Time-Varying Exponential Stabilization of a Rigid Spacecraft with Two Control Torques

P. Morin and C. Samson

Abstract— **Rigid body models with two controls cannot be locally asymptotically stabilized by continuous feedbacks which are functions of the state only. This impossibility no longer holds when the feedback is also a function of time, and time-varying asymptotically stabilizing feedbacks have already been proposed. However, due to the smoothness of the feedbacks, the convergence rate is only polynomial. In the present note, exponential convergence is obtained by considering time-varying feedbacks which are only continuous.**

Index Terms— **Altitude stabilization, continuous feedback, homogeneous system, time-varying control.**

I. Introduction

Following an idea which may be traced back to a work by Sontag and Sussmann [17] in 1980, an article by Samson [16] in 1990 has revealed that continuous time-varying feedbacks, i.e., feedbacks which depend not only on the system's state vector but also on time, can be of interest in stabilizing many systems which cannot be stabilized by continuous pure-state feedbacks. This has been confirmed by Coron's results [3], which establish that most STLC (small-time locally controllable) systems can be stabilized by continuous time-varying feedback.

It is known that a given attitude for a rigid spacecraft with only two controls cannot be asymptotically stabilized by means of continuous-state feedbacks, as pointed out for example in [1], since Brockett's necessary condition [2] for smooth feedback stabilizability is not satisfied in this case. Nevertheless, the existence of stabilizing continuous time-varying feedbacks for this problem follows from [3] and [8], with the latter reference establishing that the system is STLC.

In [11], explicit smooth time-varying feedbacks have been derived by using center manifold theory, time-averaging, and Lyapunov techniques. Similar results have independently been announced by Walsh *et al*. in [18]. However, due to the smoothness of the control laws, the asymptotical rate of convergence to zero of the closed-loop system's solutions is only polynomial in the worst case. In the present paper, we derive time-varying continuous feedbacks which locally *exponentially* asymptotically stabilize the attitude of a rigid spacecraft. Our construction relies on the properties of homogeneous systems, combined with averaging and Lyapunov techniques. It also uses a specific *cascaded high-gain control* result established here for systems homogeneous of degree zero involving controls which are not necessarily differentiable everywhere.

Another solution, also yielding exponential stabilization, has recently been proposed by Coron and Kerai in [4]. A particularity of this solution is that it consists of switching periodically between two control laws, one of which depends on time. The resulting feedback control is continuous and time-periodic. By contrast, the solution proposed here consists of a single and simpler control expression.

The paper is organized as follows. In Section II, the equations of a rigid body, when using a set of Rodrigues parameters to represent attitude errors, are recalled, and the control objective is stated. In Section III, general properties and stabilization results associated with homogeneous systems, which are useful to establishing our main result, are recalled. The aforementioned cascaded high-gain control result, for systems which are homogeneous of degree zero, is derived in Section IV. A set of continuous time-varying control laws which locally asymptotically and exponentially stabilizes the desired attitude is proposed in Section V, with the corresponding proof of stability. Finally, simulation results are given in Section VI.

The main stabilization results described in this note have been presented at the *34th IEEE CDC* [10]. They are here complemented by simulation results and by a proposition (Proposition 4) which can be used to derive an explicit Lyapunov function for the controlled system and control gain values for which asymptotic stability is ensured.

Throughout the paper, we use the following notations.

- $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product and $|\cdot|$ denotes the associated norm.
- I_3 denotes the identity matrix in $\mathbb{R}^3$.
- A function $f\colon \mathbb{R}^n \mapsto \mathbb{R}^p$ is of class C^p (respectively, C^∞) if it has continuous partial derivatives up to order p (respectively, at any order).

II. Equations of the Rigid Body

Let us consider a frame F_0 attached to the spacecraft and whose axes correspond to the principal inertia axes of the body, and a fixed frame F_1 whose attitude is the desired one for F_0. Let us also denote ω the angular velocity vector of the frame F_0 with respect to the frame F_1, expressed in the basis of F_0, J the diagonal matrix of the principal moments of inertia $(J = \mathrm{Diag}(j_1, j_2, j_3))$, and $S(\omega)$ the matrix representation of the cross product

$$S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{1}$$

If R is the rotation matrix representing the attitude of F_1 with respect to F_0 (and whose columns vectors are the basis vectors of F_1 expressed in F_0), we get the well-known equations

$$\begin{aligned} \dot{R} &= S(\omega)R \\ J\dot{\omega} &= S(\omega)J\omega + B(\tau_1, \tau_2, 0)^T \end{aligned} \tag{2}$$

where the τ_i are the torques applied to the rigid body and B represents the directions in which these torques are applied.

We make the assumption that $B = I_3$ (i.e., that the torques are applied in the direction of principal inertia axes). However, our result can be easily extended to any location of the actuators for which the spacecraft is controllable, after an adequate change of state and control variables similar to the one proposed in [4].

System (2) is a control system with two scalar inputs τ_1 and τ_2 and state space $SO(3) \times \mathbb{R}^3$. Our objective is to find a control $(\tau_1(t, R, \omega), \tau_2(t, R, \omega))$ periodic with respect to time, which locally exponentially stabilizes the point $(I_3, 0)$ of $SO(3) \times \mathbb{R}^3$.

In order to control the body rotations, a preliminary step traditionally consists of defining a minimal set of local coordinates for the parameterization of $SO(3)$ around I_3. As in [11], we choose a set of coordinates, sometimes called Rodrigues parameters. To any rotation R of angle $\theta \in]-\pi, \pi[$ and axis, the direction of which is defined by the unit vector $\vec{u}$, we associate the following three-dimensional vector: $X = (x_1, x_2, x_3)^T = -\tan(\theta/2)u$, with u denoting the coordinates of the vector $\vec{u}$ in the frame of F_0. It is shown in [11]

Manuscript received November 2, 1995; revised July 1, 1996.

The authors are with INRIA Sophia-Antipolis, 06902 Sophia Antipolis Cédex, France.

Publisher Item Identifier S 0018-9286(97)02809-2.

that (2) can be written in the coordinates (X, ω)

$$\begin{aligned}
\dot{X} &= \tfrac{1}{2}(\omega + S(\omega)X + \langle \omega, X\rangle X) \\
\dot{\omega}_1 &= c_1\omega_2\omega_3 + u_1 \\
\dot{\omega}_2 &= c_2\omega_1\omega_3 + u_2 \\
\dot{\omega}_3 &= c_3\omega_1\omega_2
\end{aligned} \tag{3}$$

with

$$c_1 = \frac{j_2 - j_3}{j_1}, \quad c_2 = \frac{j_3 - j_1}{j_2}, \quad c_3 = \frac{j_1 - j_2}{j_3}$$
$$u_1 = \frac{\tau_1}{j_1}, \quad \text{and} \quad u_2 = \frac{\tau_2}{j_2}.$$

It is of course assumed that $c_3 \neq 0$, since otherwise the system would not be controllable or stabilizable. Moreover, we may also assume that $c_3 > 0$, due to the fact that the change of variables

$$\begin{aligned}
&(x_1, x_2, x_3, \omega_1, \omega_2, \omega_3, u_1, u_2) \\
&\quad \mapsto (x_2, x_1, -x_3, \omega_2, \omega_1, -\omega_3, u_2, u_1)
\end{aligned}$$

leaves (3) unchanged, except for the parameters c_1, c_2, and c_3 which are changed into $-c_2, -c_2$, and $-c_3$.

Our objective is to find a continuous feedback control law which exponentially asymptotically stabilizes the origin of (3).

III. Homogeneity and Exponential Stabilization

Let us first recall some results and definitions about homogeneous systems. For a more complete exposition, the reader is referred to [7] or [6].

For any $\lambda > 0$ and any set of real parameters $r_i > 0$ $(i = 1, \cdots, n)$, one defines the following "dilation" operator δ_λ^r: $\mathbb{R}^n \mapsto \mathbb{R}^n$ by

$$\delta_\lambda^r(x_1, \cdots, x_n) = (\lambda^{r_1}x_1, \cdots, \lambda^{r_n}x_n).$$

A *homogeneous* norm associated with this dilation operator is

$$\rho_p^r(x) = \left(\sum_{j=1}^n |x_j|^{p/r_j}\right)^{1/p}$$

with $p > 0$.

A continuous function f: $\mathbb{R}^n \mapsto \mathbb{R}$ is homogeneous of degree $\tau \geq 0$ with respect to the dilation δ_λ^r if

$$\forall \lambda > 0, \quad f(\delta_\lambda^r(x)) = \lambda^\tau f(x).$$

A differential system $\dot{x} = f(x)$ (or a vector field f), with f: $\mathbb{R}^n \mapsto \mathbb{R}^n$ continuous, is homogeneous of degree $\tau \geq 0$ with respect to the dilation δ_λ^r if

$$\forall \lambda > 0, \quad f_i(\delta_\lambda^r(x)) = \lambda^{\tau + r_i} f_i(x) \qquad (i = 1, \cdots, n).$$

The above definitions can be extended to time-dependent functions and systems. Such an extension has already been considered in [12] and simply follows by considering the extended dilation operator:

$$\delta_\lambda^r(x_1, \cdots, x_n, t) = (\lambda^{r_1}x_1, \cdots, \lambda^{r_n}x_n, t).$$

The definitions remain unchanged.

The following result, which is a particular case of a proposition by Pomet and Samson, establishes the existence of homogeneous Lyapunov functions for time-varying asymptotically stable systems which are homogeneous of degree zero with respect to some dilation. This proposition extends a theorem by Rosier [14] on autonomous systems.

Proposition 1 (Pomet, Samson [12]): Let $f(x,t)$: $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ a T-periodic continuous function $(f(x, t+T) = f(x,t))$. Assume that the system $\dot{x} = f(x,t)$ is homogeneous of degree zero with respect to a dilation $\delta_\lambda^r(x,t)$ and that $x = 0$ is an asymptotically stable equilibrium of this system.

Then, for any $\alpha > 0$ and $p < \alpha/\max\{r_j\}$, there exists a function $V(x,t)$: $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$ such that:

- V is of class C^p on $\mathbb{R}^n \times \mathbb{R}$ and of class C^∞ on $(\mathbb{R}^n - \{0\}) \times \mathbb{R}$;
- V is T-periodic $(V(x, t+T) = V(x,t))$;
- V is homogeneous of degree α with respect to the dilation δ_λ^r: $V(\delta_\lambda^r(x,t)) = \lambda^\alpha V(x,t)$;
- $V(x,t) > 0$ if $x \neq 0, V(0,t) = 0$;
- $V(x,t)$ is "proper" with respect to x: $\forall t$: $V(x,t) \mapsto +\infty$ when $|x| \mapsto +\infty$;
- $\exists M > 0, \exists \alpha > 0$: $(\partial V/\partial t)(x,t) + (\partial V/\partial x)(x,t)f(x,t) \leq -M(\rho_p^r(x))^\alpha$.

The following properties can be viewed as a consequence of the above proposition. The first property has been stated by Kawski in [7], in the case of autonomous systems. The second has been shown by Hermes, also for autonomous systems in which case no assumption on the homogeneity degree of the vector field is needed (see [6]).

Proposition 2 (Exponential Stabilization): Consider the system

$$\dot{x} = f(x,t) \tag{4}$$

with $f(x,t)$: $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ a T-periodic continuous function $(f(x, t+T) = f(x,t))$, and $f(0,t) = 0$. Assume that (4) is homogeneous of degree zero with respect to a dilation $\delta_\lambda^r(x,t)$ and that the equilibrium point $x = 0$ of this system is locally asymptotically stable.

Then:

1) $x = 0$ is globally exponentially stable *in the sense* that there exist two strictly positive constants K and γ such that along any solution of (4)
$$\rho_p^r(x(t)) \leq Ke^{-\gamma t}\rho_p^r(x(0))$$
with $\rho_p^r(x)$ denoting a homogeneous norm associated with the dilation $\delta_\lambda^r(x,t)$;

2) the solution $x = 0$ of the "perturbed" system $\dot{x} = f(x,t) + g(x,t)$ is locally exponentially stable when $g(x,t)$: $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ is a continuous T-periodic function such that the corresponding vector field g is a sum of homogeneous vector fields of degree strictly positive with respect to δ_λ^r.

For the proof of Part 1), we refer to the proof of [12, Prop. 1]. The proof of Part 2) follows from Proposition 1 and from the proof of [14, Th. 3].

The next Proposition is a corollary of a result by M'Closkey and Murray.

Proposition 3 (M'Closkey, Murray [9]): Consider the system

$$\dot{x} = f(x, t/\epsilon) \tag{5}$$

with $f(x,t)$: $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ a continuous T-periodic function $(f(x, t+T) = f(x,t))$. Assume that (5) is homogeneous of degree zero with respect to a dilation $\delta_\lambda^r(x,t)$ and that the origin $y = 0$ of the "averaged system" $\dot{y} = \overline{f}(y)$ (with $\overline{f}(y) = 1/T\ \int_0^T\ f(y,t)\, dt$) is asymptotically stable.

Then, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, the origin $x = 0$ of (5) is exponentially stable.

The following proposition complements the previous one for a specific class of systems, in the sense that it provides us with a value of ϵ_0.

Proposition 4: Consider the system

$$\dot{x} = f_0(x) + \sum_{i=1}^{p} g_i(t/\epsilon) f_i(x) \tag{6}$$

where f_i $(i = 0, \cdots, p)$: $\mathbb{R}^n \mapsto \mathbb{R}^n$ are continuous functions which define homogeneous vector fields of degree zero with respect to a dilation $\delta_\lambda^r(x), f_i$ $(i = 1, \cdots, p)$, are functions of class C^1 on $\mathbb{R}^n - 0$, and g_i $(i = 1, \cdots, p)$: $\mathbb{R} \mapsto \mathbb{R}$ are continuous T-periodic functions such that $\int_0^T g_i(\tau)\, d\tau = 0$.

Assume that the origin $x = 0$ of the "averaged system" $\dot{x} = f_0(x)$ is asymptotically stable and that an associated Lyapunov function $V(x)$ of class C^2, homogeneous of degree β with respect to $\delta_\lambda^r(x)$, such that $V(x) \geq K_1(\rho_p^r(x))^\beta$ and $(\partial V/\partial x)(x) f_0(x) \leq -K_2(\rho_p^r(x))^\beta$, is known. Define also

$$\begin{aligned} C_i &= \sup_{t \in \mathbb{R}} |g_i(t)|, \quad I_i = \sup_{t \in \mathbb{R}} \left| \int_0^t g_i(\tau)\, d\tau \right| \\ \delta_i &= \sup_{\rho_p^r(x)=1} \left| \frac{\partial V}{\partial x}(x) f_i(x) \right| \\ \gamma_{i,j} &= \sup_{\rho_p^r(x)=1} \left| \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}(x) f_i(x) \right) f_j(x) \right| \end{aligned}$$

and

$$\epsilon_0 = \mathrm{Max} \left\{ \frac{K_1}{\sum_{i=1}^{p} I_i \delta_i}, \frac{K_2}{\sum_{i=1}^{p} I_i \left(\gamma_{i,0} + \sum_{j=1}^{p} C_j \gamma_{i,j} \right)} \right\}.$$

Then for any $\epsilon \in (0, \epsilon_0)$ the origin $x = 0$ of (6) is exponentially stable.

Proof: The proof relies on the construction of a Lyapunov function for (6).

Since the functions g_i and $(\int_0^t g_i(\tau)\, d\tau)$ are T-periodic continuous functions, the values C_i and I_i $(i = 1, \cdots, p)$ are well defined. Let us consider the following continuous periodic function, homogeneous of degree β with respect to δ_λ^r:

$$W(x,t) = V(x) - \epsilon \sum_{i=1}^{p} \left(\int_0^{t/\epsilon} g_i(\tau)\, d\tau \right) \frac{\partial V}{\partial x}(x) f_i(x). \tag{7}$$

Then for ϵ smaller than ϵ_0, and using the fact that

$$\left| \frac{\partial V}{\partial x}(x) f_i(x) \right| \leq \delta_i (\rho_p^r(x))^\beta$$

it is simple to verify that W is a positive function. Moreover, this function is of class C^1 on $(\mathbb{R}^n - 0) \times \mathbb{R}$. The time derivative of W along any trajectory of the (6) which does not pass through $x = 0$ is then given by

$$\begin{aligned} \dot{W} = \frac{\partial V}{\partial x} f_0(x) &- \epsilon \sum_{i=1}^{p} \left(\int_0^{t/\epsilon} g_i(\tau)\, d\tau \right) \\ &\cdot \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}(x) f_i(x) \right) \cdot (f_0(x) \\ &+ \sum_{j=1}^{p} g_j(t/\epsilon) f_j(x)). \end{aligned} \tag{8}$$

For $\epsilon \leq \epsilon_0$, it is simple to verify that $\dot{W} \leq -K(\rho_p^r(x))^\beta$, with $K > 0$. □

IV. Cascaded High-Gain Control for a Class of Homoeneous Systems

The next proposition concerns the classical problem of "adding integrators." For autonomous systems, the existence of asymptotically stabilizing homogeneous feedbacks, for a homogeneous asymptotically stabilizable system to which an integrator has been added at the input level, has been proved in [5]. Some (nonsystematic) constructive methods have also been developed in [13] and [15]. The following result provides a simple solution to this problem for a class of homogeneous time-periodic systems.

Proposition 5: Consider the following system:

$$\dot{x} = f(x, v(x^1, t), t) \tag{9}$$

with $f(x, y, t)$: $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}^n$ a continuous T-periodic function, $x^1 = (x_1, \cdots, x_m), m \leq n$, and $v(x^1, t)$: $\mathbb{R}^m \times \mathbb{R} \mapsto \mathbb{R}$ a continuous T-periodic function, differentiable with respect to t, of class C^1 on $(\mathbb{R}^m - \{0\}) \times \mathbb{R}$, homogeneous of degree q with respect to a dilation $\delta_\lambda^r(x, t)$.

Assume further that (9) is homogeneous of degree zero with respect to the dilation $\delta_\lambda^r(x, t)$ and that the origin $x = 0$ of this system is asymptotically stable.

Then, for k positive and large enough, the origin $(x = 0, y = 0)$ of the system

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ \dot{y} &= -k(y - v(x^1, t)) \end{aligned} \tag{10}$$

is asymptotically stable.

Proof: Let $\delta(x, t) = (\lambda^{r_1} x_1, \cdots, \lambda^{r_n} x_n, t)$ denote the dilation with respect to which (9) is homogeneous of degree zero, and $\rho(x)$ an associated homogeneous norm.

Let us also denote $\delta_e(x, y, t) = (\lambda^{r_1} x_1, \cdots, \lambda^{r_n} x_n, \lambda^q y, t)$ the dilation with respect to which (10) is homogeneous of degree zero.

Since the function $v(x^1, t)$ is, by assumption, of class C^1 on $(\mathbb{R}^m - \{0\}) \times \mathbb{R}$ and homogeneous of degree q with respect to the dilation $\delta(x, t)$, the function $v^r(x^1, t)$ is also of class C^1 on $(\mathbb{R}^m - \{0\}) \times \mathbb{R}$, and it is homogeneous of degree rq for any positive integer r. Consequently, the function $\partial v^r / \partial x_i$ is homogeneous of degree $rq - r_i$, for $i = 1, \cdots, m$. The integer r is here chosen such that $r > \max\{r_i/q, 1 \leq i \leq m\}$. In this case each partial derivative of v^r is homogeneous of strictly positive degree with respect to the dilation $\delta(x, t)$ and thus tends to zero as $|x^1|$ tends to zero. Therefore, v^r is at least of class C^1 on $\mathbb{R}^m \times \mathbb{R}$. In what follows, it is further assumed that r is odd.

We denote as $V(x, t)$ a T-periodic Lyapunov function for (9), homogeneous of degree $\beta = (r + 1)q$ with respect to the dilation $\delta(x, t)$, and of class C^1. Such a function exists by application of Proposition 1. Following the "desingularization method" proposed in [13], we consider the following function:

$$W(x, y, t) = V(x, t) + \frac{1}{\sqrt{k}} \phi(y, x^1, t) \tag{11}$$

with

$$\phi(y, x^1, t) = \int_{v(x^1, t)}^{y} s^r - v^r(x^1, t)\, ds. \tag{12}$$

In order to prove the proposition, we show that W is a Lyapunov function for (10), when k is large enough.

We first note that ϕ is positive and equal to zero if and only if $y = v(x^1, t)$. This already implies that W is positive and vanishes only at $(x, y) = (0, 0)$. It is also proper with respect to (x, y) since $V(x, t)$ is, by assumption, proper with respect to x and $\phi(y, x^1, t)$, seen as a function of y when x and t are fixed, tends to infinity when $|y|$ tends to infinity. Now, from (12), it is simple to verify that

$$\phi(y, x^1, t) = \frac{y^{r+1}}{r+1} + r\frac{v^{r+1}}{r+1} - v^r y.$$

Since v^r is of class C^1, one deduces from the above expression that $\phi(y, x^1, t)$ is also of class C^1.

Let us now calculate the time derivative $\dot{W}$ of W along any trajectory of (10). With a slight abuse in the notations, introduced here for the sake of legibility, we have

$$\begin{aligned}\dot{W} &= \dot{V} + \frac{1}{\sqrt{k}}\,\dot{\phi}\\ &= \frac{\partial V}{\partial x} f(x, v(x^1, t), t) + \frac{\partial V}{\partial t}\\ &\quad + \frac{\partial V}{\partial x}\,(f(x, y, t) - f(x, v(x^1, t), t))\\ &\quad + \frac{1}{\sqrt{k}}\left(\frac{\partial \phi}{\partial y}\,\dot{y} + \frac{\partial \phi}{\partial x^1} f^1(x, y, t) + \frac{\partial \phi}{\partial t}\right) \end{aligned} \tag{13}$$

with f^1 denoting the vector-function whose components are the m-first components of f.

Since V is a homogeneous, of degree β, Lyapunov function for (9), there exists a strictly positive constant K_1 such that

$$\frac{\partial V}{\partial x}\,(x, t) f(x, v(x^1, t), t) + \frac{\partial V}{\partial t}\,(x, t) \le -2K_1(\rho(x))^\beta. \tag{14}$$

We show next that there exists another positive constant K_2 such that

$$\begin{aligned}&\left|\frac{\partial V}{\partial x}\,(x, t)(f(x, y, t) - f(x, v(x^1, t), t))\right|\\ &\qquad \le K_1(\rho(x))^\beta + K_2(y - v(x^1, t))^{r+1}.\end{aligned} \tag{15}$$

To this purpose, let us consider the following set of functions:

$$G_p(x, y, t) = \frac{\left|\frac{\partial V}{\partial x}\,(x, t)(f(x, y, t) - f(x, v(x^1, t), t))\right|}{K_1(\rho(x))^\beta + p(y - v(x^1, t))^{r+1}} \tag{16}$$

indexed by the positive integer p. G_p is a continuous T-periodic function, homogeneous of degree zero with respect to the dilation $\delta_\epsilon(x, y, t)$, and it is well defined for $(x, y) \neq (0, 0)$. Time-periodicity of G_p allows one to consider that time lives on the compact set $S^1 = \mathbb{R}/TZ$ instead of $\mathbb{R}$. Since G_p is homogeneous of degree zero, G_p reaches its maximum at some point (x_p, y_p, t_p) in $S \times S^1$, with S denoting the unit sphere in $\mathbb{R}^{n+1}$. By compacity of $S \times S^1$, one can extract a subsequence $(x_{p_l}, y_{p_l}, t_{p_l}), l \in \mathbb{N}$ which converges to some point $(\bar{x}, \bar{y}, \bar{t}) \in S \times S^1$. Let us distinguish the following two cases.

1) $\bar{y} = v(\bar{x}^1, \bar{t})$: By continuity of f and v, the numerator of $G_{p_l}(x_{p_l}, y_{p_l}, t_{p_l})$ tends to zero as l tends to $+\infty$, and for l large enough the denominator is greater than $(K_1/2)(\rho(\bar{x}))^\beta > 0$, using the fact that $\bar{x}$ cannot be equal to zero. Indeed, if $\bar{x}$ were equal to zero, then $\bar{y} = v(0, \bar{t})$ would also be equal to zero, contradicting the fact that $(\bar{x}, \bar{y})$ belongs to S. As a consequence, $G_{p_l}(x_{p_l}, y_{p_l}, t_{p_l})$ must be smaller than one for large enough values of l.
2) $\bar{y} \neq v(\bar{x}^1, \bar{t})$: By continuity of f and v, the numerator of $G_{p_l}(x_{p_l}, y_{p_l}, t_{p_l})$ is bounded independently of l, and the denominator tends to $+\infty$ as l tends to $+\infty$. Therefore, $G_{p_l}(x_{p_l}, y_{p_l}, t_{p_l})$ tends to zero as l tends to $+\infty$.

We thus have proved the existence of an integer p for which $|G_p(x, y, t)| < 1$. By taking K_2 equal to this integer, (15) follows.

Let us now consider the term $(\partial\phi/\partial y)\dot{y}$ of (13). From (12) and (10), we have

$$\frac{\partial \phi}{\partial y}\,\dot{y} = -k(y^r - v^r(x^1, t))(y - v(x^1, t)). \tag{17}$$

We show the existence of a strictly positive constant α such that

$$\frac{\partial \phi}{\partial y}\,\dot{y} \le -k\alpha(y - v(x^1, t))^{r+1}. \tag{18}$$

To this purpose, let us consider the following function (with r odd): $h(x) = 2^{r-1}[(1+x)^r - x^r] - 1$, the positivity of which is easily established. By taking $x = v(x^1, t)/(y - v(x^1, t))$, one has

$$\frac{2^{r-1}}{(y - v(x^1, t))^r}\,(y^r - v^r(x^1, t)) - 1 \ge 0. \tag{19}$$

Multiplying each number of (19) by $(y - v(x^1, t))^{r+1}$, one obtains, in view of (17), the desired inequality (18) with $\alpha = 2^{1-r}$.

Finally, we have for some value K_3

$$\begin{aligned}&\left|\frac{\partial \phi}{\partial x^1}\,(y, x^1, t) f^1(x, y, t) + \frac{\partial \phi}{\partial t}\,(y, x^1, t)\right|\\ &\qquad \le K_3((\rho(x))^\beta + (y - v(x^1, t))^{r+1}).\end{aligned} \tag{20}$$

This inequality comes from the fact that the function

$$\frac{\frac{\partial \phi}{\partial x^1}\,(y, x^1, t) f^1(x, y, t) + \frac{\partial \phi}{\partial t}\,(y, x^1, t)}{(\rho(x))^\beta + (y - v(x^1, t))^{r+1}}$$

is homogeneous of degree zero with respect to the dilation $\delta_\epsilon(x, y, t)$, well defined outside $(x, y) = (0, 0)$, and is thus bounded. By using (13)–(15), (18), and (20), one obtains

$$\begin{aligned}\dot{W} \le &-2K_1(\rho(x))^\beta + K_1(\rho(x))^\beta + K_2(y - v(x^1, t))^{r+1}\\ &+ \frac{K_3}{\sqrt{k}}\,(\rho(x))^\beta + \frac{K_3}{\sqrt{k}}\,(y - v(x^1, t))^{r+1}\\ &- \sqrt{k}\alpha(y - v(x^1, t))^{r+1}.\end{aligned} \tag{21}$$

For any $k > \mathrm{Max}\{1, (K_3/K_1)^2, ((K_2 + K_3)/\alpha)^2\}, \dot{W}$ is negative and equal to zero if and only if $x = 0$ and $y = 0$. □

Proposition 5 can be used for a multi-input system to which an integrator has been added at each input level. More precisely, one easily deduces the following corollary.

Corollary 1: Consider the following system:

$$\dot{x} = f(x, v(x, t), t) \tag{22}$$

with $f(x, y, t)$: $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R} \mapsto \mathbb{R}^n$ a continuous T-periodic function, and $v(x, t)$: $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^p$ a continuous T-periodic vector-function whose components $v_1(x, t), \cdots v_p(x, t)$ are differentiable with respect to t, of class C^1 on $(\mathbb{R}^n - \{0\}) \times \mathbb{R}$, and homogeneous, respectively, of degree $q_1, \cdots, q_p$ with respect to a dilation $\delta_\lambda^r(x, t)$.

Assume further that (22) is homogeneous of degree zero with respect to the dilation $\delta_\lambda^r(x, t)$ and that the origin $x = 0$ of this system is asymptotically stable.

Then, for positive and large enough values of $k_1, \cdots k_p$, the origin $(x = 0, y = 0)$ of the system

$$\begin{aligned}\dot{x} &= f(x, y, t)\\ \dot{y}_1 &= -k_1(y_1 - v_1(x, t))\\ &\vdots\\ \dot{y}_p &= -k_p(y_p - v_p(x, t))\end{aligned} \tag{23}$$

is asymptotically stable.

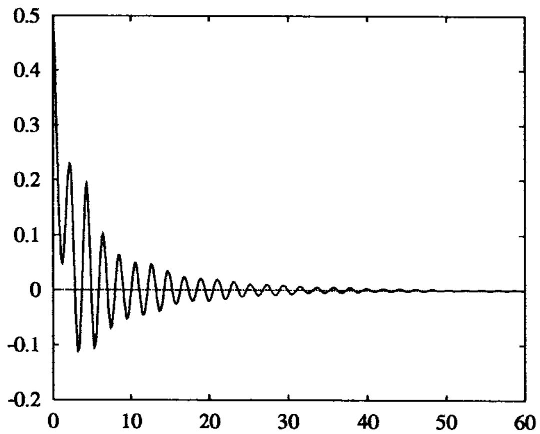

Fig. 1.

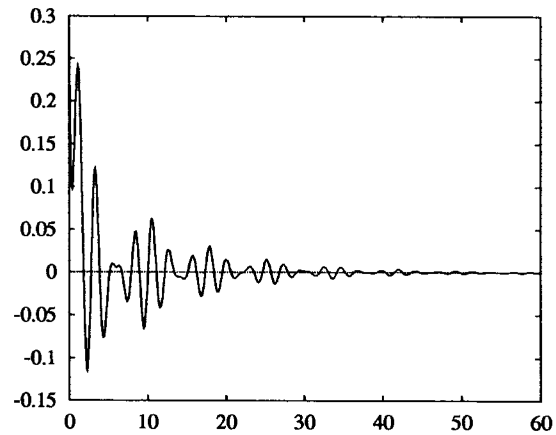

Fig. 2.

V. Exponential Stabilization of the Rigid Spacecraft

Our main result, which gives time-varying stabilizing feedbacks for the spacecraft, is stated next.

Theorem 1: Consider the functions

$$\begin{aligned} v_1(X, \omega_3, t) &= -k_1 x_1 - \rho(X, \omega_3) \sin(t/\epsilon) \\ v_2(X, \omega_3, t) &= -k_2 x_2 + \frac{1}{\rho(X, \omega_3)} (x_3 + \omega_3) \sin(t/\epsilon) \end{aligned} \tag{24}$$

with ρ, of class C^1 on $\mathbb{R}^4 - \{0\}$, a homogeneous norm associated with the dilation $\delta_\lambda^r(X, \omega_3, t) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^2 \omega_3, t)$, and the following time-varying continuous feedback:

$$\begin{aligned} u_1(X, \omega, t) &= -k_3(\omega_1 - v_1(X, \omega_3, t)) \\ u_2(X, \omega, t) &= -k_4(\omega_2 - v_2(X, \omega_3, t)). \end{aligned} \tag{25}$$

Then, for any positive parameters k_1 and k_2, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$ and large enough parameters $k_3 > 0$ and $k_4 > 0$, the feedback (25) locally asymptotically and exponentially stabilizes the origin of (3).

Proof: Let us consider the following dilation: $\delta_e^r(X, \omega, t) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda\omega_1, \lambda\omega_2, \lambda^2\omega_3, t)$.

System (3)–(25) can be rewritten as

$$\begin{pmatrix} \dot{X} \\ \dot{\omega} \end{pmatrix} = f(X, \omega, t) + g(X, \omega, t) \tag{26}$$

with

$$\begin{aligned} f(X, \omega, t) = (&\tfrac{1}{2}\omega_1, \tfrac{1}{2}\omega_2, \tfrac{1}{2}(\omega_3 + \omega_2 x_1 - \omega_1 x_2), \\ &u_1(X, \omega, t), u_2(X, \omega, t), c_3\omega_1\omega_2)^T. \end{aligned} \tag{27}$$

One easily verifies that $f(X, \omega, t)$ defines a continuous T-periodic vector field homogeneous of degree zero with respect to the dilation $\delta_e^r(X, \omega, t)$ and that $g(X, \omega, t)$ is continuous and defines a sum of homogeneous vector fields of degree strictly positive with respect to $\delta_e^r(X, \omega, t)$.

From Proposition 2, applied to (26), it is sufficient to show that the origin $(X = 0, \omega = 0)$ of the system

$$\begin{pmatrix} \dot{X} \\ \dot{\omega} \end{pmatrix} = f(X, \omega, t) \tag{28}$$

is locally asymptotically stable.

To this purpose, let us first consider the following reduced system obtained from (28)–(27) by taking $v_1 \stackrel{\text{def}}{=} \omega_1$ and $v_2 \stackrel{\text{def}}{=} \omega_2$ as control variables:

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} v_1 \\ v_2 \\ \omega_3 + v_2 x_1 - v_1 x_2 \end{pmatrix} \\ \dot{\omega}_3 &= c_3 v_1 v_2. \end{aligned} \tag{29}$$

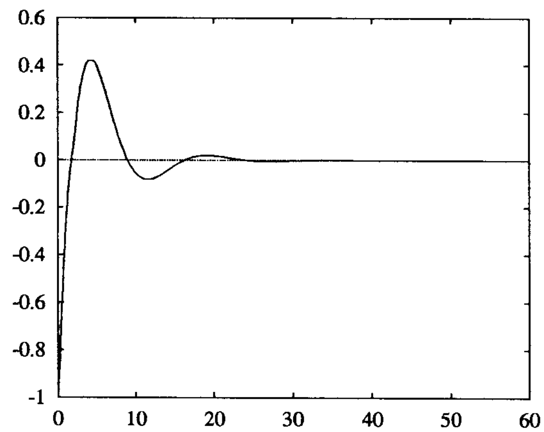

Fig. 3.

With the controls v_1 and v_2 given by (24), one verifies, by application of Proposition 3, that the origin of the controlled system is asymptotically stable for any positive k_1 and k_2 and ϵ small enough.

Indeed, the vector-valued function associated with the right-hand side of the controlled system is continuous, since $v_1(X, \omega, t)$ and $v_2(X, \omega, t)$ are homogeneous of degree one with respect to the dilation $\delta_\lambda^r(X, \omega_3, t)$, are well defined outside the origin $(X = 0, \omega_3 = 0)$, and thus tend to zero as $|(X, \omega_3)|$ tends to zero. The corresponding vector field is also periodic and homogeneous of degree zero with respect to $\delta_\lambda^r(X, \omega_3, t)$ so that the assumptions of Proposition 3 are met. Moreover, the corresponding "averaged" system is given by

$$\begin{aligned} \dot{x}_1 &= -\frac{k_1}{2} x_1 \\ \dot{x}_2 &= -\frac{k_2}{2} x_2 \\ \dot{x}_3 &= \frac{1}{2}\omega_3 + \frac{1}{2} (k_1 - k_2) x_1 x_2 \\ \dot{\omega}_3 &= c_3 (k_1 k_2 x_1 x_2 - \frac{1}{2} x_3 - \frac{1}{2}\omega_3) \end{aligned} \tag{30}$$

and the origin of this system is locally asymptotically stable, since the linear approximation of this system around the origin is obviously stable. The asymptotic stability of the origin of the system (28) follows by direct application of Corollary 1, after noticing that the functions $v_1(X, \omega_3, t)$ and $v_2(X, \omega_3, t)$ are of class C^1 on $(\mathbb{R}^3 \times \mathbb{R} - \{0, 0\}) \times \mathbb{R}$. $\square$

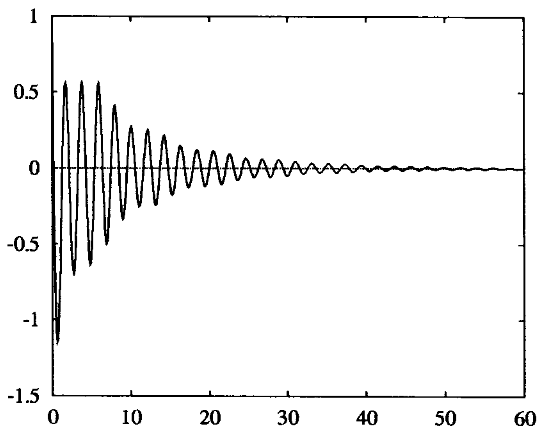

Fig. 4.

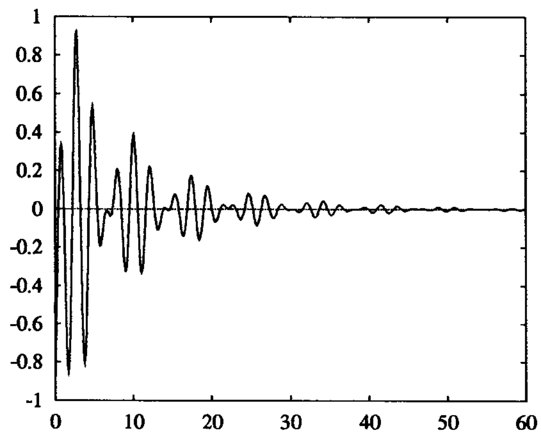

Fig. 5.

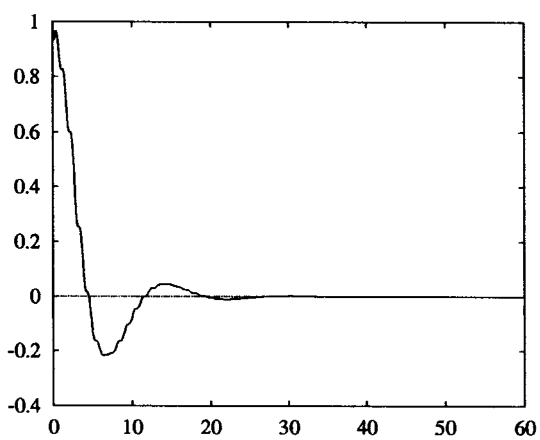

Fig. 6.

VI. Simulation Results

The feedback laws given by (24), (25) make the origin of (3) asymptotically stable for small enough values of ϵ and large enough values of k_3 and k_4. For practical purposes, it is necessary to specify values for which the stabilization is ensured. Conservative values can be determined via a complementary analysis. For instance, using the fact that $V(X, \omega_3) = 4x_1^4 + 4x_2^4 + x_3^2 + \omega_3^2 + x_3\omega_3$ is a Lyapunov function for (30) when $c_3 = k_1 = k_2 = 1$, one can deduce from Proposition 4 an upper bound for ϵ_0. Conservative values of k_3 and

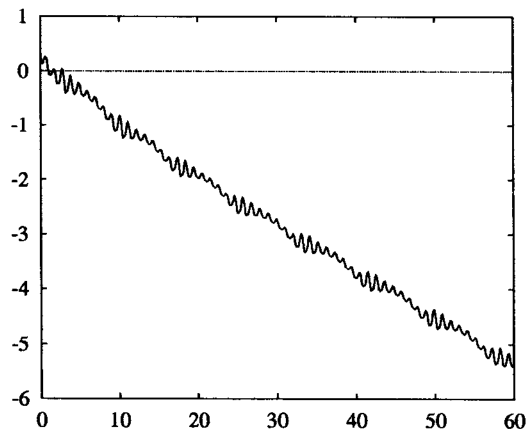

Fig. 7.

k_4 can in turn be obtained by following the proof of Proposition 5. As for now, we will illustrate by simulation that ϵ does not have to be very small, nor k_3 and k_4 very large. For example, the action of the control laws (24), (25) on (3) has been simulated with the following choice of parameters: $\epsilon = 1/3, k_1 = k_2 = 1, k_3 = k_4 = 5,$ and with the initial conditions $(x_1(0), x_2(0), x_3(0), \omega_1(0), \omega_2(0), \omega_3(0))^T = (0.5, 0.3, -1, 1, -1, 1)^T$.

Figs. 1–6 show the time evolution of the state variables $x_1, x_2, x_3, \omega_1, \omega_2, \omega_3,$ and Fig. 7 shows the linear decreasing of the *log* of the homogeneous norm $\rho(X, \omega) = (x_1^4 + x_2^4 + x_3^2 + \omega_1^4 + \omega_2^4 + \omega_3^2)^{1/4}$, in order to illustrate the exponential convergence of this norm to zero.

It has also been verified by simulation that no choice of the parameters yields stability.

References

[1] C. I. Byrnes and A. Isidori, "On the attitude stabilization of rigid spacecraft," *Automatica*, vol. 27, pp. 87–95, 1991.

[2] R. W. Brockett, "Asymptotic stability and feedback stabilization," in *Differential Geometric Control Theory*, R. W. Brockett, R. S. Millman, and H. H. Sussmann, Eds., 1983.

[3] J.-M. Coron, "On the stabilization in finite time of locally controllable systems by means of continuous time-varying feedback laws," *SIAM J. Contr. Optimization*, vol. 33, pp. 804–833, 1995.

[4] J.-M. Coron and E.-Y. Kerai, "Explicit feedbacks stabilizing the attitude of a rigid spacecraft with two control torques," *Automatica*, vol. 32, pp. 669–677, 1996.

[5] J.-M. Coron and L. Praly, "Adding an integrator for the stabilization problem," *Syst. Contr. Lett.*, vol. 17, pp. 89–104, 1991.

[6] H. Hermes, "Nilpotent and high-order approximations of vector fields systems," *SIAM Rev.*, vol. 33, no. 2, pp. 238–264, 1991.

[7] M. Kawzki, "Homogeneous stabilizing control laws," *Contr. Th. Adv. Tech.*, vol. 6, pp. 497–516, 1990.

[8] E. Kerai, "Analysis of small time local controllability of the rigid body model," in *Proc. IFAC Conf. Syst. Structure Contr.*, Nantes, France, July 5–7, 1995.

[9] R. T. M'Closkey and R. M. Murray, "Nonholonomic systems and exponential convergence: Some analysis tools," in *Proc. 32nd IEEE Conf. Decision Contr.*, 1993, pp. 943–948.

[10] P. Morin and C. Samson, "Time-varying exponential stabilization of the attitude of a rigid spacecraft with two controls," in *Proc. 34th IEEE Conf. Dec. Contr.*, New Orleans, 1995.

[11] P. Morin, C. Samson, J.-B. Pomet, and Z.-P. Jiang, "Time-varying feedback stabilization of the attitude of a rigid spacecraft with two controls," *Syst. Contr. Lett.*, vol. 25, pp. 375–385, 1995.

[12] J.-B. Pomet and C. Samson, "Exponential stabilization of nonholonomic systems in power form," in *Proc. IFAC Symp. Robust Contr. Design*, Rio de Janeiro, 1994, pp. 447–452.

[13] L. Praly, B. d'Andréa-Novel, and J.-M. Coron, "Lyapunov design of stabilizing controllers for cascaded systems," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1177–1181, 1991.

[14] L. Rosier, "Homogeneous Lyapunov function for homogeneous continuous vector field," *Syst. Contr. Lett.*, vol. 19, pp. 467–473, 1992.

[15] L. Rosier, "Stabilization of a system with integrator, "Thése de l'Ecole Normale Supérieure de Cachan," 1993.

[16] C. Samson, "Velocity and torque feedback control of a nonholonomic cart," in *Proc. Int. Wkshp. Adaptative Nonlinear Control: Issues Robotics*, Grenoble, France, 1990; also in *Proc. Advanced Robot Control*, vol. 162. New York: Springer-Verlag, 1991.

[17] E. D. Sontag and H. J. Sussmann, "Remarks on continuous feedback," in *Proc. 19th IEEE Conf. Decision Contr.*, Albuquerque, NM, 1980, pp. 916–921.

[18] G. C. Walsh, R. Montgomery, and S. S. Sastry, "Orientation control of the dynamic satellite," in *Proc. Amer. Contr. Conf.*, Baltimore, MD, 1994, pp. 138–142.

A Systematic Approach to Adaptive Observer Synthesis for Nonlinear Systems

Young Man Cho and Rajesh Rajamani

***Abstract*— Geometric techniques of controller design for nonlinear systems have enjoyed great success. A serious shortcoming, however, has been the need for access to full-state feedback. This paper addresses the issue of state estimation from limited sensor measurements in the presence of parameter uncertainty. An adaptive nonlinear observer is suggested for Lipschitz nonlinear systems, and the stability of this observer is shown to be related to finding solutions to a quadratic inequality involving two variables. A coordinate transformation is used to reformulate this inequality as a linear matrix inequality. A systematic algorithm is presented, which checks for feasibility of a solution to the quadratic inequality and yields an observer whenever the solution is feasible. The state estimation errors then are guaranteed to converge to zero asymptotically. The convergence of the parameters, however, is determined by a persistence-of-excitation-type constraint.**

***Index Terms*—Adaptive observer, interior point method, linear matrix inequality, nonlinear systems.**

I. Introduction

Observer design and adaptive control for nonlinear systems have both been very active fields of research during the last decade. The introduction of geometric techniques has led to great success in the development of controllers for nonlinear systems. Many attempts have been made to achieve results of equally wide applicability for state estimation and adaptation. The observer problem has, however, turned out to be much more difficult than the controller problem [1], [2].

An adaptive observer performs the twin tasks of state estimation and parameter identification. The two tasks are performed simultaneously and cannot be separated. The identification algorithm has to be defined using access to only the measured outputs and the estimated states. The state estimation algorithm has to work in the presence of uncertain parameters. This makes the problem very challenging.

The design of an adaptive observer for a linear time invariant system has been well analyzed [3]. In this case the order of the plant "n" is assumed to be known, nothing else about the plant need be known. The output of the plant is described as the output of a first-order differential equation whose input is a linear combination of "$2n$" signals. The coefficients of these signals represent the unknown parameters of the plant. The adaptive observer is also described by a similar first-order equation using the input and output of the plant, with its parameters being adjustable. An adaptation/estimation law is derived, and its uniform stability about the origin for the state estimation error can be shown without any further assumptions on the plant. If the plant is stable, the convergence of the state estimation error to zero can also be concluded. The parameters of the observer are adjusted using stable adaptation laws so that the error between the plant and observer outputs converges to zero. The convergence of the parameters to the desired values, however, depends on the persistent excitation of the input signals.

In the case of nonlinear systems, Sastry and Isidori presented results on the use of parameter adaptive control for obtaining asymptotically exact cancellation for the class of nonlinear systems which can be feedback linearized [4]. The full-state was assumed to be available, however, for the controller. Papers by Marino *et al.* on adaptive observers attempted to find a coordinate transformation so that the estimation error dynamics would be linearized in the new coordinates [5], [6]. They provide necessary and sufficient conditions for the existence of such a coordinate transformation. Even if these conditions are satisfied, the construction of the observer still remains a difficult task due to the need to solve a set of simultaneous partial differential equations to obtain the actual transformation function. An intuitively appealing and systematic treatment of the output feedback and adaptive observer problem for nonlinear systems has been developed by Kokotovic *et al.* [7]–[9]. Here the authors develop a set of tools which the user can attempt to customize for his specific problem. There has also been work by authors to propose adaptive observers for very special classes of nonlinear systems [10], [11].

The present work deals with a fairly general class of nonlinear systems, in which the nonlinearities are assumed to be Lipschitz. A systematic algorithm is provided which checks for the feasibility of an asymptotically stable adaptive observer. If the feasibility condition is satisfied, the algorithm provides the observer gains.

II. Background

This section presents results which will be used in the construction of our proposed observer.

A. Adaptive Observers for a Class of Nonlinear Systems

We begin with the adaptive observer proposed for a class of nonlinear systems in [15]. The class of systems we consider are linear in the unknown parameters and nonlinear in the states, with the nonlinearities assumed to be Lipschitz as described in (1) below. This is a fairly general class, since most nonlinearities can be bounded in a Lipschitz manner if the states can be assumed to be bounded. Further, many nonlinearities, like the sinusoidal terms encountered in robotics, are globally Lipschitz. The success of the adaptive observer method as outlined below, however, depends on being able to find a positive definite matrix $\boldsymbol{P}$ and an observer gain matrix $\boldsymbol{L}$ to satisfy (2) and (6). For proof of the Theorem, refer to [15].

Theorem II.1: Consider the class of nonlinear dynamical systems described by

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\Phi}(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{b}\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})\boldsymbol{\theta} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}\end{aligned} \tag{1}$$

Manuscript received April 4, 1995; revised January 29, 1996 and June 24, 1996.

The authors are with the United Technologies Research Center, East Hartford, CT 06108 USA.

Publisher Item Identifier S 0018-9286(97)02810-9.