_{m=0}^{n-1} (\sigma_m - \tau_m) \right) \) is the stationary waiting time.

We are now in position to derive the following asymptotic upper bounds for \( P(Q > x) \) and for \( \log P(Q > x) \) when \( G \) and \( G_1 \) are in \( \mathcal{S} \).

**Proposition 4.1 (Upper bounds)**

Assume that \( G, G_1 \in \mathcal{S} \). Then,

\[
\limsup_{x \to \infty} \frac{P(Q > x)}{G_1(x)} \leq \rho + \frac{\rho}{c - \rho}.
\]  

(36)

In particular, (36) implies that

\[
\limsup_{x \to \infty} \frac{\log P(Q > x)}{-\log G_1(x)} \leq -1.
\]  

(37)

**Proof.** Define

\[
a_0 = \sum_{j=1}^{b^0} (\hat{\sigma}_j + 1)
\]

(38)

\[
a_s = \sum_{j=1}^{v_s} [\sigma_j + T_j(s) - s], \quad s = 1, 2 \ldots
\]

(39)

where \( v_s \) denotes the number of arrivals in the M/G/\( \infty \) queue in the interval of time \( [s-1, s) \) and \( T_j(s) \) is the time of the \( j \)-th arrival in \( [s-1, s) \) for \( s = 1, 2, \ldots \). Since the arrival process in
this queue is Poisson with rate \( \lambda \), \( \{v_s, s \in \mathbb{N}\} \) constitutes an i.i.d. sequence of Poisson r.v.'s with intensity \( E[v_s] = \lambda \), namely \( P(a_s = k) = \lambda^k \exp(-k)/k! \) for all \( k \in \mathbb{N} \).

We first establish some preliminary results related to the r.v.'s \( a_0, a_1, \ldots \). To begin with, we observe from (8)-(9) and (38)-(39) that

(a1) \( a_0(t) \leq a_0 \) (a.s.) and \( a_s(t) \leq a_s \) for all \( t = 1, 2, \ldots, s = 1, 2, \ldots, t - 1 \);

(a2) the r.v.'s \( a_s, s = 1, 2, \ldots \) are i.i.d. and independent of the r.v. \( a_0 \),

where \( X \leq_{st} Y \) if the real-valued r.v.'s \( X \) and \( Y \) satisfy \( E[f(X)] \leq E[f(Y)] \) for all measurable and nondecreasing mappings \( f : (-\infty, \infty) \rightarrow (-\infty, \infty) \) such that the expectations exist.

To get the second inequality in (a1) note from (9) that

\[
a_s(t) = \sum_{s-1 \leq T_j < s} \sum_{i=0}^{t-1-s} I(\sigma_j + T_j - s > i) = \sum_{s-1 \leq T_j < s} \min([\sigma_j + T_j - s], t - s) = \sum_{j=1}^{u_s} [\sigma_j + T_j(s) - s] = a_s \text{ for } s = 1, 2, \ldots, t - 1,
\]

where \( X \equiv Y \) if the r.v.'s \( X \) and \( Y \) have the same probability distribution. Next, we focus on the asymptotic behavior of \( P(a_s > x) \) for \( s \in \mathbb{N} \). Under the assumptions \( G, G_1 \in \mathcal{S} \), the inclusion \( S \subset L \) (see Lemma 2.2(a)) and Lemma 2.2(c) imply that

\[
\overline{G}(x) = P(\sigma_j > x) \sim P(\sigma_j - 1 > x) \in S \tag{40}
\]

\[
\overline{G_1}(x) = P(\hat{\sigma}_j > x) \sim P(\hat{\sigma}_j + 1 > x) \in S. \tag{41}
\]

On the other hand, the inequalities \( \sigma_j - 1 \leq [\sigma_j + T_j(s) - s] \leq \sigma_j \) combined with (40) and Lemma 2.2(c) in turn yields

\[
\overline{G}(x) \sim P([\sigma_j + T_j(s) - s] > x) \in S. \tag{42}
\]

By using now (41), (42) and [14, Theorem 1.3.9] we see that

\[
P(a_0 > x) \sim \rho \overline{G_1}(x) \tag{43}
\]

\[
P(a_s > x) \sim \lambda \overline{G}(x) \quad \text{for } s = 1, 2, \ldots. \tag{44}
\]

We conclude these preliminary remarks with the computation of \( E[a_s] \) for \( s \geq 1 \). For fixed \( s \geq 1 \), the r.v. \( s - T_j(s) \) is uniformly distributed over \( (0, 1) \) (since the arrivals are Poisson and independent of \( \sigma_j \). Hence, by applying Lemma A.1 with \( X = \sigma_j \) and \( U = s - T_j \) we find that \( E[\sigma_j + T_j - s] = E[\sigma_j] \), which in turn yields

\[
E[a_s] = E[v_s] E[\sigma_j + T_j - s] = \rho \tag{45}
\]

from Wald’s identity and the definition of \( \rho \).
We are now in position to proof (36). We start from (cf. (6), (10), (a1))

\[
P(Q > x) = P \left( \sup_{t \in \mathbb{N}} \left( a_0(t) + \sum_{s=1}^{t-1} a_s(t) - ct \right) > x \right) 
\]

\[
\leq P \left( a_0 + \sup_{t \in \mathbb{N}} \left( \sum_{s=1}^{t} a_s - ct \right) > x \right) 
\]

\[
= P(a_0 + W > x) 
\]  

(46)

where \( a_0 \) and \( W := \sup_{t \in \mathbb{N}} \left( \sum_{s=1}^{t} a_s - ct \right) \) are independent r.v.'s.

To proceed, we notice that under (a2), (45) and the (stability) condition \( \rho < c \), \( P(W \leq x) \) is the probability distribution of the stationary waiting time in a stable \( D/GI/1 \) queue with interarrival times \( c \) and i.i.d. service times \( \{a_s\}_s \). Therefore, by (44) and Lemma 4.2 [with \( \sigma_n = a_n \) and \( \tau_n = c \)] we find

\[
P(W > x) \sim \frac{\rho}{c - \rho} \bar{G}_1(x). 
\]  

(47)

By using now (43), (46), (47), the independence of the r.v.'s \( a_0 \) and \( W \) (see (a2)), and Lemma 4.1 [with \( F = G_1, F^1(x) = P(a_0 \leq x) \) and \( F^2(x) = P(W \leq x) \)] we conclude that (36) holds true.

It is known that both \( G \) and \( G_1 \) belong to \( S \) when \( G \) is (i) Pareto, (ii) Weibull or (iii) lognormal. We conclude this section by specializing Proposition 4.1 to these particular probability distributions.

(i) \( G \) is Pareto. From (28) and (37) we get

\[
\limsup_{x \to \infty} \frac{1}{\log x} \log P(Q > x) \leq -\alpha + 1. 
\]  

(48)

Also note that the bound in (48) is tighter than Duffield's corresponding bound (3) when \( c - \rho \leq \alpha/(\alpha - 1) \); otherwise Duffield's is tighter.

(ii) \( G \) is Weibull. From (31) and (37) we get

\[
\limsup_{x \to \infty} \frac{1}{x^{\rho}} \log P(Q > x) \leq -1. 
\]  

(49)

(iii) \( G \) is lognormal. From (34) and (37) we get

\[
\limsup_{x \to \infty} \frac{1}{(\log x)^2} \log P(Q > x) \leq -\frac{1}{2 \delta^2}. 
\]  

(50)

We observe from (29), (48) and (35), (50) that the bounds are tight when \( c - \rho < 1 \):
Corollary 4.1 Assume that \( c - \rho < 1 \). If \( G \) is Pareto then
\[
\lim_{x \to \infty} \frac{1}{\log x} \log P(Q > x) = -\alpha + 1
\]  
and if \( G \) is lognormal then
\[
\lim_{x \to \infty} \frac{1}{(\log x)^2} \log P(Q > x) = -\frac{1}{2\delta^2}.
\]

5 Concluding Remarks

We conclude this paper by addressing the situation when the multiplexer is fed by \( N \) independent \( M/G/\infty \) input processes, with arrival rate \( \lambda_i \) and c.d.f. of the service times \( G^i \) for the system \( i \) (\( i = 1, 2, \ldots, N \)). Because the arrivals are Poisson this is equivalent to considering a single \( M/G/\infty \) queueing system with arrival intensity \( \lambda := \sum_{i=1}^{N} \lambda_i \) and c.d.f. \( G \) of the service time given by
\[
G(x) = \sum_{i=1}^{N} (\lambda_i/\lambda) G^i(x).
\]
All of the results in the paper therefore apply to this pair \((\lambda, G)\). Of particular interest is the case when one c.d.f. of the service times, say \( G^1 \), has a heavier tail than the others, namely, \( G^i(x) = o(G^1(x)) \) for all \( i = 2, 3, \ldots, N \). Then, \( \frac{G_1(x)}{G^1(x)} \sim (\lambda_1/\lambda) \), and we may conclude from the results in Sections 3-4 that the source with the heaviest tail dominates the other sources. In particular, we see from (11) and (37) that
\[
-\theta_1 \leq \liminf_{x \to \infty} \frac{\log P(Q > x)}{-\log G^1_1(x)} \leq \limsup_{x \to \infty} \frac{\log P(Q > x)}{-\log G^1_1(x)} \leq -1
\]
where the upper bound holds if \( G^1, G^1_1 \in \mathcal{S} \), with \( \theta_1 := \inf_{\beta > 0} \left\{ h(\beta) \limsup_{x \to \infty} \frac{\log G^1_1(x)}{\log G^1_1(\beta x)} \right\} \),
\[
h(\beta) := |c - \rho + \beta| + 1 \text{ and } \rho = \sum_{i=1}^{N} (\lambda_i/\lambda) \int_0^{\infty} x G^i(dx).
\]

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A Appendix

Lemma A.1 Let \( X \) and \( U \) be independent r.v.'s. We assume that \( U \) is uniformly distributed over \((0,1)\) and \( X \) is a nonnegative r.v. Then,
\[
E \left[ \lfloor X - U \rfloor \right] = E[X].
\]

Proof. Since \( \lfloor X - U \rfloor \) is a nonnegative integer, we have
\[
E \left[ \lfloor X - U \rfloor \right] = \sum_{n \geq 0} P(\lfloor X - U \rfloor > n)
\]

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\begin{align*}
&\sum_{n\geq 0} P(X - U > n) = \sum_{n\geq 0} \int_{0}^{1} P(X > n + u) \, du \\
&= \sum_{n\geq 0} \int_{n}^{n+1} P(X > u) \, du = \int_{0}^{\infty} P(X > u) \, du = E[X].
\end{align*}

References


