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A CHARACTERIZATION OF THE LIE ALGEBRA RANK CONDITION BY TRANSVERSE PERIODIC FUNCTIONS*

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Abstract. The Lie algebra rank condition plays a central role in nonlinear systems control theory. The present paper establishes that the satisfaction of this condition by a set of smooth control vector fields is equivalent to the existence of smooth transverse periodic functions. The proof here enclosed is constructive and provides an explicit method for the synthesis of such functions.

Key words. controllability, driftless system, transversality, Lie algebra

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1. Introduction.

Let $X_1, \dots, X_m$ denote smooth vector fields (v.f.) on a smooth n-dimensional manifold M. By definition, the Lie algebra rank condition at a point $p_0 \in M$ ($LARC(p_0)$) is the property that[1]

$$M_{p_0} = \text{Span}\{X(p_0) : X \in \text{Lie}(X_1, \dots, X_m)\},$$

where $\text{Lie}(X_1, \dots, X_m)$ denotes the Lie algebra of v.f. generated by $X_1, \dots, X_m$. This condition plays a major role in the study of controllability properties of nonlinear control systems, as shown in the classical works of Chow [2], Lobry [10], Hermann [4], Sussmann and Jurdjevic [18], and others. For example, the well-known "Chow's theorem" states that if $LARC(p_0)$ is satisfied for the v.f. $X_1, \dots, X_m$, then the set of points reachable from p_0 by trajectories of the control system

$$\dot{p} = \sum_{i=1}^{m} u_i X_i(p) \tag{1}$$

contains a neighborhood of p_0. While the Lie algebra rank condition provides a systematic tool to test the controllability of system (1), its use at the control design level is usually not direct. For instance, even though $LARC(p_0)$ implies the existence of elements $X_{m+1}, \dots, X_{\bar{n}}$ of $\text{Lie}(X_1, \dots, X_m)$ such that

$$\forall p \in \mathcal{V}, \quad M_p = \text{Span}\{X_1(p), \dots, X_m(p)\} + \text{Span}\{X_{m+1}(p), \dots, X_{\bar{n}}(p)\}, \tag{2}$$

where $\mathcal{V}$ denotes a neighborhood of p_0, the "generation of motion" in the direction of the v.f. $X_{m+1}, \dots, X_{\bar{n}}$ by means of the control variables u_i is not simple. Although general results have been obtained for this problem in both the open-loop [9] and closed-loop [11] contexts, their application to physical systems usually raises several difficult issues—complexity, robustness, etc.

In this paper, we present a characterization of the Lie algebra rank condition which allows us to consider the control of system (1) from a slightly different perspective. More precisely, the following result is proved.

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[1] Throughout the paper, the notation N_q is used to denote the tangent space of a manifold N at q, whereas T_qF denotes the tangent mapping of a smooth map F at q.

THEOREM 1. *Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denote the one-dimensional torus, and let $X_1, \dots, X_m$ denote smooth v.f. on a smooth n-dimensional manifold M, such that the accessibility distribution $\Delta(p) \triangleq Span\,\{X(p) : X \in Lie(X_1, \dots, X_m)\}$ is of constant dimension n_0 in a neighborhood of p_0. Then the following properties are equivalent:*

1. *$n_0 = n$; i.e., the Lie algebra rank condition at p_0, $LARC(p_0)$, is satisfied for the v.f. $X_1, \dots, X_m$.*
2. *There exist $\bar{n} \in \mathbb{N}$ and, for any neighborhood $\mathcal{U}$ of p_0, a function $F \in \mathcal{C}^\infty(\mathbb{T}^{\bar{n}-m}; \mathcal{U})$ such that*

$$\forall \theta \in \mathbb{T}^{\bar{n}-m}, \quad M_{F(\theta)} = Span\,\{X_1(F(\theta)), \dots, X_m(F(\theta))\} + T_\theta F(\mathbb{T}^{\bar{n}-m}_\theta)\,. \tag{3}$$

REMARK 1.

1. *Relation* (3) *is reminiscent of the transversality property for functions—see, e.g.,* [1, Section 3.5] *for a definition.*
2. *It is clear that $\bar{n}$ is at least equal to n. For some systems—in particular, for free systems introduced later—it can be chosen equal to n, so that the sum in the right-hand side of* (3) *becomes direct, and F is an immersion.*

Roughly speaking, by comparison with (2), equality (3) implies that at any point $F(\theta) \in M$, the directions $X_{m+1}(F(\theta)), \dots, X_{\bar{n}}(F(\theta))$, which are not directly available for control, are spanned by the partial derivatives of the smooth function F. An important property of this characterization is that the function F can be directly used for control design purposes. In order to briefly illustrate this fact (for more details on potential applications, the reader is referred to [13]), let us consider the well-known chain system on $\mathbb{R}^3$, where $p = (p_1, p_2, p_3)^T \in \mathbb{R}^3$:

$$\dot{p} = u_1 X_1(p) + u_2 X_2\,, \quad X_1(p) = (1, 0, p_2)^T\,, X_2 = (0, 1, 0)^T \tag{4}$$

for which LARC(0) is clearly satisfied. For this system, (3) is satisfied with $\bar{n} = 3$—so that $\mathbb{T}^{\bar{n}-m} = \mathbb{T}$—and, for example, any function F_ϵ ($\epsilon > 0$) defined by

$$F_\epsilon(\theta) = \begin{pmatrix} \epsilon \sin\theta \\ \epsilon\cos\theta \\ \dfrac{\epsilon^2}{4}\sin 2\theta \end{pmatrix}.$$

Indeed, (3) is in this case equivalent to the condition

$$\forall \theta \in \mathbb{T}\,, \quad \mathrm{Det}\left(H(\theta) \triangleq \left[X_1(F_\epsilon(\theta)) \;\; X_2 \;\; -\frac{\partial F_\epsilon}{\partial \theta}(\theta)\right]\right) \neq 0, \tag{5}$$

the satisfaction of which is readily verified. Let us now introduce a new state vector φ defined by

$$\varphi(p, \theta) \triangleq \begin{pmatrix} p_1 - F_{\epsilon,1}(\theta) \\ p_2 - F_{\epsilon,2}(\theta) \\ p_3 - F_{\epsilon,3}(\theta) - p_1\,(p_2 - F_{\epsilon,2}(\theta)) \end{pmatrix}.$$

A direct calculation shows that for any function of time $\theta(.)$ the time derivative of φ along any solution to (4) satisfies

$$\dot{\varphi}(p, \theta) = C(p)H(\theta)(u_1, u_2, \dot{\theta})^T$$

with

$$C(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -p_1 & 1 \end{pmatrix}.$$

Since both matrices $C(p)$ and $H(\theta)$ are invertible for any $p \in \mathbb{R}^3$ and any $\theta \in \mathbb{T}$, it is straightforward, by considering $(u_1, u_2, \dot{\theta})$ as a new control vector, to globally asymptotically stabilize φ to zero. For instance, uniform exponential stabilization of $\varphi = 0$ is obtained by setting

$$(u_1, u_2, \dot{\theta})^T = -kH^{-1}(\theta)C^{-1}(p)\varphi(p,\theta), \qquad k > 0\,.$$

In terms of the state p, this yields a control law which globally stabilizes a neighborhood of the origin, the size of which can be made arbitrarily small by choosing ϵ as small as needed. Let us remark that, although it was not formalized in this way, this idea has been used implicitly in [3] for the problem of tracking a unicycle-type vehicle.

Based on this simple example, potential applications of Theorem 1 to various control problems are easily envisioned. Direct applications concern practical feedback stabilization of either systems without drift—as illustrated in the above example—or systems with a nonvanishing drift v.f. (see, e.g., [13], where potential application to nonholonomic motion planning is also briefly discussed). Other applications in the domain of nonlinear observer design or control of PDEs might also be considered.

This paper is organized as follows: Theorem 1 is proved[2] in section 2, and an example to illustrate the construction of transverse functions F is provided in section 3. Let us finally indicate that a presentation of Theorem 1 was accepted at the IEEE Conference on Decision and Control 2000 [12] in the form of a regular paper which did not contain the proof.

The following notation is used throughout the paper.

- δ_i^j denotes the Kronecker delta.
- $B_n(0,\delta)$ denotes the closed ball in $\mathbb{R}^n$ centered at zero and of radius δ.
- For $h \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R}^m)$ and $g \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{R})$ with $g(x) \neq 0$ for $x \neq 0$, we write $h = o(g)$ when $|h(x)|/|g(x)| \longrightarrow 0$ as $x \longrightarrow 0$.
- $\mathbf{d}$ denotes the exterior derivative.

2. Proof of Theorem 1. By considering a system of local coordinates $x = (x_1, \dots, x_n)$ on M, which maps p_0 to $0 \in \mathbb{R}^n$, and a—globally defined—frame[3] $\{\frac{\partial}{\partial\theta_{m+1}}, \dots, \frac{\partial}{\partial\theta_{\bar{n}}}\}$ on $\mathbb{T}^{\bar{n}-m}$, Theorem 1 rewrites as follows.

COROLLARY 1. *Let $g_1, \dots, g_m$ denote smooth v.f. on $\mathbb{R}^n$ such that the accessibility distribution is of constant dimension in a neighborhood of the origin. Then the following properties are equivalent:*

1. *LARC(0): the system*

$$S: \qquad \dot{x} = \sum_{i=1}^{m} u_i g_i(x)$$

satisfies the Lie algebra rank condition at the origin.

[2] Note added in proof: A simpler proof has recently been obtained. More details are available from the authors.

[3] The dual basis—coframe—will be denoted $(\mathrm{d}\theta_{m+1}, \dots, \mathrm{d}\theta_{\bar{n}})$.

2. *TC(0): there exist* $\bar{n} \in \mathbb{N}$ *and a family of functions* $f_\epsilon \in \mathcal{C}^\infty(\mathbb{T}^{\bar{n}-m}; B_n(0,\epsilon))$ $(\epsilon > 0)$ *such that, for any* $\epsilon > 0$*, the following transversality condition holds:*

$$(6) \quad \forall \theta \in \mathbb{T}^{\bar{n}-m}, \quad \text{Rank}\begin{pmatrix} g_1(f_\epsilon(\theta)) & \dots & g_m(f_\epsilon(\theta)) & \dfrac{\partial f_\epsilon}{\partial \theta_{m+1}}(\theta) & \dots & \dfrac{\partial f_\epsilon}{\partial \theta_{\bar{n}}}(\theta) \end{pmatrix} = n\,.$$

We now focus on the proof of this equivalent formulation of Theorem 1.

2.1. TC(0) $\Longrightarrow$ LARC(0). We assume that $LARC(0)$ is not satisfied and show that $TC(0)$ cannot be satisfied either. By assumption, the accessibility distribution is of constant dimension n_0 in a neighborhood of the origin. Therefore, if $n_0 < n$, the Frobenius theorem guarantees the existence of local coordinates $\phi(x)$ such that ϕ_n is constant along the trajectories of S, i.e., for some neighborhood $\mathcal{U}$ of the origin,

$$(7) \qquad \forall x \in \mathcal{U},\ \forall i = 1, \dots, m, \qquad \frac{\partial \phi_n}{\partial x}(x) \neq 0, \quad \text{and} \quad \frac{\partial \phi_n}{\partial x}(x) g_i(x) = 0\,.$$

Now assume that $TC(0)$ is satisfied, and choose any f_ϵ satisfying (6) and such that $B_n(0,\epsilon) \subset \mathcal{U}$. By the compactness of $\mathbb{T}^{\bar{n}-m}$, the smooth function $\theta \longmapsto \phi_n(f_\epsilon(\theta))$ from $\mathbb{T}^{\bar{n}-m}$ to $\mathbb{R}$ attains its maximum value for some $\bar{\theta}$, i.e.,

$$(8) \qquad \forall i = m+1, \dots \bar{n}, \qquad \frac{\partial \phi_n}{\partial x}(f_\epsilon(\bar{\theta})) \frac{\partial f_\epsilon}{\partial \theta_i}(\bar{\theta}) = 0\ .$$

From (8) and from (7) evaluated at $x = f_\epsilon(\bar{\theta})$, we obtain

$$\frac{\partial \phi_n}{\partial x}(f_\epsilon(\bar{\theta})) \begin{pmatrix} g_1(f_\epsilon(\bar{\theta})) & \dots & g_m(f_\epsilon(\bar{\theta})) & \dfrac{\partial f_\epsilon}{\partial \theta_{m+1}}(\bar{\theta}) & \dots & \dfrac{\partial f_\epsilon}{\partial \theta_{\bar{n}}}(\bar{\theta}) \end{pmatrix} = 0\ ,$$

which is in contradiction with $TC(0)$. $\square$

2.2. LARC(0) $\Longrightarrow$ TC(0).

2.2.1. Notation and recalls. Prior to addressing the proof itself, we specify some notation and recall a few basic definitions and results that are extensively used in what follows. These recalls are about homogeneity on one hand and free Lie algebras on the other hand. For a more complete survey about these issues, we refer the reader to [5, 6] for the properties associated with homogeneity, and to [7, 17] for the role of free Lie algebras in control theory.

About homogeneity. Given $\mu > 0$ and a *weight vector* $r = (r_1, \dots, r_n)$ $(r_i > 0\ \forall i)$, a *dilation* Δ_μ^r on $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ defined by $\forall z = (z_1, \dots, z_n) \in \mathbb{R}^n$, $\Delta_\mu^r z \stackrel{\Delta}{=} (\mu^{r_1} z_1, \dots, \mu^{r_n} z_n)$. A function $f \in \mathcal{C}^0(\mathbb{R}^n; \mathbb{R})$ is *homogeneous of degree* l *with respect to the family of dilations* $(\Delta_\mu^r)_{\mu > 0}$ or, more concisely, Δ^r*-homogeneous of degree* l if $\forall \mu > 0$, $f(\Delta_\mu^r z) = \mu^l f(z)$. A Δ^r*-homogeneous norm* is defined as a positive definite function on $\mathbb{R}^n$, Δ^r-homogeneous of degree one. A smooth v.f. X on $\mathbb{R}^n$ is Δ^r*-homogeneous of degree* d if, $\forall i = 1, \dots, n$, the function $x \longmapsto X_i(x)$ is Δ^r-homogeneous of degree $d + r_i$. The system

$$(9) \qquad S_{ap}: \qquad \dot{z} = \sum_{i=1}^m b_i(z) u_i$$

is a Δ^r-*homogeneous approximation* of S if there exists a change of coordinates $\phi:\ x \longmapsto z$ which transforms S into

$$\dot{z} = \sum_{i=1}^{m} (b_i(z) + h_i(z))\, u_i\,, \tag{10}$$

where b_i is Δ^r-homogeneous of degree -1, and h_i denotes higher-order terms; i.e., for any j, the jth component $h_{i,j}$ of h_i satisfies $h_{i,j} = o(\rho^{r_j-1})$, where ρ is any Δ^r-homogeneous norm.

The main motivation for introducing such approximations comes from the following result.

PROPOSITION 1 (see [5, 15]). *For any system S of smooth v.f. which satisfies $LARC(0)$, there exists a Δ^r-homogeneous approximation S_{ap} which also satisfies $LARC(0)$.*

Finally, we say that a set $\{b_1, \dots, b_m\}$ of v.f., or the associated system (9), is *nilpotent of order* $d+1$ if any Lie bracket of these v.f. of length larger than, or equal to, $d+1$ is identically zero. It is simple to verify that any set $\{b_1, \dots, b_m\}$ of smooth v.f. with the b_i's Δ^r-homogeneous of degree -1 is nilpotent of order $1 + \mathrm{Max}\{r_i : i = 1, \dots, n\}$.

About free Lie algebras. Let us consider a finite set of indeterminates $X_1, \dots, X_m$ and denote by $\mathrm{Lie}(X)$ the free Lie algebra over $\mathbb{R}$ generated by the X_i's. We also denote by $\mathcal{F}(X)$ the set of formal brackets in the X_i's. For any set $\mathbf{b} \triangleq \{b_1, \dots, b_m\}$ of smooth v.f. and any $B \in \mathcal{F}(X)$, we denote by $\mathrm{Ev}_{\mathbf{b}}(B)$ the *evaluation map*, i.e., $\mathrm{Ev}_{\mathbf{b}}(X_i) = b_i$, and

$$\mathrm{Ev}_{\mathbf{b}}([B_\lambda, B_\rho]) = [\mathrm{Ev}_{\mathbf{b}}(B_\lambda), \mathrm{Ev}_{\mathbf{b}}(B_\rho)]\,.$$

The definition of a (generalized) P. Hall basis of $\mathrm{Lie}(X)$ is recalled below.

DEFINITION 1. *A P. Hall basis $\mathcal{B}$ of Lie(X) is a totally ordered subset of $\mathcal{F}(X)$ such that*

1. *each X_i belongs to $\mathcal{B}$;*
2. *if $B = [B_\lambda, B_\rho] \in \mathcal{F}$ with $B_\lambda, B_\rho \in \mathcal{F}$, then $B \in \mathcal{B}$ if and only if $B_\lambda, B_\rho \in \mathcal{B}$ with $B_\lambda < B_\rho$, and either* (i) *B_ρ is one of the X_i's or* (ii) *$B_\rho = [B_{\lambda\rho}, B_{\rho^2}]$ with $B_{\lambda\rho} \le B_\lambda$;*
3. *if $B \in \mathcal{B}$ is a bracket of length $\ell(B) \ge 2$, i.e., $B = [B_\lambda, B_\rho]$, with $B_\lambda, B_\rho \in \mathcal{B}$, then $B_\lambda < B$.*

In order to simplify the forthcoming analysis we choose a specific P. Hall basis $\mathcal{B}$ associated with a specific total order. The P. Hall basis so obtained is in fact a Hall basis in the original (narrow) sense (see, e.g., [14, Section IV.5]).

Specific order.

$$\begin{cases} \ell(B) < \ell(B') \Longrightarrow B < B', \\ X_i < X_j \Longleftrightarrow i < j, \\ \text{For } \ell(B) = \ell(B') > 1, B < B' \Longleftrightarrow B_\lambda < B'_\lambda, \text{ or } B_\lambda = B'_\lambda \text{ and } B_\rho < B'_\rho\,. \end{cases} \tag{11}$$

We denote by

$$\mathcal{B} = \{B_1, B_2, \dots, B_q, \dots\},\ B_1 < B_2 < \cdots < B_q < \cdots\,, \tag{12}$$

the P. Hall basis associated with the total order (11), and also by $\ell(i)$ the length of any bracket B_i of this basis. From (11) and the definition of a P. Hall basis, we deduce the following properties which will be extensively used in what follows:

$$i \in \{1, \dots, m\} \Longleftrightarrow \ell(i) = 1 \Longleftrightarrow B_i = X_i, \tag{13}$$

$$i > m \Longleftrightarrow \ell(i) > 1 \Longleftrightarrow B_i = [B_{\lambda(i)}, B_{\rho(i)}], \tag{14}$$

where $\lambda(i)$ and $\rho(i)$ are uniquely defined integers. By extension of this notation, and whenever this will make sense, we will use the symbols $\lambda^2(i), \lambda\rho(i), \rho^2(i), \dots$, to index the elements of $\mathcal{B}$. For instance, if $\ell(\rho(i)) \geq 2$, we can write $B_{\rho(i)} = [B_{\lambda\rho(i)}, B_{\rho^2(i)}]$. Finally, it also follows from (11) and the definition of a P. Hall basis that

$$\ell(i) > 1 \Longrightarrow \lambda(i) < \rho(i) < i\,.$$

Letting $0 < d \in \mathbb{N}$, we denote by $\mathrm{Lie}_d(X)$ the subspace of $\mathrm{Lie}(X)$ generated by brackets of length at most equal to d. Then the subset of $\mathcal{B}$ composed of all brackets B_j such that $\ell(j) \leq d$ is a basis of $\mathrm{Lie}_d(X)$ denoted as $\mathcal{B}_d$. Let $n(d)$ denote the dimension of $\mathrm{Lie}_d(X)$ so that

$$\mathcal{B}_d = \{B_1, \dots, B_{n(d)}\} \quad \text{and } \ell(n(d)) = d\,.$$

One can associate the following *free system* with $\mathcal{B}_d$:

$$\left\{ \begin{array}{rcll} \dot{x}_i & = & u_i, & i = 1, \dots, m, \\ \dot{x}_i & = & x_{\lambda(i)}\dot{x}_{\rho(i)}, & i = m+1, \dots, n(d). \end{array} \right. \tag{15}$$

REMARK 2. *Since there is a one-to-one correspondence between the components of the state vector x associated with the free system* (15) *and the element of $\mathcal{B}_d$, it would be natural to index each component of x by the corresponding element of $\mathcal{B}_d$, as done, for example, in* [7]. *We have preferred here to write B_i for an element of $\mathcal{B}_d$ and x_i for the corresponding component of x in order to lighten the notation.*

It is straightforward to verify that (15) defines a control-affine driftless system:

$$S(m,d): \qquad \dot{x} = \sum_{i=1}^{m} u_i b_i(x), \tag{16}$$

where the components $b_{i,j}$ of the v.f. b_i are defined by

$$b_{i,j}(x) = \left\{ \begin{array}{ll} \delta_i^j & \text{if } \ell(j) = 1, \\ x_{\lambda(j)} b_{i,\rho(j)} & \text{otherwise.} \end{array} \right. \tag{17}$$

The following properties of free systems will be used in what follows. For the first two properties, we refer to [7]. The third property has been proved in [8, Section 3] in a formal algebraic framework. A proof of the fourth property is given in the appendix.

LEMMA 1. *For $i = m+1, \dots, n(d)$, let b_i denote the v.f. $Ev_{\mathbf{b}}(B_i)$, where $\mathbf{b} = \{b_1, \dots, b_m\}$. Then the following properties hold.*

1. *For any $i \in \{1, \dots, n(d)\}$, $b_i = a_i \partial/\partial x_i + \sum_{j>i} b_{i,j} \partial/\partial x_j$ for some nonzero constant a_i and some smooth functions $b_{i,j}$ so that $S(m,d)$ satisfies $LARC(x)$ for any $x \in \mathbb{R}^{n(d)}$.*

2. *The v.f.* b_i *are* Δ*-homogeneous of degree* $-\ell(i)$ *with* Δ_μ $(\mu > 0)$*, the dilation defined by*

$$\Delta_\mu x = (\mu^{\ell(1)} x_1, \dots, \mu^{\ell(n(d))} x_{n(d)}) \tag{18}$$

so that $S(m,d)$ *is nilpotent of order* $d+1$*.*

3. *For any* $p \in \mathcal{C}^\infty(\mathbb{R}^{n(d)};\mathbb{R})$*,* Δ*-homogeneous of degree* $d' < d$*, and any* $j \in \{1,\dots,m\}$*, there exists* $q^j \in \mathcal{C}^\infty(\mathbb{R}^{n(d)};\mathbb{R})$*,* Δ*-homogeneous of degree* $d'+1$*, such that*

$$\forall i \in \{1,\dots,m\}, \qquad L_{b_i} q^j = \begin{cases} p & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

4. *For any* $i \in \{1,\dots,n(d)\}$ *and any* $p \in \mathcal{C}^\infty(\mathbb{R}^{n(d)};\mathbb{R})$*,* Δ*-homogeneous of degree* $d' - \ell(i)$ *with* $d' \leq d$*, there exist* h_1 *and* $h_{2,j}$ $(1 < \ell(j) \leq d')$ *in* $\mathcal{C}^\infty(\mathbb{R}^{n(d)};\mathbb{R})$*,* Δ*-homogeneous of degree* d' *and* $d' - \ell(j)$*, respectively, such that*

$$p(x)\mathrm{d}x_i = \mathbf{d}h_1 + \sum_{j:1<\ell(j)\leq d'} h_{2,j}(x)\left(\mathrm{d}x_j - x_{\lambda(j)}\mathrm{d}x_{\rho(j)}\right). \tag{20}$$

REMARK 3.

1. *The functions* p*,* q^j*,* h_1*, and* $h_{2,j}$ *in properties* 3 *and* 4 *are polynomial in* x *because they are smooth and homogeneous.*
2. *Since the smooth functions* q^j *in property* 3 *are homogeneous of degree* $d'+1$*, it can depend only on the* $n(d'+1)$ *first components of* x*.*

After these preliminary recalls, we can now proceed with the proof of Theorem 1. It is composed of three steps which are summarized in the following three propositions.

PROPOSITION 2. *If* $TC(0)$ *holds for a homogeneous approximation* S_{ap} *of a system* S*, then* $TC(0)$ *holds for* S *also.*

PROPOSITION 3. *If, for any* $d \in \mathbb{N}-\{0\}$*,* $TC(0)$ *holds for the free system* $S(m,d)$ *with* $\bar{n} = n(d)$*, then* $TC(0)$ *holds for any smooth driftless system* S_{hom} *which satisfies* $LARC(0)$ *and whose control v.f. are* Δ^r*-homogeneous of degree* -1 *for some dilation* Δ^r_μ*.*

PROPOSITION 4. *For any* $d \in \mathbb{N}-\{0\}$*,* $TC(0)$ *holds for the free system* $S(m,d)$ *with* $\bar{n} = n(d)$*.*

From Proposition 1, if S satisfies $LARC(0)$, it has a homogeneous approximation which also satisfies $LARC(0)$. This property, combined with the three propositions above, clearly implies that $LARC(0) \Longrightarrow TC(0)$. There remains to prove these three propositions.

2.2.2. Proof of Proposition 2. S rewrites, in some coordinates $z = \phi(x)$, as

$$\dot{z} = \sum_{i=1}^m u_i \left(\tilde{b}_i(z) + h_i(z)\right), \tag{21}$$

where the $\tilde{b}_i$'s, Δ^r-homogeneous of degree -1 (for some dilation Δ^r), are the v.f. of the homogeneous approximation S_{ap}, and h_i denotes higher-order terms, i.e.,

$$h_{i,j} = o(\rho^{r_j - 1}), \tag{22}$$

with ρ denoting any Δ^r-homogeneous norm. We want to show that if $TC(0)$ holds for S_{ap}, then it also holds for S. Since $TC(0)$ is independent of the system of coordinates, it is sufficient to show that $TC(0)$ holds in the coordinates z. Let $\bar{n}$ and $(f_\epsilon)_{\epsilon>0}$ denote an integer and a family of functions which satisfy (6) with the v.f. of the approximation S_{ap}. We show below that S satisfies $TC(0)$ by considering the same integer $\bar{n}$ and the family of functions $(\bar{f}_\epsilon)_{\epsilon>0}$ defined by

$$\bar{f}_\epsilon(\theta) = \Delta^r_{\mu(\epsilon)} f_1(\theta)\,, \tag{23}$$

with $\mu(\epsilon)$ denoting a strictly positive number which is (i) smaller than some adequately chosen $\mu_0 > 0$ and (ii) such that $\sup_{\theta\in\mathbb{T}^{\bar{n}-m}} |\Delta^r_{\mu(\epsilon)} f_1(\theta)| \leq \epsilon$. Note that $\mu(\epsilon)$ always exists because $f_1(\mathbb{T}^{\bar{n}-m})$ is a compact set so that $\lim_{\mu\to 0}\sup_{\theta\in\mathbb{T}^{\bar{n}-m}} |\Delta^r_\mu f_1(\theta)| = 0$.

With z denoting a vector in $\mathbb{R}^n$, one deduces from (22) that

$$\lim_{\mu\to 0} \frac{h_{i,j}(\Delta^r_\mu z)}{\rho^{r_j-1}(\Delta^r_\mu z)} = \lim_{\mu\to 0} \frac{h_{i,j}(\Delta^r_\mu z)}{\mu^{r_j-1}\rho^{r_j-1}(z)} = 0.$$

Therefore,

$$h_{i,j}(\Delta^r_\mu z) = c_{i,j}(\mu, z)\mu^{r_j-1},$$

where $|c_{i,j}(\mu, z)|$ tends to zero as μ tends to zero. Moreover, the convergence is uniform with respect to the z variable when $z \in B_n(0,1)$. The above equation can also be written in vectorial form as

$$h_i(\Delta^r_\mu z) = \mu^{-1}\Delta^r_\mu c_i(\mu, z) \tag{24}$$

with $c_i = (c_{i,1}, \dots , c_{i,n})^T$.

Let us now evaluate the rank of the matrix

$$A(\epsilon,\theta) \triangleq \left((\tilde{b}_1 + h_1)(\bar{f}_\epsilon(\theta)) \quad \dots \quad (\tilde{b}_m + h_m)(\bar{f}_\epsilon(\theta)) \quad \frac{\partial \bar{f}_\epsilon}{\partial\theta_{m+1}}(\theta) \quad \dots \quad \frac{\partial \bar{f}_\epsilon}{\partial\theta_{\bar{n}}}(\theta) \right).$$

Using (23), (24), and the fact that each $\tilde{b}_i$ is homogeneous of degree -1,

$$A(\epsilon,\theta) \;=\; \bar{A}(\epsilon,\theta)\; D(\mu(\epsilon))$$

with

$$\begin{aligned}
&\bar{A}(\epsilon,\theta)\\
&\quad\triangleq \left(\Delta^r_{\mu(\epsilon)}\tilde{b}_1(f_1(\theta)) \quad \dots \quad \Delta^r_{\mu(\epsilon)}\tilde{b}_m(f_1(\theta)) \quad \Delta^r_{\mu(\epsilon)}\frac{\partial f_1}{\partial\theta_{m+1}}(\theta) \quad \dots \quad \Delta^r_{\mu(\epsilon)}\frac{\partial f_1}{\partial\theta_{\bar{n}}}(\theta) \right)\\
&\quad + \left(\Delta^r_{\mu(\epsilon)} c_1(\mu(\epsilon), f_1(\theta)) \quad \dots \quad \Delta^r_{\mu(\epsilon)} c_m(\mu(\epsilon), f_1(\theta)) \quad 0 \quad \dots \quad 0 \right),
\end{aligned}$$

and

$$D(\mu(\epsilon)) \triangleq \mathrm{diag}\{1/\mu(\epsilon), \dots , 1/\mu(\epsilon), 1, \dots , 1\}\,.$$

Since $D(\mu(\epsilon))$ is nonsingular, it readily follows that

$$\begin{aligned}
&\text{Rank } A(\epsilon,\theta) = \text{Rank } \Big(\tilde{b}_1(f_1(\theta)) + c_1(\mu(\epsilon), f_1(\theta)) \quad \dots \quad \tilde{b}_m(f_1(\theta)) + c_m(\mu(\epsilon), f_1(\theta))\\
&\qquad\qquad\qquad \frac{\partial f_1}{\partial\theta_{m+1}}(\theta) \quad \dots \quad \frac{\partial f_1}{\partial\theta_{\bar{n}}}(\theta) \Big)\,.
\end{aligned} \tag{25}$$

Now, by assumption,

$$\forall\theta \in \mathbb{T}^{\bar n-m}, \qquad \operatorname{Rank}\left(\tilde b_1(f_1(\theta)) \quad \ldots \quad \tilde b_m(f_1(\theta)) \quad \frac{\partial f_1}{\partial\theta_{m+1}}(\theta) \quad \ldots \quad \frac{\partial f_1}{\partial\theta_{\bar n}}(\theta)\right) = n\,. \tag{26}$$

In view of (25) and (26) and using the facts that $f_1(\theta) \in B_n(0,1)$ and that $|c_{i,j}(\mu,z)|$ tends uniformly (with respect to $z \in B_n(0,1)$) to zero as μ tends to zero, there exists a strictly positive number μ_0 such that

$$\mu(\epsilon) \le \mu_0 \Longrightarrow \forall\theta \in \mathbb{T}^{\bar n-m}\,,\ \operatorname{Rank} A(\epsilon,\theta) = n.$$

This concludes the proof of Proposition 2. $\square$

REMARK 4. *The previous analysis implies—by setting* $\forall i,\ h_i \equiv 0$ *in* (21)*—that for a homogeneous system, if a function* $f \in \mathcal{C}^\infty(\mathbb{T}^{\bar n-m};\mathbb{R}^n)$ *satisfies* (6)*, then, for any* $\mu > 0$*,* $\Delta_\mu f$ *also satisfies* (6)*. Therefore,* $TC(0)$ *is satisfied for this homogeneous system with the functions* $f_\epsilon \stackrel{\Delta}{=} \Delta_{\mu(\epsilon)} f$*, where* $\mu(\epsilon)$ *is any strictly positive value such that* $\sup_{\theta\in\mathbb{T}^{\bar n-m}} |\Delta_{\mu(\epsilon)} f(\theta)| \le \epsilon$*.*

2.2.3. Proof of Proposition 3. Consider a smooth driftless system

$$S_{hom}: \qquad \dot z = \sum_{i=1}^{m} \tilde b_i(z) u_i, \tag{27}$$

whose v.f. $\tilde b_i$ $(i = 1,\ldots,m)$ are Δ^r-homogeneous of degree -1 for some dilation Δ^r_μ and satisfy $LARC(0)$. Since S_{hom} is nilpotent of some order $d+1$, it can be associated with the free system $S(m,d)$ whose v.f. b_i are defined in (17). We show below that any family $(f_\epsilon)_{\epsilon>0}$ which satisfies $TC(0)$ for the free system $S(m,d)$ induces a family $(\tilde f_\epsilon)_{\epsilon>0}$ which satisfies $TC(0)$ for S_{hom}. In fact, from Remark 4 above, we need only to show the existence of a single function $\tilde f \in \mathcal{C}^\infty(\mathbb{T}^{n(d)-m};\mathbb{R}^n)$, which satisfies the transversality condition (6) for S_{hom}.

Let f denote any of the functions f_ϵ which satisfy the transversality condition for $S(m,d)$. From property 1 of Lemma 1, the vectors $b_1(x),\ldots,b_{n(d)}(x)$ are linearly independent at any $x \in \mathbb{R}^{n(d)}$. Therefore, there exist (unique) smooth functions $u_{i,j}$ such that

$$\forall j = m+1,\ldots,n(d),\ \forall\theta \in \mathbb{T}^{n(d)-m}\,, \qquad \frac{\partial f}{\partial\theta_j}(\theta) = \sum_{i=1}^{n(d)} u_{i,j}(\theta) b_i(f(\theta)). \tag{28}$$

Also, using the fact that f satisfies the transversality condition (6) for $S(m,d)$,

$$\forall\theta \in \mathbb{T}^{n(d)-m}, \qquad \operatorname{Det} U(\theta) \ne 0 \quad \text{with} \quad U(\theta) \stackrel{\Delta}{=} \big(u_{i,j}(\theta)\big)_{i,j=m+1,\ldots,n(d)}. \tag{29}$$

Let us now define the function $\tilde f$. To this purpose, let us pick an arbitrary couple $(\theta_0, z_0) \in (\mathbb{T}^{n(d)-m} \times \mathbb{R}^n)$ and consider an element θ of $\mathbb{T}^{n(d)-m}$. Consider also a smooth path $\gamma : t \in [0,1] \longrightarrow \gamma(t) \in \mathbb{T}^{n(d)-m}$ which connects θ_0 to θ, i.e., such that $\gamma(0) = \theta_0$ and $\gamma(1) = \theta$. Let $z_\gamma(t)$ denote the solution, for $t \in [0,1]$, of

$$\dot z = \sum_{i=1}^{n(d)} \bar U_i(\gamma(t), \dot\gamma(t))\, \tilde b_i(z), \qquad z(0) = z_0\,, \tag{30}$$

where

$$\bar{U}_i(\gamma,\dot{\gamma}) = \sum_{j=m+1}^{n(d)} u_{i,j}(\gamma)\mathrm{d}\theta_j(\dot{\gamma})\,, \tag{31}$$

and, for $i = m+1,\dots,n(d)$, $\tilde{b}_i \overset{\Delta}{=} Ev_{\tilde{\mathbf{b}}}(B_i)$. Note that $z_\gamma(t)$ is well defined for $t \in [0,1]$. Indeed, finite-time escape is not possible because the v.f. $\tilde{b}_i$ are homogeneous of negative degree (by assumption). Let us show that $z_\gamma(1)$ is independent of the path γ chosen to connect θ_0 to θ. To this purpose, consider two paths γ_i $(i = 1,2)$ which map 0 to θ_0 and 1 to θ. We must show that the solution $z_{\gamma_1}(1)$ of (30) at $t = 1$ with $\gamma = \gamma_1$ is the same as the solution $z_{\gamma_2}(1)$ of (30) at $t = 1$ with $\gamma = \gamma_2$. To show this, we will use the properties stated in the following lemma, which are easily derived from well-known results. (See the appendix for details.)

LEMMA 2. *Consider the P. Hall basis $\mathcal{B}$ of Lie$(X_1,\dots,X_m)$ defined by* (12). *Then there exist mappings $(T,u) \longmapsto c_i(T,u)$ such that, for any set $\mathbf{g} = \{g_1,\dots,g_m\}$ of v.f. nilpotent of order $d+1$, and any $u \in \mathcal{C}^\infty([0,T];\mathbb{R}^{n(d)})$, the solution at time T of*

$$\dot{x} = \sum_{i=1}^{n(d)} u_i(t)\, g_i(x)\,, \qquad x(0) = x_0, \tag{32}$$

is

$$x(T) = \exp\left(\sum_{i=1}^{n(d)} c_i(T,u)\, g_i\right) x_0, \tag{33}$$

where $g_i \overset{\Delta}{=} Ev_{\mathbf{g}}(B_i)\,(i = m+1,\dots,n(d))$. Furthermore, if $g_1,\dots,g_m$ are the control v.f. of the ($n(d)$-dimensional) free system $S(m,d)$, then for any $x_0 \in \mathbb{R}^{n(d)}$ the mapping

$$(c_1,\dots,c_{n(d)}) \longmapsto \exp\left(\sum_{i=1}^{n(d)} c_i\, g_i\right) x_0 \tag{34}$$

from $\mathbb{R}^{n(d)}$ to $\mathbb{R}^{n(d)}$ is one-to-one.

Applying the first result stated in the lemma to (30) yields

$$\forall k = 1,2, \qquad z_{\gamma_k}(1) = \exp\left(\sum_{i=1}^{n(d)} c_i\left(1,\bar{U}(\gamma_k,\dot{\gamma}_k)\right)\, \tilde{b}_i\right) z_0\,. \tag{35}$$

Consider now the following equation parameterized by $k = 1,2$ (compare with (30)):

$$\dot{x} = \sum_{i=1}^{n(d)} \bar{U}_i(\gamma_k(t),\dot{\gamma}_k(t))\; b_i(x), \qquad x(0) = f(\theta_0)\,. \tag{36}$$

From (28) and (31), $f(\gamma_k(.))$ is a solution to (36). Therefore, applying the first result stated in the lemma to this equation and using the fact that $f(\theta) = f(\gamma_k(1))$ for $k = 1,2$ yields

$$\exp\left(\sum_{i=1}^{n(d)} c_i\left(1,\bar{U}(\gamma_1,\dot{\gamma}_1)\right)\; b_i\right) f(\theta_0) = \exp\left(\sum_{i=1}^{n(d)} c_i\left(1,\bar{U}(\gamma_2,\dot{\gamma}_2)\right)\; b_i\right) f(\theta_0)\,.$$

The second result stated in the lemma then implies that

$$\forall i = 1, \dots, n(d), \qquad c_i\left(1, \bar{U}(\gamma_1, \dot{\gamma}_1)\right) = c_i\left(1, \bar{U}(\gamma_2, \dot{\gamma}_2)\right), \tag{37}$$

and it follows, in view of (35), that $z_{\gamma_1}(1) = z_{\gamma_2}(1)$. This in turn establishes that the mapping $(\theta, \gamma) \to z_\gamma(1)$ is a function of θ solely. This is the function $\tilde{f}$ which we were looking for. At this point, it remains only to verify that the function $\tilde{f}$ so defined satisfies the transversality condition (6) for S_{hom}. Recalling that $\tilde{f}(\theta)$ is obtained as the solution of (30) at $t = 1$ and that this solution does not depend on the path γ which passes thru θ at time $t = 1$, one deduces that along any smooth curve $\theta(.)$ the mapping $t \longmapsto \tilde{f}(\theta(t))$ is differentiable with

$$\frac{d}{dt}\tilde{f}(\theta(t)) = \sum_{i=1}^{n(d)} \bar{U}_i(\theta(t), \dot{\theta}(t))\, \tilde{b}_i(\tilde{f}(\theta(t)))\,.$$

This in turn implies that $\tilde{f}$ is smooth and satisfies

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \quad \frac{\partial \tilde{f}}{\partial \theta_j}(\theta) = \sum_{i=1}^{n(d)} u_{i,j}(\theta)\, \tilde{b}_i(\tilde{f}(\theta))\,. \tag{38}$$

This implies that

$$\begin{aligned}\left(\tilde{b}_1(\tilde{f}(\theta)), \dots, \tilde{b}_m(\tilde{f}(\theta)), \frac{\partial \tilde{f}}{\partial \theta_{m+1}}(\theta), \dots, \frac{\partial \tilde{f}}{\partial \theta_{n(d)}}(\theta)\right)& \\ = \left(\tilde{b}_1(\tilde{f}(\theta)), \dots, \tilde{b}_{n(d)}(\tilde{f}(\theta))\right)&\begin{pmatrix} I_m & \star \\ 0 & U(\theta)\end{pmatrix},\end{aligned}$$

where $I_m \in \mathbb{R}^{m\times m}$ is the identity matrix. Using (29) and the fact that S_{hom} satisfies $LARC(x)$ for $x \in \mathbb{R}^n$—indeed, it satisfies $LARC(0)$ so that, by continuity it satisfies $LARC(x)$ in a neighborhood of the origin and therefore, by homogeneity, in $\mathbb{R}^n$ itself—one easily deduces from the above equality that $\tilde{f}$ satisfies the transversality condition (6) for S_{hom}. □

2.2.4. Proof of Proposition 4. From Remark 4 and property 2 of Lemma 1, it is sufficient to prove the existence of a single function $f \in \mathcal{C}^\infty(\mathbb{T}^{n(d)-m}; \mathbb{R}^{n(d)})$ for which the transversality condition (6) is satisfied. In order to simplify some of the forthcoming analysis, we will use the formalism of differential forms, from which condition (6) can be written as

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \quad (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{n(d)})\left(b_1, \dots, b_m, \frac{\partial f}{\partial \theta_{m+1}}, \dots, \frac{\partial f}{\partial \theta_{n(d)}}\right)_{|x=f(\theta)} \neq 0.$$

By skew-symmetry of the wedge product, this is equivalent to the condition that

$$\begin{aligned}&\forall \theta \in \mathbb{T}^{n(d)-m}, \\ &(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m \wedge \omega^x_{m+1} \wedge \cdots \wedge \omega^x_{n(d)})\left(b_1, \dots, b_m, \frac{\partial f}{\partial \theta_{m+1}}, \dots, \frac{\partial f}{\partial \theta_{n(d)}}\right)_{|x=f(\theta)} \neq 0,\end{aligned} \tag{39}$$

where $\omega_i^x = \mathrm{d}x_i - x_{\lambda(i)}\mathrm{d}x_{\rho(i)}$ $(i = m+1, \dots, n(d))$. From (17),

$$\forall j = 1, \dots, m \quad \begin{cases} \mathrm{d}x_i(b_j) &= \delta_i^j \quad \text{if } i \in \{1, \dots, m\}, \\ \omega_i^x(b_j) &= 0 \quad \text{if } i \in \{m+1, \dots, n(d)\} \end{cases}$$

so that one easily rewrites (39) as

$$\forall \theta \in \mathbb{T}^{n(d)-m}, \quad \left(\omega_{m+1} \wedge \cdots \wedge \omega_{n(d)}\right)(\theta) \neq 0, \tag{40}$$

with ω_i the differential one-form on $\mathbb{T}^{n(d)-m}$ defined by

$$\omega_i = \mathbf{d}f_i - f_{\lambda(i)}\mathbf{d}f_{\rho(i)}\,. \tag{41}$$

Design algorithm. The function f is defined by setting $f \triangleq f^{n(d)}$, with the function $f^{n(d)}$ denoting the last function obtained via a recursive construction which starts with some function f^{m+1}. For each $k = m+1, \dots, n(d)$, the function $f^k \in \mathcal{C}^\infty(\mathbb{T}^{k-m}; \mathbb{R}^{n(d)})$ is required to verify the following property:

$$\forall \theta^k = (\theta_{m+1}, \dots, \theta_k) \in \mathbb{T}^{k-m}, \quad \left(\omega_{m+1}^k \wedge \cdots \wedge \omega_k^k\right)(\theta^k) \neq 0\,, \tag{42}$$

with ω_i^k the differential one-form on $\mathbb{T}^{k-m}$ defined by

$$\omega_i^k = \mathbf{d}f_i^k - f_{\lambda(i)}^k\mathbf{d}f_{\rho(i)}^k\,. \tag{43}$$

$\boldsymbol{f^{m+1}}$. A possible choice for f^{m+1} is as follows:

$$f_i^{m+1}(\theta_{m+1}) = \begin{cases} \sin\theta_{m+1} & \text{for } i = \lambda(m+1), \\ \cos\theta_{m+1} & \text{for } i = \rho(m+1), \\ \dfrac{1}{4}\sin 2\theta_{m+1} & \text{for } i = m+1, \\ 0 & \text{otherwise}\,. \end{cases} \tag{44}$$

Indeed, it readily follows from this definition that

$$\forall \theta^{m+1} \in \mathbb{T}, \quad \omega_{m+1}^{m+1}(\theta^{m+1}) = \frac{1}{2}\,.$$

$\boldsymbol{f^{k-1} \longrightarrow f^k}$. Assume now that, for some $k-1 \in \{m+1, \dots, n(d)-1\}$, a function $f^{k-1} \in \mathcal{C}^\infty(\mathbb{T}^{k-1-m}; \mathbb{R}^{n(d)})$ which verifies the property (42) for $k-1$ has been obtained. We show below how to construct from this function a new function $f^k \in \mathcal{C}^\infty(\mathbb{T}^{k-m}; \mathbb{R}^{n(d)})$ which verifies the property (42).

Let Δ_μ^k $(\mu > 0)$ denote the dilation defined on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^{n(d)}$ by

(45)
$$\Delta_\mu^k(s, c, f) = \left(\mu^{\ell(\lambda(k))}s, \mu^{\ell(\rho(k))}c, \Delta_\mu(f)\right) \text{ with } \Delta_\mu(f) \triangleq \left(\mu^{\ell(1)}f_1, \dots, \mu^{\ell(n(d))}f_{n(d)}\right).$$

Denote also p_i^k $(i = 1, \dots, n(d))$ the functions defined on $\mathbb{R} \times \mathbb{R}$ by

$$p_i^k(s, c) = s\,\delta_i^{\lambda(k)} + c\,\delta_i^{\rho(k)} + \frac{m_k^k}{2}sc\,\delta_i^k \tag{46}$$

with

$$m_i^k = \begin{cases} 0 & \text{if } \ell(i) \le \ell(\lambda(k)) \text{ or } \lambda(i) \ne \lambda(k), \\ 1 + m_{\rho(i)}^k & \text{otherwise}. \end{cases} \tag{47}$$

The next step consists in finding polynomial functions $q_{i,j}^k \in \mathcal{C}^\infty(\mathbb{R}^{n(d)};\mathbb{R})$ for $i = 1,\dots,n(d)$ and $j = 1,\dots,j_{i,k} \triangleq \max\{j : \ell(i) - j\ell(\lambda(k)) \ge 0\}$) such that the two following properties are verified.

P1(i) (for $i = 1,\dots,n(d)$). Each function $q_{i,j}^k$ is Δ-homogeneous of degree $\ell(i) - j\ell(\lambda(k))$.

P2(i) (for $i = m+1,\dots,k$).

$$\bar{\omega}_i^k = \left(\mathrm{d}f_i - f_{\lambda(i)}\mathrm{d}f_{\rho(i)} + \bar{\gamma}_i^k\right) + \sum_{j=m+1}^{i-1} t_{i,j}(s,f)\left(\mathrm{d}f_j - f_{\lambda(j)}\mathrm{d}f_{\rho(j)} + \bar{\gamma}_j^k\right), \tag{48}$$

where

$$\forall i = m+1,\dots,k, \qquad \bar{\omega}_i^k \triangleq \mathbf{d}\bar{f}_i^k - \bar{f}_{\lambda(i)}^k \mathbf{d}\bar{f}_{\rho(i)}^k, \tag{49}$$

$$\bar{f}_i^k : (s,c,f) \longmapsto f_i + p_i^k(s,c) + \sum_{j=1}^{j_{i,k}} s^j q_{i,j}^k(f), \tag{50}$$

the $t_{i,j}$'s are smooth functions, and $\bar{\gamma}_i^k$ is a differential one-form on $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{n(d)}$ such that

$$\bar{\gamma}_i^k = \bar{\gamma}_{i,1}^k \mathrm{d}s + \bar{\gamma}_{i,2}^k \mathrm{d}c$$

with $\bar{\gamma}_{i,1}$, $\bar{\gamma}_{i,2}$, Δ^k-homogeneous of degree $\ell(i)-\ell(\lambda(k))$ and $\ell(i)-\ell(\rho(k))$, respectively, and

$$\begin{cases} \bar{\gamma}_{i,1}^k \equiv 0 & \text{if } i < \lambda(k), \\ \bar{\gamma}_{i,1}^k \equiv 1 & \text{if } i = \lambda(k), \\ \bar{\gamma}_{i,1}^k(s,c,0) = 0 & \text{if } \lambda(k) < i < k, \\ \bar{\gamma}_{i,1}^k(s,c,0) = \dfrac{m_k^k}{2}\, c & \text{for } i = k, \end{cases} \qquad \begin{cases} \bar{\gamma}_{i,2}^k \equiv 0 & \text{if } i < \rho(k), \\ \bar{\gamma}_{i,2}^k \equiv 1 & \text{if } i = \rho(k), \\ \bar{\gamma}_{i,2}^k(s,c,0) = 0 & \text{if } \rho(k) < i < k, \\ \bar{\gamma}_{i,2}^k(s,c,0) = -\dfrac{m_k^k}{2}\, s & \text{for } i = k. \end{cases} \tag{51}$$

LEMMA 3. *There exist functions $q_{i,j}^k$, which are solutions to the problems* **P1(i)** *and* **P2(i)**. *In particular, one can always choose*

$$\begin{cases} q_{i,j}^k \equiv 0 & \textit{if } i \in \{1,\dots,\mathrm{Max}\{m,\lambda(k)\}\} \cup \{k+1,\dots,n(d)\} \\ & \textit{and} \quad j \in \{1,\dots,j_{i,k}\}, \\ q_{i,1}^k \equiv 0 & \textit{if } \mathrm{Max}\{m,\lambda(k)\} < i \le k \textit{ and } \quad \lambda(i) < \lambda(k), \\ q_{i,1}^k(f) = m_i^k f_{\rho(i)} & \textit{if } \mathrm{Max}\{m,\lambda(k)\} < i \le k \textit{ and } \quad \lambda(i) = \lambda(k). \end{cases} \tag{52}$$

Once suitable functions $q_{i,j}^k$ are determined so that the functions $\bar{f}_i^k$ in (50) are also defined, we set

$$f^k \triangleq \bar{f}^k \circ \bar{g}_\eta^k \quad \text{with} \quad \bar{g}_\eta^k(\theta^k) \triangleq \left(\eta^{\ell(\lambda(k))}\sin\theta_k, \eta^{\ell(\rho(k))}\cos\theta_k, f^{k-1}(\theta^{k-1})\right). \tag{53}$$

LEMMA 4. *For η larger than some positive value η_0,* (42) *is satisfied with the function f^k defined by* (53)*.*

Therefore, Proposition 4 is proved once Lemmas 3 and 4 are proved.

REMARK 5. *It is simple to verify that each function $\bar{f}_i^k$ in* (53) *is polynomial in its arguments and Δ^k-homogeneous of degree $\ell(i)$ with respect to the dilation defined by* (45)*. The proof of the lemmas much relies on this property.*

Proof of Lemma 3. We distinguish three cases.

Case 1. $1 \leq i \leq \mathrm{Max}\{m, \lambda(k)\}$. We define $q_{i,j}^k$ according to (52) so that $\mathbf{P1}(\boldsymbol{i})$ is clearly verified for these values of i. If $i \leq m$, $\mathbf{P2}(\boldsymbol{i})$ is irrelevant. If $m+1 \leq i \leq \lambda(k)$, it readily follows from (46), (49), (50), and (52) that

$$\bar{\omega}_i^k = \mathrm{d}f_i - f_{\lambda(i)}\mathrm{d}f_{\rho(i)} + \bar{\gamma}_i^k, \tag{54}$$

where $\bar{\gamma}_i^k \equiv 0$ if $i < \lambda(k)$ and $\bar{\gamma}_i^k = \mathrm{d}s$ if $i = \lambda(k)$. Therefore, $\mathbf{P2}(\boldsymbol{i})$ is also verified.

Case 2. $\mathrm{Max}\{m, \lambda(k)\} < i \leq k$. We define $q_{i,1}^k$ according to (52), which is consistent with $\mathbf{P1}(\boldsymbol{i})$. To define the other functions $q_{i,j}^k$, we consider a construction which is recursive in the index i. More precisely, let us assume that functions $q_{1,j}^k, \ldots, q_{i-1,j}^k$ have been defined so that $\mathbf{P1(1)}, \ldots, \mathbf{P1}(\boldsymbol{i}\text{-}\mathbf{1})$ and $\mathbf{P2(1)}, \ldots, \mathbf{P2}(\boldsymbol{i}\text{-}\mathbf{1})$ are verified. We show below how to obtain functions $q_{i,j}^k$ so that $\mathbf{P1}(\boldsymbol{i})$ and $\mathbf{P2}(\boldsymbol{i})$ are also verified.

We first note that

$$\lambda(i) < \rho(k)\,. \tag{55}$$

Assume, on the contrary, that $\lambda(i) \geq \rho(k)$. Then, from the definition of a P. Hall basis, $\lambda(i) < \rho(i)$. This implies that

$$\ell(i) = \ell(\lambda(i)) + \ell(\rho(i)) \geq 2\ell(\rho(k)) \geq \ell(k)\,.$$

If $\ell(i) > \ell(k)$, then $i > k$, and this contradicts the assumption. Otherwise, $\ell(i) = \ell(k)$, and we also get $i > k$ because of (11) and the fact that $\lambda(i) \geq \rho(k) > \lambda(k)$.

We introduce the following definitions for the sake of simplifying some aspects of the forthcoming analysis.

DEFINITION 2. *A differential one-form $r = r_s\mathrm{d}s + r_c\mathrm{d}c + \sum_{j=1}^{n(d)} r_j\mathrm{d}f_j$, with r_s, r_c, r_j homogeneous of degree $\ell(i) - \ell(\lambda(k))$, $\ell(i) - \ell(\rho(k))$, and $\ell(i) - \ell(j)$, respectively, is said to be of*

- *type* 1 *if $r_j \equiv 0$ for each j, and both r_s and r_c are identically zero at $f = 0$;*
- *type* 2 *if $r_c \equiv r_j \equiv 0$ for each j, and $r_s = as^\kappa$ with $a \in \mathbb{R}$ and $1 \leq \kappa \in \mathbb{N}$;*
- *type* 3 *if $r_s \equiv r_c \equiv 0$ and, for each j, $r_j(s,c,f)$ is in the form $r_j(s,c,f) = s^{2+\kappa_j}r'_j(f)$ with $\kappa_j \in \mathbb{N}$.*

An upper-left index i for a one-form will indicate its type, e.g., ${}^2r$ indicates that ${}^2r$ is of type 2*.*

Next, we develop $\bar{\omega}_i^k$ and examine the terms involved in this development. From (49) and (50), we have

(56)

$$\begin{aligned}\bar{\omega}_i^k = \quad & \mathrm{d}f_i + \mathbf{d}p_i^k + \mathbf{d}\left(\sum_{j=1}^{j_{i,k}} s^j q_{i,j}^k\right)\\ & - \left(f_{\lambda(i)} + p_{\lambda(i)}^k + \sum_{j=1}^{j_{\lambda(i),k}} s^j q_{\lambda(i),j}^k\right)\left(\mathrm{d}f_{\rho(i)} + \mathbf{d}p_{\rho(i)}^k + \mathbf{d}\left(\sum_{j=1}^{j_{\rho(i),k}} s^j q_{\rho(i),j}^k\right)\right)\end{aligned}$$

and, by rearranging the terms in the right-hand side of this equality,

$$\bar{\omega}_i^k = \mathrm{d}f_i - f_{\lambda(i)}\mathrm{d}f_{\rho(i)} + \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q_{i,j}^k\right) + \alpha_1 + \alpha_2 + \alpha_3 + {}^1 r + {}^2 r + {}^3 r \tag{57}$$

with

$$\begin{cases} \alpha_1 \triangleq \mathbf{d}p_i^k - p_{\lambda(i)}^k \mathbf{d}p_{\rho(i)}^k, \\ \alpha_2 \triangleq s\mathbf{d}q_{i,1}^k - sq_{\lambda(i),1}^k \mathrm{d}f_{\rho(i)} - sf_{\lambda(i)}\mathbf{d}q_{\rho(i),1}^k - p_{\lambda(i)}^k \mathrm{d}f_{\rho(i)}, \\ \alpha_3 \triangleq -\mathbf{d}p_{\rho(i)}^k \displaystyle\sum_{j=2}^{j_{\lambda(i),k}} s^j q_{\lambda(i),j}^k\,. \end{cases} \tag{58}$$

In (57), ${}^1r$, ${}^2r$, and ${}^3r$ just correspond to terms which do not need to be specified further and are of type 1, 2, and 3, following Definition 2. In order to obtain (57), we have used the following two arguments: (i) each function $q_{j,1}^k$ ($j \le i$) vanishes at the origin—this follows from (52) if $\lambda(j) \le \lambda(k)$; otherwise, $\lambda(j) > \lambda(k)$ so that $\ell(j) > \ell(\lambda(k))$, and this follows from the fact that $q_{j,1}^k$ is Δ^k-homogeneous of positive degree; (ii) from (55), $\lambda(i) < \rho(k)$ so that (46) implies that $p_{\lambda(i)}^k(s,c)$ is either s or zero. Note also that the homogeneity properties of the components of ${}^1r$, ${}^2r$, and ${}^3r$ follow directly from the homogeneity of $\bar{f}_i^k$ (see Remark 5).

Let us now focus our attention on the terms α_i which are specified in (58). We first note that

$$\alpha_3 \equiv 0\,. \tag{59}$$

Indeed, assume on the contrary that α_3 is not the null function. Then, in view of (52), it is necessary that $\lambda(i) > \lambda(k)$. (Otherwise, $q_{\lambda(i),j}^k$, and thus α_3, would be equal to zero.) Since $\lambda(i) < \rho(i)$ (from the definition of a P. Hall basis), we also have $\rho(i) \ge \rho(k)$. (Otherwise, $p_{\rho(i)}^k$, and thus α_3, would be equal to zero.) This implies that $i > k$, which is in contradiction with the assumption.

We now consider the term α_2 in (58). We have

$$\lambda(i) < \lambda(k) \Longrightarrow \alpha_2 \equiv 0\,. \tag{60}$$

This follows from (46) and (52) after noticing that either $\ell(\rho(i)) = 1$ so that $q_{\rho(i),1}^k \equiv 0$, or $\ell(\rho(i)) > 1$ and $\lambda\rho(i) \le \lambda(i) < \lambda(k)$ (from the definition of a P. Hall basis), so that we still obtain $q_{\rho(i),1}^k \equiv 0$. Then

$$\left.\lambda(i) = \lambda(k) \quad \text{with} \quad \begin{matrix} \ell(\rho(i)) = 1 \\ \text{or} \\ \lambda\rho(i) < \lambda(k) \end{matrix}\right\} \Longrightarrow \alpha_2 \equiv 0\,. \tag{61}$$

Indeed, if the left-hand side of the above implication holds, then (46), (47), and (52) imply

$$\begin{aligned} \alpha_2 &= s\left(m_i^k \mathrm{d}f_{\rho(i)} - f_{\lambda(i)}\mathbf{d}q_{\rho(i),1}^k - \mathrm{d}f_{\rho(i)}\right) \\ &= s\left(m_i^k \mathrm{d}f_{\rho(i)} - \mathrm{d}f_{\rho(i)}\right) \\ &\equiv 0\,. \end{aligned} \tag{62}$$

From the definition of a P. Hall basis, $\lambda\rho(i) \leq \lambda(i)$ so that the case where $\lambda(i) = \lambda(k)$ with $\lambda\rho(i) > \lambda(k)$ cannot happen. Therefore, if $\lambda(i) = \lambda(k)$, the last possible case is $\lambda\rho(i) = \lambda(k)$. We have

$$\left.\lambda(i) = \lambda(k) \text{ and } \lambda\rho(i) = \lambda(k)\right\} \Longrightarrow \alpha_2 = s\, m^k_{\rho(i)} \left(\mathrm{d}f_{\rho(i)} - f_{\lambda\rho(i)}\mathrm{d}f_{\rho^2(i)}\right). \tag{63}$$

Indeed, from (52),

$$\begin{aligned}\alpha_2 &= s\left(m_i^k \mathrm{d}f_{\rho(i)} - f_{\lambda\rho(i)}\mathrm{d}q^k_{\rho(i),1} - \mathrm{d}f_{\rho(i)}\right)\\ &= s\left(m_i^k \mathrm{d}f_{\rho(i)} - f_{\lambda\rho(i)} m^k_{\rho(i)}\mathrm{d}f_{\rho^2(i)} - \mathrm{d}f_{\rho(i)}\right),\end{aligned}$$

and (63) follows from (47). Concerning α_2, there remains only to examine the case where $\lambda(i) > \lambda(k)$. In this case $p^k_{\lambda(i)} \equiv 0$—since, by (55), $\lambda(i) < \rho(k)$—so that

$$\alpha_2 = s\left(\mathbf{d}q^k_{i,1} - q^k_{\lambda(i),1}\mathrm{d}f_{\rho(i)} - f_{\lambda(i)}\mathbf{d}q^k_{\rho(i),1}\right). \tag{64}$$

Each term within the above parentheses is a sum of terms $p_{i,j}(f)\mathrm{d}f_j$, where each $p_{i,j}$ is homogeneous of degree $\ell(i) - \ell(\lambda(k)) - \ell(j)$. By applying property 4 in Lemma 1 to the term $q^k_{\lambda(i),1}\mathrm{d}f_{\rho(i)} + f_{\lambda(i)}\mathbf{d}q^k_{\rho(i),1}$ and by replacing x with f in Lemma 1, we obtain

$$\alpha_2 = s\left(\mathbf{d}q^k_{i,1} - \mathbf{d}h_1 + \sum_{1<\ell(j)\leq\ell(i)-\ell(\lambda(k))} h_{2,j}(f)\left(\mathrm{d}f_j - f_{\lambda(j)}\mathrm{d}f_{\rho(j)}\right)\right)$$

for some functions h_1 and $h_{2,j}$ Δ^k-homogeneous of degree $\ell(i) - \ell(\lambda(k))$ and $\ell(i) - \ell(\lambda(k)) - \ell(j)$, respectively. Furthermore, by choosing

$$q^k_{i,1} = h_1 \tag{65}$$

(this choice is clearly consistent with $\mathbf{P1}(\boldsymbol{i})$), we get

$$\alpha_2 = s \sum_{1<\ell(j)\leq\ell(i)-\ell(\lambda(k))} h_{2,j}(f)\left(\mathrm{d}f_j - f_{\lambda(j)}\mathrm{d}f_{\rho(j)}\right). \tag{66}$$

From what precedes, we finally obtain

(67)

$$\alpha_2 = \begin{cases} s\displaystyle\sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h_{2,j}(f)\left(\mathrm{d}f_j - f_{\lambda(j)}\mathrm{d}f_{\rho(j)} + \bar{\gamma}^k_j\right) - s\sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h_{2,j}(f)\bar{\gamma}^k_j & \text{if } i < k,\\ s\, m^k_{\rho(k)}\left(\mathrm{d}f_{\rho(k)} - f_{\lambda\rho(k)}\mathrm{d}f_{\rho^2(k)} + \bar{\gamma}^k_{\rho(k)}\right) - s\, m^k_{\rho(k)}\bar{\gamma}^k_{\rho(k)} & \text{if } i = k.\end{cases}$$

The second equation is a consequence of (63) when $\lambda\rho(k) = \lambda(k)$, and of (47) and (61) otherwise. As for the first equation, we argue as follows. If $\lambda(i) < \lambda(k)$, the result follows directly from (60) with $h_{2,j} \equiv 0$. If $\lambda(i) = \lambda(k)$ so that $\rho(i) < \rho(k)$, the result follows from (61) or (63). Finally, if $\lambda(i) > \lambda(k)$, then, by (11) and the assumption $i < k$, $\ell(i) < \ell(k)$, so that $\ell(i) - \ell(\lambda(k)) < \ell(\rho(k))$, and the result follows from (66).

Let us now consider the term ${}^3r$ in (57). From Definition 2, ${}^3r$ is a sum of one-forms $s^{2+\kappa_j} r'_j \mathrm{d}f_j$, where each r'_j is a polynomial function of f, Δ^k-homogeneous of degree

$$\ell(i) - \ell(j) - (2+\kappa_j)\ell(\lambda(k)) < \min\{\ell(i) - \ell(j), \ell(\rho(k)) - \ell(j)\}.$$

By applying property 4 in Lemma 1 to each one-form $r'_j \mathrm{d}f_j$, we get

$$(68)\quad {}^3r = s^2\left(\sum_j s^{\kappa_j}\mathbf{d}h_{1,j} + \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h'_{2,j}(s,f)\left(\mathrm{d}f_j - f_{\lambda(j)}\mathrm{d}f_{\rho(j)} + \bar{\gamma}_j^k\right) - \sum_{j=m+1}^{\min\{i,\rho(k)\}-1} h'_{2,j}(s,f)\bar{\gamma}_j^k\right),$$

where the functions $h_{1,j}$ are Δ^k-homogeneous of positive degree and therefore vanish at the origin.

We can now define the functions $q_{i,j}^k$. Let us note that $q_{i,1}^k$ has already been defined by (52) if $\lambda(i) \leq \lambda(k)$ and by (65) otherwise. For the definition of $q_{i,j}^k$ with $j > 1$, we distinguish two cases according to whether i is smaller than or equal to k.

If $i < k$, by using (59), (67), and (68), relation (57) can be rewritten in the form (48), with

(69)

$$\bar{\gamma}_i^k = \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q_{i,j}^k\right) + \alpha_1 + {}^1r + {}^2r - s\sum_{j=m+1}^{\min\{i,\rho(k)\}-1}\left(h_{2,j} + sh'_{2,j}\right)(s,f)\bar{\gamma}_j^k + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j}$$

and smooth functions $t_{i,j}$ which we do not need to specify further. The functions $h_{2,j}$ and $sh'_{2,j}$, involved in the above expression of $\bar{\gamma}_i^k$, are polynomial in s and f. From the induction hypothesis and (51), the $\bar{\gamma}_j^k$'s in the right-hand side of (69) are such that $\bar{\gamma}_j^k = \bar{\gamma}_{j,1}^k \mathrm{d}s$ because $j < \rho(k)$. Furthermore, $\bar{\gamma}_{j,1}^k$ depends on s and f only because it is homogeneous of degree $\ell(j) - \ell(\lambda(k)) \leq \ell(\rho(k))$, and $\bar{\gamma}_{j,1}^k(s,c,0) = 0$. As a consequence, we have

$$(70)\qquad -s\sum_{j=m+1}^{\min\{i,\rho(k)\}-1}\left(h_{2,j} + sh'_{2,j}\right)(s,f)\bar{\gamma}_j^k = sh'(s,f)\mathrm{d}s = a_0 s^{\kappa'}\mathrm{d}s + h''\mathrm{d}s$$

with $a_0 \in \mathbb{R}$, $1 \leq \kappa' \in \mathbb{N}$, h' and h'' functions of s and f only, and h'' identically zero when $f = 0$. From Definition 2, (70) can be rewritten as

$$(71)\qquad -s\sum_{j=m+1}^{\min\{i,\rho(k)\}-1}\left(h_{2,j} + sh'_{2,j}\right)(s,f)\bar{\gamma}_j^k = {}^1r' + {}^2r'.$$

From (46), (58), and the fact that $i < k$ implies that either $\lambda(i) < \lambda(k)$ or $\lambda(k) \leq \lambda(i) < \rho(i) < \rho(k)$, we deduce that $\alpha_1 = \mathbf{d}p_i^k$. Therefore, by using (71) in (69),

$$(72)\qquad \bar{\gamma}_i^k = \mathbf{d}\left(\sum_{j=2}^{j_{i,k}} s^j q_{i,j}^k\right) + \mathbf{d}p_i^k + {}^1r'' + \mathbf{d}(as^{2+\kappa}) + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j},$$

where we have used the fact that any function of type 2 is the differential of a polynomial as^q with $q \geq 2$. From there, the functions $q_{i,j}^k$ ($j > 1$) are uniquely defined by setting

$$(73)\qquad \sum_{j=2}^{j_{i,k}} s^j q_{i,j}^k \triangleq -as^{2+\kappa} - \sum_j s^{2+\kappa_j} h_{1,j}.$$

It is simple to check that $\mathbf{P1(}\boldsymbol{i}\mathbf{)}$ is verified with this choice. This yields, in view of (72),

$$\bar{\gamma}_i^k = \mathbf{d}p_i^k +^1 r'' - \sum_j h_{1,j}\mathbf{d}\left(s^{2+\kappa_j}\right) = \mathbf{d}p_i^k +^1 r''',$$

where the last equality comes from the fact that $h_{1,j}(0) = 0$, as mentioned after (68). By using the definition of one-forms of type 1, it follows that (51) is satisfied and thus that $\mathbf{P2(}\boldsymbol{i}\mathbf{)}$ is verified—note that, if ${}^1r''' = r_s\mathrm{d}s + r_c\mathrm{d}c$ and $i \leq \rho(k)$, then r_c is homogeneous of nonpositive degree so that it is necessarily a constant, which in fact is equal to zero since r_c vanishes at $f = 0$.

For the last case, $i = k$, we proceed similarly. By using (59), (67), and (68), relation (57) can again be rewritten in the form (48), this time with

$$\bar{\gamma}_k^k = \mathbf{d}\left(\sum_{j=2}^{j_{k,k}} s^j q_{k,j}^k\right) + \alpha_1 - s m_{\rho(k)}^k \bar{\gamma}_{\rho(k)}^k +^1 r +^2 r - s^2 \sum_{j=m+1}^{\rho(k)-1} h_{2,j}(s,f)\bar{\gamma}_j^k + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j} \tag{74}$$

instead of (69). From (46), (47), (58), and the induction hypothesis $\mathbf{P2(}\rho(\boldsymbol{k})\mathbf{)}$ if $\rho(k) > m$,

$$\begin{aligned} \alpha_1 - s\, m_{\rho(k)}^k \bar{\gamma}_{\rho(k)}^k &= \alpha_1 - s\, m_{\rho(k)}^k \mathrm{d}c - s\, m_{\rho(k)}^k \bar{\gamma}_{\rho(k),1}^k \mathrm{d}s \\ &= \frac{m_k^k}{2}(c\mathrm{d}s - s\mathrm{d}c) - s\, m_{\rho(k)}^k \bar{\gamma}_{\rho(k),1}^k \mathrm{d}s\,. \end{aligned} \tag{75}$$

If $\rho(k) \leq m$ so that $\lambda(k) < \rho(k) \leq m < k$, these equalities are still valid since (47) implies that $m_{\rho(k)}^k = 0$. Using (75), (74) rewrites as

$$\begin{aligned} \bar{\gamma}_k^k = \mathbf{d}\left(\sum_{j=2}^{j_{k,k}} s^j q_{k,j}^k\right) + \frac{m_k^k}{2}(c\mathrm{d}s - s\mathrm{d}c) +^1 r +^2 r - s^2 \sum_{j=m+1}^{\rho(k)-1} h_{2,j}(s,f)\bar{\gamma}_j^k \\ - s\, m_{\rho(k)}^k \bar{\gamma}_{\rho(k),1}^k \mathrm{d}s + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j}. \end{aligned} \tag{76}$$

From here, we proceed as for the previous case in order to rewrite the above equation as (compare with (72))

$$\bar{\gamma}_k^k = \mathbf{d}\left(\sum_{j=2}^{j_{k,k}} s^j q_{k,j}^k\right) + \frac{m_k^k}{2}(c\mathrm{d}s - s\mathrm{d}c) +^1 r'' + \mathbf{d}(a s^{2+\kappa}) + \sum_j s^{2+\kappa_j}\mathbf{d}h_{1,j}\,. \tag{77}$$

Using again (73) to define the functions $q_{k,j}^k$ $(j > 1)$ yields

$$\bar{\gamma}_k^k = \frac{m_k^k}{2}(c\mathrm{d}s - s\mathrm{d}c) +^1 r''',$$

and it is simple to check that the one-form $\bar{\gamma}_k^k$ satisfies (51) so that $\mathbf{P2(}\boldsymbol{i}\mathbf{)}$ is verified. This ends the study of Case 2.

Case 3. $k < i \leq n(d)$. We define $q_{i,j}^k \equiv 0$ according to (52) so that both $\mathbf{P1(}\boldsymbol{i}\mathbf{)}$ and $\mathbf{P2(}\boldsymbol{i}\mathbf{)}$ are readily verified. This ends the proof of Lemma 3.

Proof of Lemma 4. Since $f^k = \bar{f}^k \circ \bar{g}^k_\eta$, we deduce from (43), (48), and (49) that, for $i \in \{m+1, \dots, k\}$,

$$\begin{aligned} \omega_i^k &= \bar{\omega}_i^k \circ \mathbf{d}\bar{g}^k_\eta \\ &= \left(\omega_i^{k-1} + \gamma_i^k \mathrm{d}\theta_k\right) + \sum_{j=m+1}^{i-1} t'_{i,j}\left(\omega_j^{k-1} + \gamma_j^k \mathrm{d}\theta_k\right), \end{aligned} \tag{78}$$

where

$$\gamma_i^k(\theta^k) = \bar{\gamma}^k_{i,1}(\bar{g}^k_\eta(\theta^k))\eta^{\ell(\lambda(k))}\cos\theta_k - \bar{\gamma}^k_{i,2}(\bar{g}^k_\eta(\theta^k))\eta^{\ell(\rho(k))}\sin\theta_k\,. \tag{79}$$

By skew-symmetry of the wedge product, it follows from (78) that

$$\omega^k_{m+1} \wedge \cdots \wedge \omega^k_k = \left(\omega^{k-1}_{m+1} + \gamma^k_{m+1}\mathrm{d}\theta_k\right) \wedge \cdots \wedge \left(\omega^{k-1}_k + \gamma^k_k \mathrm{d}\theta_k\right).$$

Since each ω_i^{k-1} is a one-form on $\mathbb{T}^{k-m-1}$, we deduce from the above equation (using multilinearity and skew-symmetry of the wedge product) that

$$\omega^k_{m+1} \wedge \cdots \wedge \omega^k_k = \sum_{i=m+1}^{k} \gamma_i^k \left(\omega^{k-1}_{m+1} \wedge \cdots \wedge \omega^{k-1}_{i-1} \wedge \mathrm{d}\theta_k \wedge \omega^{k-1}_{i+1} \wedge \cdots \wedge \omega^{k-1}_k\right). \tag{80}$$

From (45) and (53),

$$\begin{aligned} \bar{\gamma}^k_{i,1}(\bar{g}^k_\eta(\theta^k)) &= \bar{\gamma}^k_{i,1}(\Delta^k_\eta(\sin\theta_k, \cos\theta_k, \Delta_{1/\eta} f^{k-1}(\theta^{k-1}))) \\ &= \eta^{\ell(i)-\ell(\lambda(k))}\bar{\gamma}^k_{i,1}(\sin\theta_k, \cos\theta_k, \Delta_{1/\eta} f^{k-1}(\theta^{k-1})) \\ &= \eta^{\ell(i)-\ell(\lambda(k))}\bar{\gamma}^k_{i,1}(\sin\theta_k, \cos\theta_k, 0) + \sum_{j<\ell(i)-\ell(\lambda(k))} \eta^j \bar{\beta}_{i,j}(\theta^k), \end{aligned} \tag{81}$$

where the $\bar{\beta}_{i,j}$'s denote smooth functions on $\mathbb{T}^{k-m}$. The second equality in the above equation comes from the fact that $\bar{\gamma}^k_{i,1}$ is Δ^k-homogeneous of degree $\ell(i) - \ell(\lambda(k))$, and the third one from the fact that $\bar{\gamma}^k_{i,1}(s, c, f)$ is polynomial in s, c, and f. A similar calculation yields

$$\bar{\gamma}^k_{i,2}(\bar{g}^k_\eta(\theta^k)) = \eta^{\ell(i)-\ell(\rho(k))}\bar{\gamma}^k_{i,2}(\sin\theta_k, \cos\theta_k, 0) + \sum_{j<\ell(i)-\ell(\rho(k))} \eta^j \bar{\bar{\beta}}_{i,j}(\theta^k). \tag{82}$$

From (51), (79), (81), and (82),

$$\gamma_i^k(\theta^k) = \begin{cases} \eta^{\ell(k)}\dfrac{m_k^k}{2} + \displaystyle\sum_{1<j<\ell(k)} \eta^j \beta_{k,j}(\theta^k) & \text{if } i = k, \\ \displaystyle\sum_{1<j<\ell(k)} \eta^j \beta_{i,j}(\theta^k) & \text{otherwise} \end{cases} \tag{83}$$

for some smooth functions $\beta_{i,j}$ on $\mathbb{T}^{k-m}$. In view of (80) and (83),

$$\left(\omega^k_{m+1} \wedge \cdots \wedge \omega^k_k\right)(\theta^k) = \eta^{\ell(k)}\frac{m_k^k}{2}\left(\omega^{k-1}_{m+1} \wedge \cdots \wedge \omega^{k-1}_{k-1}\right)(\theta^{k-1}) + \sum_{1\le j<\ell(k)} \eta^j \beta'_{k,j}(\theta^k)$$

for some other smooth functions $\beta'_{k,j}$ on $\mathbb{T}^{k-m}$. By the compacity of $\mathbb{T}^{k-m}$ and the induction hypothesis, (42) follows when η is larger than some $\eta_0 > 0$. $\square$

3. Example. We illustrate the construction of transverse functions, as specified in the proof of Proposition 4, in the case of the free system $S(2,3)$ on $\mathbb{R}^5$. The associated truncated P. Hall basis is $\mathcal{B}_3 = \{B_1, \dots, B_5\}$, where

$$B_1 \triangleq X_1, B_2 \triangleq X_2, B_3 \triangleq [B_1, B_2] = [X_1, X_2], B_4 \triangleq [B_1, B_3], B_5 \triangleq [B_2, B_3]\,. \tag{84}$$

We have to compute $f = f^{n(d)} = f^5$, starting from $f^{m+1} = f^3$. From (14) and (84), $\lambda(3) = 1$ and $\rho(3) = 2$. Therefore, in view of (44),

$$f^3(\theta_3) = \left(\sin\theta_3, \cos\theta_3, \frac{\sin 2\theta_3}{4}, 0, 0\right)^T\,. \tag{85}$$

Let us now compute f^4 from f^3. From (14) and (84), $\lambda(4) = 1$ and $\rho(4) = 3$. Then (46), (47), (50), and (52) give

$$\bar{f}^4(s,c,x) = x + \begin{pmatrix} s \\ 0 \\ c \\ s\,c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ sq^4_{3,1}(x) + s^2 q^4_{3,2}(x) \\ sq^4_{4,1}(x) + s^2 q^4_{4,2}(x) + s^3 q^4_{4,3}(x) \\ 0 \end{pmatrix}. \tag{86}$$

From (52)

$$\begin{cases} q^4_{3,1}(x) = m^4_3\, x_{\rho(3)} = x_{\rho(3)} = x_2, \\ q^4_{4,1}(x) = m^4_4\, x_{\rho(4)} = 2x_{\rho(4)} = 2x_3. \end{cases} \tag{87}$$

Let us now proceed with the determination of $\bar{\omega}^4_3$, as defined by (49). Since $q^4_{3,2}$ is by definition homogeneous of degree $\ell(3) - 2\ell(1) = 0$, it is a constant function. A direct calculation gives

$$\bar{\omega}^4_3 = \mathrm{d}x_3 - x_1 \mathrm{d}x_2 + (x_2 + 2sq^4_{3,2})\mathrm{d}s + \mathrm{d}c\,.$$

With the simple choice

$$q^4_{3,2} \equiv 0\,, \tag{88}$$

consistent with **P1(3)**, it follows that (48) is verified with $\bar{\gamma}^4_3 \triangleq x_2\mathrm{d}s + \mathrm{d}c$, a one-form which satisfies the conditions in **P2(3)**. There remains to determine $q^4_{4,2}$ and $q^4_{4,3}$. Again, $q^4_{4,3}$ is homogeneous of degree zero, and thus it is a constant function. A simple calculation gives

$$\begin{aligned} \bar{\omega}^4_4 =& \mathrm{d}x_4 - x_1\mathrm{d}x_3 + s(\mathrm{d}x_3 - x_1\mathrm{d}x_2 + \bar{\gamma}^4_3) + s^2(\mathbf{d}q^4_{4,2} - \mathrm{d}x_2) - (x_1 + s)\mathrm{d}c \\ &+ (c + 2x_3 + 2sq^4_{4,2} + 3s^2 q^4_{4,3} - x_1x_2 - 2sx_2)\mathrm{d}s\,. \end{aligned}$$

The choice

$$q^4_{4,2}(x) = x_2\,, \quad q^4_{4,3} \equiv 0\,, \tag{89}$$

is clearly consistent with **P1(4)** and allows us to rewrite $\bar{\omega}_4^4$ in the form (48), with $\bar{\gamma}_4^4 \triangleq (c + 2x_3 - x_1x_2)\mathrm{d}s - (x_1 + s)\mathrm{d}c$ a one-form which satisfies the conditions in **P2(4)**. We finally obtain the following from (86), (87), (88), and (89):

$$\bar{f}^4(s, c, x) = x + \begin{pmatrix} s \\ 0 \\ c + sx_2 \\ s\,c + 2sx_3 + s^2x_2 \\ 0 \end{pmatrix}. \tag{90}$$

The expression of f^4 is then obtained by application of (53). As for the parameter η_4, it must be chosen large enough so that (42) is satisfied for $k = 4$. By inspection the (conservative) condition $\eta_4 \geq 5/2$ can be derived.

The determination of f^5 from f^4 is performed in the same way. We obtain (details are left to the reader)

$$\bar{f}^5(s, c, x) = x + (0, s, c, 0, s\,c/2 + sx_3)^T. \tag{91}$$

Then, (53) gives the expression of $f = f^5$. One verifies from (85), (90), and (91) that

$$f(\theta^5) = \begin{pmatrix} \sin\theta_3 + \eta_4 \sin\theta_4 \\ \cos\theta_3 + \eta_5 \sin\theta_5 \\ \frac{1}{4}\sin 2\theta_3 + \eta_4^2 \cos\theta_4 + \eta_4 \sin\theta_4 \cos\theta_3 + \eta_5^2 \cos\theta_5 \\ \frac{\eta_4^3}{2}\sin 2\theta_4 + \frac{\eta_4}{2}\sin\theta_4 \sin 2\theta_3 + \eta_4^2 \sin^2\theta_4 \cos\theta_3 \\ \frac{\eta_5^3}{4}\sin 2\theta_5 + \eta_5 \sin\theta_5 (f_3(\theta_5) - \eta_5^2 \cos\theta_5) \end{pmatrix}.$$

For practical purposes, adequate values for the parameters η_4 and η_5 must be specified. In this respect, let us mention only that numerical computations tend to indicate that for $\eta_4 = 3$ any value $\eta_5 \geq 7$ guarantees the satisfaction of (42).

Appendix.

Proof of Lemma 1 (*property* 4). We assume that $i \in \{1, \ldots, m\}$, since otherwise a simple algebraic manipulation yields

$$\begin{aligned} \mathrm{d}x_i = (\mathrm{d}x_i - x_{\lambda(i)}\mathrm{d}x_{\rho(i)}) + \sum_{r=1}^{\bar{r}} x_{\lambda(i)}x_{\lambda\rho(i)} \ldots x_{\lambda\rho^{r-1}(i)}(\mathrm{d}x_{\rho^r(i)} - x_{\lambda\rho^r(i)}\mathrm{d}x_{\rho^{r+1}}) \\ + x_{\lambda(i)}x_{\lambda\rho(i)} \ldots x_{\lambda\rho^{\bar{r}}(i)}\mathrm{d}x_{\rho^{\bar{r}+1}(i)}, \end{aligned}$$

where $\bar{r}$ is the smallest integer such that $\rho^{\bar{r}+1}(i) \in \{1, \ldots, m\}$. It is sufficient to specify some functions h_1 and $h_{2,j}$ such that equality (20) holds when each side is applied to any element of the basis $\{b_r, \partial/\partial x_s,\ r = 1, \ldots, m,\ s = m + 1, \ldots, n(d)\}$ of the tangent space to $\mathbb{R}^n$. From (17),

$$\forall i = 1, \ldots, m \quad \left\{ \begin{array}{rcll} \mathrm{d}x_j(b_i) & = & \delta_i^j & \text{if } j \in \{1, \ldots, m\}, \\ \omega_j(b_i) & = & 0 & \text{if } j \in \{m + 1, \ldots, n(d)\}, \end{array} \right. \tag{92}$$

where $\omega_j = \mathrm{d}x_j - x_{\lambda(j)}\mathrm{d}x_{\rho(j)}$. Therefore, (20) applied to any b_r holds by setting $h_1 = q^i$ defined by (19). Finally, the functions $h_{2,j}$ are defined recursively, for $\ell(j)$ decreasing from d' to 2, by setting

$$\begin{cases} h_{2,j} = -\dfrac{\partial h_1}{\partial x_j}, & \ell(j) = d', \\ h_{2,j} = -\dfrac{\partial h_1}{\partial x_j} + \displaystyle\sum_{\ell(j)<\ell(j')\leq d'} h_{2,j'} x_{\lambda(j')}\mathrm{d}x_{\rho(j')}(\partial/\partial x_j), & 1 < \ell(j) < d'. \end{cases} \qquad \square$$

Proof of Lemma 2. Since the set $\{g_1, \dots, g_m\}$ is nilpotent of order $d+1$, it follows from the definition of the P. Hall basis that $\{g_1, \dots, g_{n(d)}\}$ is a basis of Lie$\{g_1, \dots, g_m\}$. Therefore, it is clearly a basis of Lie$\{g_1, \dots, g_{n(d)}\}$. Then, (33) follows from the well-known fact that the solution of (32) is an exponential Lie series (see, e.g., [16] for details).

Let us finally prove that the mapping defined by (34) is one-to-one. Consider the system

$$\dot{x} = \sum_{i=1}^{n(d)} c_i g_i(x). \tag{93}$$

From property 2 of Lemma 1, each v.f. g_i is smooth and Δ-homogeneous of strictly negative degree. Therefore, its kth component $g_{i,k}$ can depend only on the components x_j of x such that $j < k$. From this and property 1 of Lemma 1, we deduce that the kth component of (93) can be written as

$$\dot{x}_k = c_k a_k + h_k(x_k^-, c_k^-), \tag{94}$$

where the notation y_k^- for a vector $y \in \mathbb{R}^n$ denotes the subvector $(y_1, \dots, y_{k-1})$, and h_k is some smooth function. Using (94), one easily proves by induction on k that any solution to (93) satisfies

$$\forall k = 1, \dots, n, \forall t, \qquad x_k(t) = x_k(0) + tc_k a_k + f_k(x_k^-(0), c_k^-, t)$$

for some smooth function f_k. Therefore,

$$\forall k = 1, \dots, n, \qquad \left[\exp\left(\sum_{i=1}^{n(d)} c_i g_i\right) x_0\right]_k = x_{0,k} + c_k a_k + f_k(x_{0,k}^-, c_k^-, 1),$$

and one easily infers from these equalities that

$$(c_1, \dots, c_{n(d)}) \neq (c'_1, \dots, c'_{n(d)}) \Longrightarrow \exp\left(\sum_{i=1}^{n(d)} c_i g_i\right) x_0 \neq \exp\left(\sum_{i=1}^{n(d)} c'_i g_i\right) x_0 \,. \qquad \square$$

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