Relaying in Mobile Ad Hoc Networks: The Brownian Motion Mobility Model

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Abstract. Mobile ad hoc networks are characterized by a lack of a fixed infrastructure and by node mobility. In these networks data transfer can be improved by using mobile nodes as relay nodes. As a result, transmission power and the movement pattern of the nodes have a key impact on the performance. In this work we focus on the impact of node mobility through the analysis of a simple one-dimensional ad hoc network topology. Nodes move in adjacent segments with reflecting boundaries according to Brownian motions. Communications (or relays) between nodes can occur only when they are within transmission range of each other. We determine the expected time to relay a message and compute the probability density function of relaying locations. We also provide an approximation formula for the expected relay time between any pair of mobiles.

1 Introduction

Mobile ad hoc networks can be deployed when a fixed network structure is not available. As a consequence of the absence of a fixed infrastructure, the mobile components (or nodes) of an ad hoc network need to behave as routers by relaying messages in order to improve the performance of a network [7]. Instances of nodes in ad hoc networks are laptops, planes [12], cars, electronic tags on animals [11], mobile phones, et cetera. This has led to the design of protocols that take advantage of the node mobility (e.g. messaging applications in [8]).

Data relaying cuts down transmission power, interferences and increases battery usage. On the other hand, it may increase latency—since the existence at any time of a "path" between two mobiles is not guaranteed—even if (intermediary) nodes can be used as routers to convey a message from its source to its destination.

In this paper we study the impact of mobility on the latency in the case of nodes acting as relay nodes. This is done for one-dimensional ad hoc network topologies and under the assumption that nodes move according to (independent) Brownian motions.

A natural approach (but not the only one, see [10] for another approach) to modeling a mobile ad hoc network with relaying nodes consists of looking down at the earth and representing it as a finite two-dimensional plane. If two mobiles are within a fixed transmission range of each other then a message can be relayed/transmitted (see Figure 1). Furthermore, mobiles move according to a certain movement pattern. Unfortunately, this simple model of an ad hoc network (no physical restrictions in the area covered by the nodes, nodes are homogeneous, etc.) is extremely difficult to analyze, even with simple movement patterns such as, for example, the Random Waypoint Mobility (RWM) model [14]. For instance, finding the stationary distribution of the location of the mobiles under the RWM is, to the best of our knowledge, an open problem.

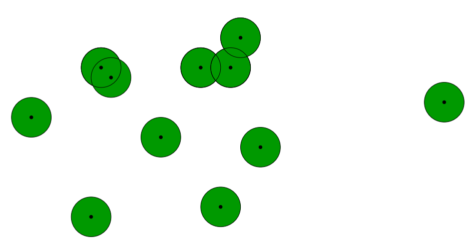

Figure 1. Graphical representation of an ad hoc network. Nodes can transfer a message only if they are within each others transmission range. In this case only two nodes can communicate with each other.

Obtaining any results characterizing the first instance of time when two mobiles come within transmission range of each other is a problem of even greater complexity. For this reason, this paper focuses on a one-dimensional topology—a model that already reveals interesting properties. Its extension to two dimensions is an open problem.

When analyzing a mobile ad hoc network, an important consideration is the movement pattern. Are mobiles restricted in their movement by roads, physical objects, waterways, or mountains? Do they roam around a central point? It has been shown that the latter is the case for the RWM, where there is a higher concentration of mobiles around a central region [2].

The first version of this paper was presented in [6]. However, after its publication mistakes have been found which lead to the derivation of formulas which were a constant ($\sqrt{2}$) off from the correct result. This paper contains the correct expressions.

The following scenarios are addressed in this paper. In Section 2 we consider the situation where two mobiles move along a segment with reflecting boundaries (see Figure 2). Both mobiles move along the segment according to independent Brownian motions. We are interested in computing the expected time until both mobiles come within communication range of each other. This quantity is computed for any given initial locations (Proposition 1) as well as for the case where each Brownian motion is initially in steady-state (Proposition 3). It is known (see Section 3) that the latter assumption implies that both mobiles are uniformly distributed over the segment. The uniform spatial distribution over the coverage area has attracted attention lately and several fundamental results [1, 7] have been obtained in this setting. However, our model is different from the models considered in those papers.

In Section 3, we consider I mobiles and I segments, one mobile per segment, as depicted in Figure 5. The mobiles move along their respective segment (with reflecting boundaries) according to independent Brownian motions. The goal is to determine the expected transfer time between the first and last mobile in the sequence (Proposition 5). As an additional result, we identify the probability density function (pdf) of the position of a mobile at a relay epoch (Proposition 4). Numerical results are reported in Section 4. These results suggest an accurate and scalable approximation for the expected transfer time (see equation (15)). The possible extensions of the model are discussed in Section 5.

2 Two mobiles moving along a line segment

We consider two mobiles (say mobiles X and Y) moving along segment $[0, L]$. See Figure 2. Communications between these two mobiles occur only when the distance between them is less than or equal to $r \leq L$. The objective of this section is to determine the expected *transfer time*, defined as the first time when both mobiles come with a distance r of each other.

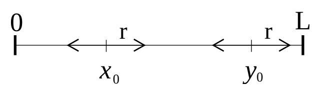

Figure 2. Two mobiles moving along $[0, L]$ with transmission range r.

Let $x(t)$ and $y(t)$ be the position of mobiles X and Y, respectively, at time t. We assume that $\mathbf{X} = \{x(t), t \geq 0\}$ and $\mathbf{Y} = \{y(t), t \geq 0\}$ are identical and independent Brownian motions with drift 0 and diffusion coefficient[1] D, both moving along the segment $[0, L]$ with *reflecting* boundaries at the edges. Let $T_{L,r}$ be the transfer time, namely,

$$T_{L,r} = \inf\{t \geq 0 \,:\, |y(t) - x(t)| \leq r\}. \tag{1}$$

Set $x(0) = x_0$ and $y(0) = y_0$. By convention we assume that $T_{L,r} = 0$ if $|y_0 - x_0| \leq r$. From now on we assume that $|y_0 - x_0| > r$.

We are interested in

$$T_{L,r}(x_0, y_0) := \mathbb{E}[T_{L,r} \,|\, x(0) = x_0, y(0) = y_0], \quad 0 < x_0, y_0 < L,$$

the expected transfer time given that mobiles X and Y are located at position x_0 and y_0, respectively, at time $t = 0$. The following result holds:

[1] i.e $x(t+h) - x(t)$ (respectively $y(t+h) - y(t)$) is normally distributed with mean 0 and variance $2Dh$ for all $h > 0$, and non-overlapping time intervals are independent of each other.

Proposition 1 (Expected transfer time with given initial positions).

For $0 \le x_0 < y_0 \le L$ *with* $x_0 + r < y_0$ *and* $0 \le r \le L$

$$T_{L,r}(x_0, y_0) = \frac{32(L-r)^2}{D\pi^4} \sum_{\substack{m\geq 1\\ m \text{ odd}}}^{\infty} \sum_{\substack{n\geq 1\\ n \text{ odd}}}^{\infty} \frac{\sin\left(\frac{m\pi(y_0+x_0-r)}{2(L-r)}\right)\sin\left(\frac{n\pi(y_0-x_0-r)}{2(L-r)}\right)}{mn(m^2+n^2)}. \tag{2}$$

◇

The proof of Proposition 1 is based on the following intermediary result that gives the expected time for a two-dimensional Brownian motion $\mathbf{Z}$ evolving in a R by R square to hit any boundary of the square.

Proposition 2 (Two Brownian motions in a square).

Consider two independent and identical one-dimensional Brownian motions $\{u(t), t \geq 0\}$ *and* $\{v(t), t \geq 0\}$, *with zero drift and diffusion coefficient* D. *Define the two-dimensional Brownian motion* $\mathbf{Z} = \{z(t) = (u(t), v(t)), t \geq 0\}$. *Set* $u_0 = u(0)$ *and* $v_0 := v(0)$ *and assume that* $0 \le u_0 \le R$ *and* $0 \le v_0 \le R$.

Let

$$\tau_R := \inf\{t \geq 0 \,:\, u(t) \in \{0, R\} \text{ or } v(t) \in \{0, R\}\}$$

be the first time when the process $\mathbf{Z}$ *hits the boundary of a square of size* R *by* R.

Define $\tau_R(u_0, u_0) = \mathbb{E}[\tau_R \,|\, z(0) = (u_0, v_0)]$. *Then,*

$$\tau_R(u_0, v_0) = \frac{16R^2}{D\pi^4} \sum_{\substack{m\geq 1\\ m \text{ odd}}}^{\infty} \sum_{\substack{n\geq 1\\ n \text{ odd}}}^{\infty} \frac{\sin\left(\frac{m\pi u_0}{R}\right)\sin\left(\frac{n\pi v_0}{R}\right)}{mn(m^2+n^2)}. \tag{3}$$

◇

The proof of Proposition 2 is given in Appendix A. We are now in a position to prove Proposition 1.

Proof of Proposition 1.

Let $x_0 + r < y_0 \le L$. An equivalent way to view the Brownian motions $\mathbf{X}$ and $\mathbf{Y}$ at time $t = 0$ is to consider that the point (x_0, y_0) is located in the upper triangle in

Figure 3 delimited by the lines $x = 0$, $y = L$ and $y = x + r$. If we assume that the boundaries $x = 0$ and $y = L$ are reflecting boundaries in Figure 3, then we see that $T_{L,r}(x_0, y_0)$ is nothing but the expected time needed for the two-dimensional Brownian motion $\{(x(t), y(t)), t \geq 0\}$ to hit the diagonal of the triangle (i.e. to hit the line $y = x+r$) given that $(x(0), y(0)) = (x_0, y_0)$. (The process $\{(x(t), y(t)), t \geq 0\}$ is a two-dimensional Brownian motion since $\{x(t),\, t \geq 0\}$ and $\{y(t), t \geq 0\}$ are both independent Brownian motions.)

By using the classical method of images (see e.g. [9, p. 81]), it can be seen that this time is itself identical to the expected time needed to hit the boundary of the square of size $\sqrt{2}(L - r)$ by $\sqrt{2}(L - r)$ shown in Figure 4 given that $(x(0), y(0)) = (x_0, y_0)$. This is due to the reflecting boundaries at $x = 0$ and $y = L$ acting as mirrors.

In order to apply the result in Proposition 2, we need to compute the coordinates (x'_0, y'_0) of (x_0, y_0) in a new system of coordinates (x', y') depicted in Figure 4 and which is rotated 45^o from the original coordinate system. We find $(x'_0, y'_0) = ((y_0 + x_0 - r)/\sqrt{2}$ and $(y_0 - x_0 - r)/\sqrt{2})$ and we may conclude, from Proposition 2, that

$$T_{L,r}(x_0, y_0) = \tau_{\sqrt{2}(L-r)}\left((y_0 + x_0 - r)/\sqrt{2}, (y_0 - x_0 - r)/\sqrt{2}\right). \tag{4}$$

By using (3) in the right hand side of (4) we see that (2) holds. ∎

An example of the expected transfer time $T_{L,r}(x_0, y_0)$ is displayed in Figure 6 (see Section 4 for comments).

We conclude this section by giving the expected transfer time when both mobiles are uniformly distributed over the segment $[0, L]$ at time $t = 0$. We will see in the next section that this case corresponds to the situation where both Brownian motions $\mathbf{X}$ and $\mathbf{Y}$ are in steady-state at time $t = 0$.

Proposition 3 (Expected transfer time for uniform initial positions).

Assume that both mobiles X and Y are uniformly distributed over $[0, L]$ at time $t = 0$ and $0 \leq r \leq L$. The expected transfer time $\mathbb{E}[T_{L,r}]$ is

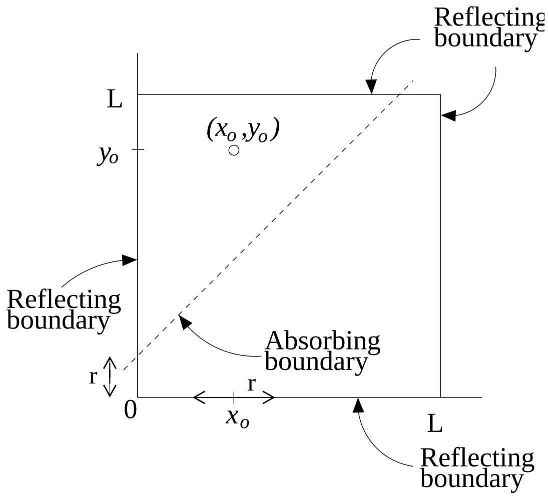

Figure 3. When mobiles X and Y are at a distance r of each other they are located on the line $y = x + r$ $(y_0 > x_0 + r)$.

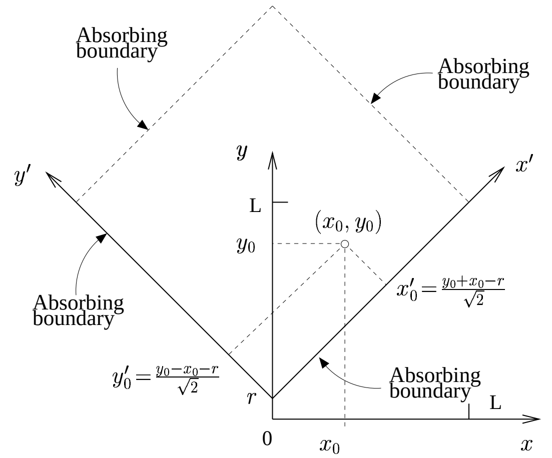

Figure 4. Since reflecting barriers at $x = 0$ and $y = L$ act as mirrors, the method of images turns the problem into a 2D Brownian motion inside four absorbing barriers.

$$\mathbb{E}[T_{L,r}] = \frac{128(L-r)^4}{D\pi^6 L^2} C_0, \tag{5}$$

where C_0 is a constant given by $C_0 = \sum_{\substack{m=1 \\ m \ odd}}^{\infty} \sum_{\substack{n=1 \\ n \ odd}}^{\infty} \frac{1}{m^2 n^2 (m^2+n^2)} \approx 0.52792664.$ ⋄

Proof. Since $\mathbf{X}$ and $\mathbf{Y}$ are uniformly distributed at $t = 0$, we have

$$\begin{aligned}
\mathbb{E}[T_{L,r}] &= \frac{1}{L^2} \int_0^L \int_0^L \mathbb{E}[T_{L,r} \mid x(0) = x_0, y(0) = y_0] dx_0 dy_0 \\
&= \frac{1}{L^2} \int_{x_0+r<y_0 \le L} T_{L,r}(x_0, y_0)\, dx_0 dy_0 + \frac{1}{L^2} \int_{y_0+r<x_0 \le L} T_{L,r}(y_0, x_0)\, dx_0 dy_0 \\
&= \frac{2}{L^2} \int_{x_0+r<y_0 \le L} T_{L,r}(y_0, x_0)\, dx_0 dy_0 \\
&= \frac{64(L-r)^2}{D\pi^4 L^2} \int_{x_0+r<y_0 \le L} h(y_0+x_0-r, y_0-x_0-r) dx_0\, dy_0.
\end{aligned}$$

where

$$h(u,v) := \sum_{\substack{m \ge 1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n \ge 1 \\ n \text{ odd}}}^{\infty} \frac{\sin(mu\beta)\sin(nv\beta)}{mn(m^2+n^2)}, \qquad \beta := \frac{\pi}{\sqrt{2}(L-r)}.$$

Define the new variables $u = (y_0 + x_0 - r)/\sqrt{2}$ and $v = (y_0 - x_0 - r)/\sqrt{2}$. We find

$$\mathbb{E}[T_{L,r}] = \frac{64(L-r)^2}{D\pi^4 L^2}\left[\int_{u=0}^{\frac{L-r}{\sqrt{2}}}\int_{v=0}^{u} h(u,v)|J(u,v)|\,dv\,du + \int_{u=\frac{L-r}{\sqrt{2}}}^{\sqrt{2}(L-r)}\int_{v=0}^{\sqrt{2}(L-r)-u} h(u,v)|J(u,v)|\,dv\,du\right] \tag{6}$$

where $|J(u,v)|$ (=1) is the determinant of the Jacobian matrix

$$J(u,v) = \begin{pmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \\ \frac{dy}{du} & \frac{dy}{dv} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

It remains to evaluate the two double integrals in equation (6). By making use of the identity $h(u,v) = h(\sqrt{2}(L-r)-u, v)$ we see that both integrals in the right hand side of (6) are equal, since

$$\int_{u=\frac{L-r}{\sqrt{2}}}^{\sqrt{2}(L-r)}\int_{v=0}^{\sqrt{2}(L-r)-u} h(u,v)dvdu = \int_{u=\frac{L-r}{\sqrt{2}}}^{\sqrt{2}(L-r)}\int_{v=0}^{\sqrt{2}(L-r)-u} h(\sqrt{2}(L-r)-u,v)dvdu = \int_{u=0}^{\frac{L-r}{\sqrt{2}}}\int_{v=0}^{u} h(u,v)dvdu.$$

The first integral can be evaluated by using the symmetry $h(u,v) = h(v,u)$. This gives

$$\begin{aligned}\int_{u=0}^{\frac{L-r}{\sqrt{2}}}\int_{v=0}^{u} h(u,v)dvdu &= \int_{u=0}^{\frac{L-r}{\sqrt{2}}}\int_{v=0}^{u} h(v,u)dvdu = \int_{v=0}^{\frac{L-r}{\sqrt{2}}}\int_{u=v}^{\frac{L-r}{\sqrt{2}}} h(v,u)dudv \\ &= \int_{u=0}^{\frac{L-r}{\sqrt{2}}}\int_{v=u}^{\frac{L-r}{\sqrt{2}}} h(u,v)dvdu.\end{aligned}$$

Hence,

$$\int_{u=0}^{\frac{L-r}{\sqrt{2}}}\int_{v=0}^{u} h(u,v)dvdu = \frac{1}{2}\int_{u=0}^{\frac{L-r}{\sqrt{2}}}\int_{v=0}^{\frac{L-r}{\sqrt{2}}} h(u,v)dvdu$$

so that

$$\mathbb{E}[T_{L,r}] = \frac{64(L-r)^2}{D\pi^4 L^2}\int_{0}^{\frac{L-r}{\sqrt{2}}}\int_{0}^{\frac{L-r}{\sqrt{2}}} h(u,v)dvdu. \tag{7}$$

Since the double series in $h(u, v)$ are uniformly bounded in the variables $u, v \in [0, \sqrt{2}(L-r)]$ (its absolute value is bounded from above by $(\sum_{k\geq 1} 1/k^2)^2 = \pi^4/36$), we may invoke the bounded convergence theorem to interchange the integral and summation signs in (7). This gives

$$\begin{aligned}\mathbb{E}[T_{L,r}] =& \frac{64(L-r)^2}{D\pi^4 L^2} \sum_{\substack{m\geq 1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n\geq 1 \\ n \text{ odd}}}^{\infty} \frac{1}{mn(m^2+n^2)} \int_{u=0}^{\frac{L-r}{\sqrt{2}}} \sin(mu\beta)du \int_{v=0}^{\frac{L-r}{\sqrt{2}}} \sin(nv\beta)dv \\ =& \frac{128(L-r)^4}{D\pi^6 L^2} \sum_{\substack{m\geq 1 \\ m \text{ odd}}}^{\infty} \sum_{\substack{n\geq 1 \\ n \text{ odd}}}^{\infty} \frac{1}{m^2 n^2(m^2+n^2)}.\end{aligned}$$

The last line follows because $cos(j\pi/2) = 0$ for j odd. ∎

3 A chain of relaying mobiles

We consider the situation depicted in Figure 5. There are I adjacent segments, each of length L, and there is a single mobile per segment. We denote by X_i the mobile in segment i. Let $0 \leq x_i(t) \leq L$ $(i = 1, \ldots, I)$ be the *relative* position of the i-th mobile in its segment. We assume that the process $\mathbf{X}_i = \{x_i(t),\, t \geq 0\}$ is a Brownian motion with zero drift and diffusion coefficient D and that $\mathbf{X}_1, \ldots, \mathbf{X}_I$ are mutually independent processes. Last, we assume that each segment has reflecting boundaries at the ends. Let

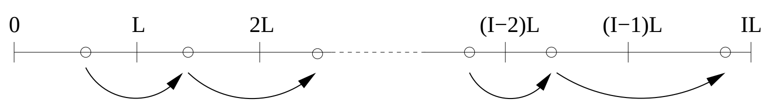

Figure 5. A chain of relaying mobiles.

$T_1 = \inf\{t \geq 0 \,:\, x_1(t) + r \geq L + x_2(t)\}$ be the transfer time between mobiles X_1 and X_2, that is T_1 is the first time when X_1 and X_2 are located at a distance less than or equal to r from each other. The relay times $T_2 \leq \cdots \leq T_{I-1}$ between mobiles X_2 and X_3, ..., X_{I-1} and X_I, respectively, are recursively defined by

$$T_i = \inf\{t \geq T_{i-1} \,:\, x_i(t) + r \geq L + x_{i+1}(t)\}, \quad i = 2, \ldots, I-1.$$

Our objective in this section is to compute $\mathbb{E}[T_i]$ for $i = 1, \ldots, I-1$.

Throughout this section we assume that $L \leq r \leq 2L$. This assumption is made for the sake of mathematical tractability. Indeed, a few seconds of reflection will convince the reader that when[2] $L \leq r \leq 2L$ and $(x(0), y(0)) = (x_0, y_0)$ the transfer time needed to transfer a message between two adjacent segments is the same as $T_{2L,r}(x_0, y_0 + L)$, the expected transfer time obtained in Section 2 for a segment of length $2L$ (with the given initial conditions). This observation allows us to find at once the expected transfer time between mobiles X_1 and X_2 for any initial conditions $x_1(0)$ and $x_2(0)$. We find

$$\mathbb{E}[T_1 \,|\, x_1(0) = x, x_2(0) = y] = \mathbb{E}[T_{2L,r}(x, y + L)]. \tag{8}$$

The difficulty arises when trying to find the expected transfer time between mobiles X_i and X_{i+1} for $i = 2, \ldots, I-1$, since the position of X_i when the transfer between X_{i-1} and X_i takes place is not uniform in $[iL, (i+1)L]$.

To overcome this difficulty, we assume that the Brownian motions $\mathbf{X}_1, \ldots, \mathbf{X}_I$ are all in steady-state at time $t = 0$. This assumption implies,[3] in particular, that the position of each mobile at time $t = 0$ is uniformly distributed over its segment (i.e. the pdf of $x_i(0)$ is uniform over $[0, L]$). The same holds of course at any arbitrary time (i.e. the pdf of $x_i(t)$ is uniform over $[0, L]$ if t is arbitrary).

Another consequence of this assumption is that the position of mobile X_{i+1} at time T_{i-1} (i.e. when X_i receives a message from X_{i-1}) is still uniformly distributed over $[0, L]$. This property will be used later on.

Proposition 4 below addresses the location of a mobile at the time when a relay occurs. For later reference, we state the result in a general form. Consider two adjacent segment, each of length L, with a single mobile in each segment (mobile X in the first segment

[2] When $L \leq r \leq 2L$ the reflecting boundaries conditions are at $x(t) = 0$ and at $y(t) = 2L$. For $0 \leq r \leq L$ there are two additional reflecting boundary conditions at $x(t) = L$ and at $y(t) = L$ which lead to a much more difficult function which we were not able to solve.

[3] Hint: let $p(x)$ be the stationary density probability that the mobile is in position $x \in [0, L]$. Solving the diffusion equation $D\partial^2 p(x)/dx^2 = 0$ with the reflecting conditions $dp(x)/dx = 0$ for $x \in \{0, L\}$ and the normalizing condition $\int_0^L p(x)dx = 1$ yields $p(x) = 1/L$ for $x \in [0, L]$ – see e.g. [3, p. 223].

and Y in the second segment). Both mobiles move in their segment (with reflecting boundaries) according to independent and identical Brownian motions with zero drift and coefficient diffusion D. We assume that the Brownian motion representing the movement of Y is in steady state at time $t = 0$. As usual, a relay will occur the first time when both mobiles come within a distance r of each other, with $L \leq r \leq 2L$.

Proposition 4 (Pdf of location at relay epoch).

Fix $L \leq r \leq 2L$. Let $q(y; x)$, $y \in [0, L]$, be the pdf of the (relative) position of mobile Y at the relay epoch, given that at time $t = 0$ the mobile X is at position x and the position of mobile Y is uniform.

We have

$$q(y; x) = \frac{\mathbf{1}_{\{y \leq x + r - L\}} + f(x, y)\, \mathbf{1}_{\{y \geq r - L,\ x < 2L - r\}}}{L}, \tag{9}$$

where

$$f(x, y) = \frac{4}{\pi^2} \sum_{m \geq 1}^{\infty} \sum_{\substack{n \geq 1 \\ n \neq m}}^{\infty} \frac{n\,(a_{m,n} + b_{m,n} + c_{m,n})}{m^2 + n^2} \sin\left(\frac{m\pi(y - r + L)}{2L - r}\right)$$
$$+ \frac{2}{\pi(2L - r)} \sum_{m \geq 1}^{\infty} \frac{d_m + e_m}{m} \sin\left(\frac{m\pi(y - r + L)}{2L - r}\right),$$

and

$$a_{m,n} = \frac{2m \sin(n\theta) - 2n \sin(m\theta)}{m^2 - n^2}, \quad b_{m,n} = \frac{\sin\Big((m - n)\pi + n\theta\Big) + \sin\Big((m - n)\pi - m\theta\Big)}{m - n}$$
$$c_{m,n} = -\frac{\sin\Big((m + n)\pi - n\theta\Big) + \sin\Big((m + n)\pi - m\theta\Big)}{m + n}, \quad d_m = 2(2L - r - x)\cos(m\theta)$$
$$e_m = \frac{2L - r}{m\pi}\left(\sin(m\theta) - \sin\Big(2m\pi - m\theta\Big)\right), \qquad \theta = \frac{\pi x}{2L - r}. \qquad \diamond$$

The proof of Proposition 4 is given in Appendix B. We are now in a position to compute the expected transfer times $\mathbb{E}[T_i]$ for $i = 1, \ldots, I - 1$.

Define $f_i(x)$ ($0 \leq x \leq L$) as the pdf of $x_i(T_{i-1})$ for $i = 1, \ldots, I - 1$ (that is, $P(x_i(T_{i-1}) < y) = \int_0^y f_i(x)\, dx$). Note that $f_1(x) = 1/L$ for $x \in [0, L]$ thanks to the assumption that mobile X_1 is in steady-state at time $t = 0$ (recall that $T_0 = 0$ by convention). Let us first compute $\mathbb{E}[T_1]$. We find

$$\begin{aligned}\mathbb{E}[T_1] &= \frac{1}{L^2}\int_0^L\int_0^L \mathbb{E}\left[T_1 \,|\, x_1(0)=x, x_2(0)=y\right]\, dx\, dy \\ &= \frac{1}{L^2}\int_{\{x+r<y+L\}} T_{2L,r}(x,y+L)\, dx\, dy, \end{aligned} \tag{10}$$

by using (8) and $T_{2L,r}(x,y+L)=0$ if $x+r\le y+L$. Similar to the derivation of (5) we get

$$\mathbb{E}[T_1] = \frac{64\,(2L-r)^4}{D\pi^6 L^2}\, C_0. \tag{11}$$

We now compute $\mathbb{E}[T_i]$ for $i=2,\ldots,I-1$. We have

$$\mathbb{E}[T_i] = \mathbb{E}[T_{i-1}] + \frac{1}{L}\int_0^L\int_0^L \mathbb{E}\left[T_i - T_{i-1}\,|x_i(T_{i-1})=x, x_{i+1}(T_{i-1})=y\right] f_i(x)\, dx\, dy \tag{12}$$

$$= \mathbb{E}[T_{i-1}] + \frac{1}{L}\int_{\{x+r<y+L\}} T_{2L,r}(x,y+L)\, f_i(x)\, dx\, dy, \tag{13}$$

where we have used equation (8) to derive (13). To derive (12) we have used the fact that the position of mobile X_{i+1} is uniformly distributed over its segment at time T_{i-1} (i.e. when the relay between mobiles X_{i-1} and X_i occurs), and that it is independent of the position of mobile X_{i-1} at time T_{i-1}. It remains to evaluate the functions $f_i(x)$ for $i=2,\ldots,I-1$. Differentiating in y on both sides of the identity

$$P(x_i(T_{i-1})<y) = \int_0^L P(x_i(T_{i-1})<y\,|\,x_{i-1}(T_{i-2})=x)\, f_{i-1}(x)\, dx,$$

and then using Proposition 9, gives

$$f_i(y) = \int_0^L q(y;x)\, f_{i-1}(x)\, dx, \quad 0\le y\le L, \tag{14}$$

for $i=2,\ldots,I-1$. These results are summarized in the next proposition.

Proposition 5 (Expected transfer times).

The expected transfer times $\mathbb{E}[T_i]$ *for* $i = 1, \ldots, I-1$, *are given by equations (11) and (13), where the functions* $f_i(x)$, $i = 2, \ldots, I-1$, *satisfy the recursion (14) with* $f_1(x) = 1/L$. *In particular,*

$$\mathbb{E}[T_1] = \frac{64\,(2L-r)^4}{D\pi^6 L^2}\,C_0.$$

◇

4 Numerical results and discussion

The expected transfer time $T_{L,r}(x_0, y_0)$ is displayed in Figure 6 as a function of the initial position x_0 and y_0 of the mobiles, for $L = 30$, $r = 5$ and $D = 1/4$ (recall that D is the diffusion coefficient of the Brownian motions $\mathbf{X}$ and $\mathbf{Y}$). The figure shows that the expected transfer time grows (roughly) linearly as the initial distance between both mobiles increases and neither of the mobiles is near the boundaries of the interval $[0, L]$. We used equation (14) to determine the mapping $x \to f_2(x)$ for $0 \leq x \leq L$, the pdf of the

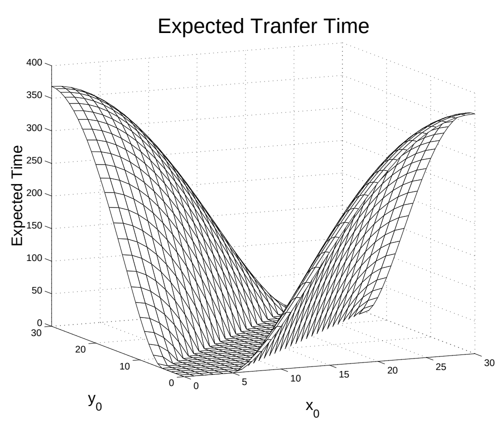

Figure 6. The mapping $(x_0, y_0) \to T_{L,r}(x_0, y_0)$ (expected transfer time between mobiles X and Y starting from x_0 and y_0, respectively, at $t = 0$. See equation (2)) for $L = 30$, $r = 5$, $D = 1/4$.

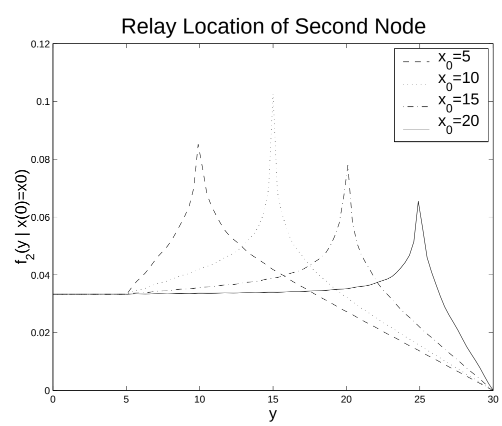

Figure 7. Mapping $x \to f_2(x)$ (pdf of location of mobile X_2 at the relay epoch) for when mobile X_1 is at position $x_0 \in \{5, 10, 15, 20\}$ at time $t = 0$, for $D = 1/4$, $L = 30$, $r = 35$.

location of mobile X_2 when the relay with X_1 occurs. This mapping is plotted in Figure 7 for different values of the starting position of mobile X_1 ($x_1(0) = 5, 10, 15, 20$) and for

$L = 30$, $r = 35$, $D = 1/4$. It is interesting to observe that $f_2(x)$ is uniform in $[0, x_1(0)]$. This is easily explained by the fact that if X_2 is located in $[0, r - L]$ at time T_1 then it was necessarily located in this interval prior to time T_1, since otherwise the relay would have occurred before T_1. Each peak corresponds to the most likely value y in $[0, L]$ where mobile X_2 will be located at time T_1. This value is given by $y = x_1(0) + r$.

Figure 8 displays mappings $x \to f_i(x)$ for $i \in \{2, 3, 100\}$ (evaluated from (14) with uniformly distributed initial locations). It is worth observing that these functions converge very rapidly (already $f_3(x)$ and $f_{100}(x)$ are extremely close to each other).

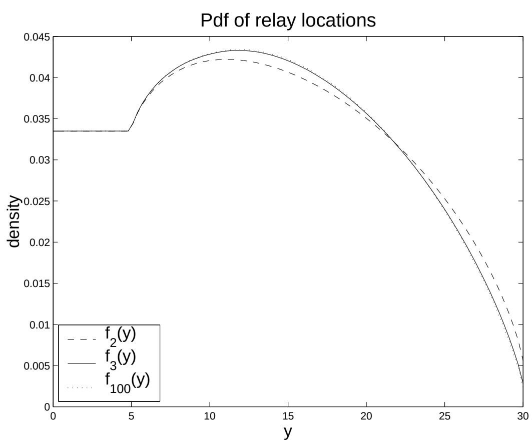

Figure 8. Mappings $x \to f_i(x)$ for $i \in \{2, 3, 100\}$ (pdf of starting location of mobiles) for $L = 30, r = 35$, and $D = 1/4$.

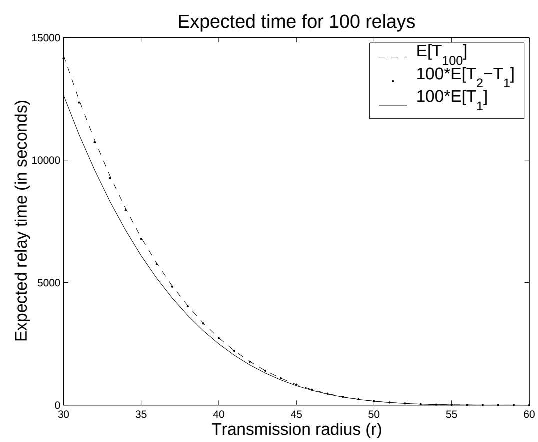

Figure 9. Comparison of mappings $r \to \mathbb{E}[T_{100}]$, $r \to 100 \times \mathbb{E}[T_2 - T_1]$, $r \to 100 \times \mathbb{E}[T_1]$ for $L = 30$ and $D = 1/4$.

Figure 9 displays mappings $r \to \mathbb{E}[T_{100}]$, $r \to 100 \times \mathbb{E}[T_2 - T_1]$ and $r \to 100 \times \mathbb{E}[T_1]$. This figure carries two important messages. First, it shows for different values of the transmission range r, that the approximation $\mathbb{E}[T_{100}] \sim 100 \times \mathbb{E}[T_2 - T_1]$ is very close to the exact result $\mathbb{E}[T_{100}]$ (derived from Proposition 5)), thereby suggesting the approximation

$$\mathbb{E}[T_i] \sim i \times \mathbb{E}[T_2 - T_1] \tag{15}$$

for the expected time to relay a message from mobile X_1 to mobile X_{i+1}. This approximation is based on the fact that the relay location convergences extremely rapidly and, with the exception of the first relay, the relay locations, and therefore also the relay times, of the consecutive relays are very similar. We have indeed checked that equations (15) is accurate for small values of i as well as for large values (i.e. larger than 100). Second, it shows that the approximation $\mathbb{E}[T_{100}] \sim 100 \times \mathbb{E}[T_1]$ may not be accurate for small transmission ranges, thereby ruling out the approximation $\mathbb{E}[T_i] \sim i \times \mathbb{E}[T_1]$. This is so because the latter approximation does not account for the fact that mobile X_i does not start from a "uniform location" at time T_{i-1} (as opposed to mobile X_1 whose position is uniformly distributed over $[0, L]$ at time $t = 0$).

5 Extensions to the model

In this paper the message delay over a one dimensional network was analysed for mobiles which move as Brownian motions in adjacent segments. The extension of this theory to more than two mobiles per segment, or with leakage from one domain to the next, does not seem to be mathematically tractable. The reason for this is the following. If there are N mobiles in a segment then their positions needs to be mapped to a single N-dimensional Brownian motion. So far no problem. The model starts becoming more complex though when one has to take into account the positions of the N-dimensional Brownian motion which correspond to two nodes being in each others transmission range. For two one-dimensional Brownian motions this resulted in a diagonal line (see Figure 3). For three nodes this leads to three intersecting planes in a three-dimensional space. On top of this one has to take into account the (reflecting) border conditions for each of the mobiles. Although writing down these conditions is still feasible, the "real" problem lies in keeping track of which nodes have or have not received a copy of the message, the correlation between these nodes their positions, and finding an expression for the function which corresponds to the expected transfer time and which satisfies all of the necessary conditions.

A similar problem arises for N two-dimensional Brownian motions. In this case the positions of the mobiles can be mapped to a single $2N$-dimensional Brownian motion. Since the area in which two nodes can communicate is given by a circle, it means that with the method of images the corresponding boundary conditions are no longer in the simple form of a square. Once again, finding a function which corresponds to the expected message delay, while keeping track of which mobile have a copy of the message, does not seem to be feasible.

As a side remark it is worth mentioning that the process which models the distance between two two-dimensional Brownian motions in free space (i.e. with no boundaries) is known as a *Bessel process*. For this process various results exist, and, just as for two Brownian motions on an infinite line, the expected time until two Brownian motions in free space come within each others range is infinite. In this paper a bounded region was assumed to ensure a finite transfer time.

6 Concluding remarks

Besides message delay, the question of power control is also central in ad hoc networking. Ongoing research is concerned with determining the minimum transmission range that will ensure communication between mobiles (within a certain probability) before the battery power runs out, and with introducing utility functions into the model.

With a certain amount of overlap to this work, in a forthcoming paper the situation is considered where instead of mobiles moving as Brownian motions they move as Random walkers over a discrete state space. This model also corresponds to messages being passed around a sensor network. The results thus obtained have been verified through simulations and are in correspondence to the results presented in this paper (in the limit for an infinite number of states).

As soon as two nodes come within each others communication range there is the important issue of how long their *contact time* is. If these times are too short then a successfull transfer of a message can not be guaranteed. This issue is discussed in detail for a number of different mobility models in [5].

As mentioned earlier, the problem with keeping track of which mobiles do or do not have a copy of the message in combination with their positions greatly increases the complexity and limits the extendibility of the theory presented in this paper. However, by assuming that $r << L$ and by working in (at least) two dimensions it has been found that the number of copies in the network can be decoupled from the positions of the copies. This greatly simplifies the analysis and has led to generic results and formulas which hold under a variety of mobility models. These results will be published in a forthcoming paper. Oddly enough, the theory developed there can not be used for the analysis of one-dimensional mobility models and hence there remains a need to study the one- and the two-dimensional settings separately.

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Appendix A - Proof of Proposition 2

The density probability $q(x, t; u_0)$ that the Brownian motion $\{u(t),\ t \geq 0\}$ is in position $x \in (0, R)$ at time t, given that $u(0) = u_0$ and that the Brownian motion has not been absorbed up to time t, is [9, p. 255, formula (8.2.1)] [13, page 177]

$$w(x, t; u_0) = \frac{2}{R} \sum_{n \geq 1} e^{-(n\pi/R)^2 Dt} \sin\left(\frac{n\pi x}{R}\right) \sin\left(\frac{n\pi u_0}{R}\right).$$

Since $\{u(t),\ t \geq 0\}$ and $\{v(t),\ t \geq 0\}$ are independent and identical Brownian motions, we deduce from the above that the density probability $p(x, y, t; u_0, v_0)$ that the two-dimensional Brownian motion $\mathbf{Z}$ is in position (x, y) at time t, without having hit one of the sides of the squares up to time t, is given by

$$p(x, y, t; u_0, v_0) = w(x, t; u_0)\, w(y, t; v_0). \quad 0 < x, y < R. \tag{16}$$

Conditioned on $z(0) = (u_0, v_0)$, the probability $S(t; u_0, v_0) = P(\tau_R > t)$ that the process has not hit the boundaries at time t (often called the survival probability [9]) is given by

$$S(t; u_0, v_0) = \int_0^R \int_0^R p(x, y, t; u_0, v_0) dx dy.$$

Therefore,

$$\begin{aligned} S(t; u_0, v_0) &= \int_0^R w(x, t; u_0) dx \int_0^R w(y, t; v_0) dy \\ &= \frac{4}{R^2} \sum_{m \geq 1} e^{-(m\pi/R)^2 Dt} \sin\left(\frac{m\pi u_0}{R}\right) \int_0^R \sin\left(\frac{m\pi x}{R}\right) dx \times \\ &\qquad \sum_{n \geq 1} e^{-(n\pi/R)^2 Dt} \sin\left(\frac{n\pi v_0}{R}\right) \int_0^R \sin\left(\frac{n\pi y}{R}\right) dy \\ &= \frac{16}{\pi^2} \sum_{\substack{m \geq 1 \\ m \text{ odd}}} \sum_{\substack{m \geq 1 \\ n \text{ odd}}} \frac{\sin\left(\frac{m\pi u_0}{R}\right) \sin\left(\frac{n\pi v_0}{R}\right)}{mn} e^{-\frac{\pi^2}{R^2}(m^2+n^2)Dt}, \end{aligned} \tag{17}$$

where the uniform convergence of the series $w(x, t; \cdot)$ in $x \in [0, \infty)$ (because $|w(x, t; \cdot)| \leq 1/(1 - \exp(-(\pi/R)^2 Dt)))$ allows one to interchange integral and summation signs in equation (17). Note that, as expected, $S(0; u_0, v_0) = 1$ since $\sum_{i\geq 1} \sin((2i-1)x)/(2i-1) = \pi/4$ for all x [4, Formula 1.442.1].

Finally,

$$
\begin{aligned}
\tau_R(u_0, v_0) &= \int_0^\infty S(t; u_0, v_0)\, dt \\
&= \frac{16}{\pi^2} \int_0^\infty \sum_{\substack{m\geq 1\\ m \text{ odd}}} \sum_{\substack{m\geq 1\\ n \text{ odd}}} \frac{\sin\left(\frac{m\pi u_0}{R}\right) \sin\left(\frac{n\pi v_0}{R}\right)}{mn}\, e^{-\frac{\pi^2}{R^2}(m^2+n^2)Dt}\, dt \qquad (18) \\
&= \frac{16}{\pi^2} \sum_{\substack{m\geq 1\\ m \text{ odd}}} \sum_{\substack{m\geq 1\\ n \text{ odd}}} \frac{\sin\left(\frac{m\pi u_0}{R}\right) \sin\left(\frac{n\pi v_0}{R}\right)}{mn} \int_0^\infty e^{-\frac{\pi^2}{R^2}(m^2+n^2)Dt} dt \qquad (19) \\
&= \frac{16R^2}{D\pi^4} \sum_{\substack{m\geq 1\\ m \text{ odd}}}^\infty \sum_{\substack{n\geq 1\\ n \text{ odd}}}^\infty \frac{\sin\left(\frac{m\pi u_0}{R}\right) \sin\left(\frac{n\pi v_0}{R}\right)}{mn(m^2+n^2)},
\end{aligned}
$$

where we have used the property that the series $S(t; \cdot, \cdot)$ is uniformly convergent in $[0, \infty)$ (since $S(t; \cdot, \cdot) \leq 1$ for all $t \geq 0$ by definition of $S(t; \cdot, \cdot)$) to interchange the summation and the integral signs in (18) to give (19). This concludes the proof. ∎

Appendix B - Proof of Proposition 4

Let $x(t)$ and $y(t)$ be the relative positions at time t of mobiles X and Y in $[0, L]$ and $[L, 2L]$, respectively. Let T the first time when $x(t) + r \geq y(t) + L$. Observe that $T = 0$ if $x(0) + r \geq y(0) + L$. We have

$$
\begin{aligned}
P(y(T) < y \mid x(0) = x_0) &= \frac{1}{L} \int_0^L P(y(T) < y \,|\, x(0) = x_0, y(0) = y_0)\, dy_0 \\
&= \frac{1}{L} \int_0^L \mathbf{1}_{\{x_0+r\geq L+y_0\}} \mathbf{1}_{\{y>y_0\}}\, dy_0 \\
&\quad + \frac{1}{L} \int_0^L \mathbf{1}_{\{x_0+r<L+y_0, y\geq r-L\}} P(y(T) < y \,|\, x(0) = x_0, y(0) = y_0)\, dy_0 \\
&= \frac{1}{L} \min(x_0 + r - L, y) \\
&\quad + \frac{1}{L} \mathbf{1}_{\{y\geq r-L, x_0<2L-r\}} \int_{x_0+r-L}^L P(y(T) < y \,|\, x(0) = x_0, y(0) = y_0)\, dy_0,
\end{aligned}
$$

where the indicator function $\mathbf{1}_{\{y\geq r-L\}}$ in the second integral in the second equality accounts for the fact that if the transfer does not take place at $t = 0$ (under the condition

$x_0 + r < y + L$ then necessarily $T > 0$) then mobile Y can not be located in $[L, L-r)$ at time T as otherwise the relay would have occurred before time T. Differentiating both sides of the above relation with regards to y gives

$$q(y; x_0) = \frac{1}{L}\mathbf{1}_{\{y \le x_0 + r - L\}} + \frac{1}{L}\mathbf{1}_{\{y \ge r - L, x_0 < 2L - r\}} \int_{x_0+r-L}^{L} g(y; x_0, y_0)\, dy_0, \tag{20}$$

with $g(y; x_0, y_0) := (\partial/\partial y) P(y(T) < y \,|\, x(0) = x_0, y(0) = y_0)$. It remains to evaluate $g(y; x_0, y_0)$. To this end, we will use again the method of images (see proof of Proposition 1).

Consider a square of size R by R, with $R = \sqrt{2}(2L - r)$, delimited by the (absorbing) boundaries $x' = 0$, $x' = R$, $y' = 0$ and $y' = R$. Starting from position (x'_0, y'_0) at time $t = 0$, the pdf $p(x', y', t; x'_0, y'_0)$ of the location of a two-dimensional Brownian motion at time t, given that the mobile has not been absorbed up to time t, is given by (see eq.(16))

$$p(x', y', t; x'_0, y'_0) = \frac{4}{R^2} \sum_{n \ge 1} \sum_{n \ge 1} e^{-(m^2+n^2)(\pi/R)^2 Dt} \times \\ \sin\left(\frac{m\pi x'}{R}\right) \sin\left(\frac{n\pi y'}{R}\right) \sin\left(\frac{m\pi x'_0}{R}\right) \sin\left(\frac{n\pi y'_0}{R}\right). \tag{21}$$

This expression will be used later on to derive the pdf of the location where the Brownian motion hits the side of the square for the first time.

Let $\xi(x', y'; x'_0, y'_0)$ $(0 \le x', y', x'_0, y'_0 \le R)$, be the pdf of the absorption occurring at point (x', y'). Since we have applied the method of images we find that $g(y; x_0, y_0)$ is the sum of four of these components. Namely, with $x' = \sqrt{2}(y + L - r)$, it is the sum of the densities of hitting the side of the square $R \times R$ at the points $(x', 0)$, $(0, x')$, $(R - x', R)$, and $(R, R - x')$. With $x'_0 = (y_0 + x_0 + L - r)/\sqrt{2}$ and $y'_0 = (y_0 - x_0 + L - r)/\sqrt{2}$ this gives

$$g(y; x_0, y_0) = \xi\Big(x', 0; x'_0, y'_0\Big) + \xi\Big(0, x'; x'_0, y'_0\Big) + \xi\Big(R - x', R; x'_0, y'_0\Big) + \xi\Big(R, R - x'; x'_0, y'_0\Big).$$

Onward calculations can be simplified slightly by making use of symmetry arguments. Continuous rotation of the square by 90^o means that each of the terms can be replaced by the density of the probability of hitting the side of the square at $(x', 0)$ while starting from, respectively, (x_0', y_0'), (y_0', x_0'), $(R - x_0', R - y_0')$, or $(R - y_0', R - x_0')$. This gives

$$
\begin{aligned}
g(y; x_0, y_0) = & \xi\left(x', 0; x_0', y_0'\right) + \xi\left(x', 0; y_0', x_0'\right) \\
& + \xi\left(x', 0; R - x_0', R - y_0'\right) + \xi\left(x', 0; R - y_0', R - x_0'\right). \qquad (22)
\end{aligned}
$$

Note that although $\xi(x', 0; \cdot, \cdot)$ no longer contains y', it still depends on y, x_0, and y_0 through $x' = \sqrt{2}(y + L - r)$, $x_0' = (y_0 + x_0 + L - r)/\sqrt{2}$, and $y_0' = (y_0 - x_0 + L - r)/\sqrt{2}$. It remains to solve $\xi\left(x', 0; x_0', y_0'\right)$ for any set of initial conditions (x_0', y_0'). We shall do this through the help of the first-passage probability of the point $(x', 0)$.

If $j(x', t)$ is the pdf of the first-passage probability of hitting the absorbing boundary of the square for the first time in the point $(x', 0)$ at time t, then naturally

$$
\xi(x', 0; x_0', y_0') = \int_0^\infty j(x', t) dt, \qquad (23)
$$

since it is the probability density of hitting the boundary for the first time in $(x', 0)$ over all time.

It is known [9, p. 25, p. 45] that $j(x', t)$ is equal to the flux going out from the point $(x', 0)$, i.e.

$$
j(x', t) = D \frac{\partial p(x', y', t; x_0', y_0')}{\partial y'}|_{y'=0},
$$

with $p(x', y', t; x_0', y_0')$ the pdf of the location of the Brownian motion at time t given by equation (21). Combining this with (23) gives

$$
\begin{aligned}
\xi(x', 0; x_0', y_0') = & D \int_0^\infty \frac{\partial p(x', y', t; x_0', y_0')}{\partial y'}|_{y'=0} \, dt \\
= & \frac{4}{R\pi} \sum_{n \geq 1} \sum_{n \geq 1} \frac{n}{m^2 + n^2} \sin\left(\frac{m\pi x'}{R}\right) \sin\left(\frac{m\pi x_0'}{R}\right) \sin\left(\frac{n\pi y_0'}{R}\right). \qquad (24)
\end{aligned}
$$

Finally, plugging (22) and (24) into (20) yields (9) after some tedious algebra. ∎