**Matthieu Fruchard**
**Pascal Morin**
**Claude Samson**

INRIA
2004, Route des Lucioles
06902 Sophia-Antipolis Cedex, France
Claude.Samson@sophia.inria.fr

# A Framework for the Control of Nonholonomic Mobile Manipulators

## Abstract

*A general framework for the feedback control of mobile manipulators is proposed. Its main originality lies in its capacity to address in a unified manner both cases of omnidirectional and nonholonomic mobile platforms. It also allows for the execution of the desired manipulation task without the manipulator colliding into its joint limits, whenever these two objectives are compatible. This is obtained without having to rely upon trajectory planning. In the nonholonomic case this feature, which is not demonstrated for previous feedback control schemes proposed in the literature, is instrumental for performing reflex maneuvers automatically. It relies upon the concept, used in the control design, of a* companion omnidirectional frame *associated with the physical platform. The generality of the approach can be appreciated from its applicability to various nonholonomic platforms, for example unicycle-type and car-like.*

KEY WORDS—mobile manipulator, nonholonomic system, motion coordination, feedback control, transverse function

## 1. Introduction

A *mobile manipulator*, composed of a manipulator arm mounted on a mobile platform, is far more versatile than a conventional manipulator whose base is fixed, as a consequence of its enlarged operational space. The diversity of robotic devices of this kind (see Bayle, Fourquet, and Renaud 2001 for a survey) reflects how this latter feature can be exploited in various ways.

This paper is devoted to the feedback control of mobile manipulators and, more specifically, to the control of mobile manipulators whose platform is subjected to nonholonomic kinematic constraints, e.g., unicycle-like or car-like platforms. From now on, such devices will, for short, be referred to as *nonholonomic mobile manipulators*. While the platform's motion and the manipulation task can sometimes be performed separately, i.e., one after the other, there are also applications for which *motion coordination* between the two subsystems is necessary. The main difficulty of controlling a nonholonomic mobile manipulator, in comparison with the case of a classical manipulator arm, or a manipulator mounted on an omnidirectional platform, comes from the kinematic constraints which limit the platform's motion capabilities. In general, these constraints do not prevent the platform from being controllable, and an important literature has been devoted to the design of open-loop controls for trajectory planning purposes (see for example Desai et al. 1996; Lee and Cho 1997; Perrier, Dauchez, and Pierrot 1998; Bayle, Fourquet, and Renaud 2001). However, Brockett's (1983) theorem implies that the asymptotic stabilization of a nonholonomic platform at a specified fixed configuration cannot be obtained by using smooth (or even continuous) pure-state feedback. This result has important implications. For instance, it points out the impossibility, with classical feedbacks, of controlling the position *and* the orientation of a nonholonomic vehicle simultaneously. While several techniques have been developed in the last fifteen years for the stabilization of nonholonomic systems, via more general types of feedbacks (see Morin and Samson 2004 for a survey), the utilization of these techniques for mobile manipulators has seldom been explored.

Let us review some of the methods proposed so far in the literature for the feedback control of nonholonomic mobile manipulators (the reader is referred to Bayle, Fourquet, and Renaud (2003), Galicki (2003), Tchon (2002), and Tchon and Jakubiak (2003), for discussions and references related to other issues, such as motion planning, dynamic modeling, repeatability, or robot cooperation). One approach, used for example in Yamamoto and Yun (1994) and Chung and Velinsky (1998), is based on input–output linearization techniques. It is indeed well known that for unicycle-like platforms, the kinematic equations associated with the coordinates of a point not located on the driving wheels' axle can be linearized by a

The International Journal of Robotics Research
Vol. 25, No. 8, August 2006, pp. 745-780
DOI: 10.1177/0278364906068374

static state feedback. This allows the design of feedback laws which drive the platform to positions at which the manipulation task can be carried out by the onboard manipulator. While this type of approach is appealing for its simplicity, and applies well in a certain number of cases, it does not permit direct control of the platform's *orientation*. This may prevent the manipulator's joint values remaining inside a given domain. For example, in the case of a tracking application such as the one considered in Section 4, the platform will tend to make a half-turn when the target moves towards it. The compensation of this rotation at the manipulator's level will in turn induce large joint values and, subsequently, risks of collisions between the arm's links and the platform's body. Another class of solutions, inherited from techniques developed for the control of redundant manipulator arms, is based on the introduction of complementary tasks and the application of inverse/pseudo-inverse Jacobian techniques, in order to address issues such as non-collision with joint limits and/or obstacle avoidance. One method (Seraji 1998; Galicki 2003) involves complementing the $m$-dimensional task vector associated with the manipulation objective with $\delta_m - m$ components, where $\delta_m$ is the total number of degrees of freedom of the mobile manipulator. Notwithstanding eventual singularities of the task vector Jacobian, the problem of stabilizing the complete task vector to a given reference value is then well-posed, and it can be solved by applying classical inverse Jacobian techniques. However, the choice of the additional task components, in relation to the problem of avoiding task singularities, is a difficult issue. A control strategy is proposed in Galicki (2003) to avoid singular configurations. But it is not clear that this strategy is always compatible with the realization of the manipulation objective. Alternatively, or in a complementary way, redundancy can be used and addressed via the minimization of a (secondary) potential function, under the constraint that the manipulation task must be perfectly executed, see Wang and Kumar (1993), Bayle et al. (2002), and Bayle, Fourquet, and Renaud (2003). This function is typically commensurable with the inverse of a distance to joint limits and/or obstacles. The control design then relies on the use of a pseudo-inverse of the manipulator's task vector Jacobian. With respect to the former approach, the singularity problem might be less sensitive. However, with this strategy the impossibility of guaranteeing the boundedness of the potential function is a sign that problems such as joint limits and/or obstacles avoidance cannot be safely treated in this way. As remarked in Bayle et al. (2002, Sec. 3.2), another difficulty concerns the extension of these techniques to car-like platforms. The difficulty is twofold. First, because the platform's degree of mobility is reduced to one in this case, in comparison with two for a unicycle-like vehicle, the number of complementary task components associated with secondary objectives is also reduced by one. This is not consistent with the fact that both types of vehicles are equally locally controllable in position and orientation. Second, this type of approach does not provide a method for determining the steering control of the vehicle.

We believe that the core of these difficulties is directly related to the problem of controlling the complete situation (i.e., position and orientation) of the mobile platform, and that this problem has not been fully addressed in previous contributions, for the case when the platform is nonholonomic. Accordingly, this paper aims to present a unified control design framework for executing a manipulation task and, *at the same time*, stabilizing/controlling the situation of the supporting platform, which may be holonomic or nonholonomic. The realization of the manipulation task is set as the prime objective. As such, it usually requires a very precise positioning of the manipulator's end-effector. By contrast, such a precision is seldom necessary (or even desirable) for the positioning of the platform, because small errors at this level can be compensated for by using the manipulator's degrees of freedom. For this reason, the control objective for the platform is expressed here in the form of a secondary cost whose exact minimization is not a strict requirement. A nominal role devoted to this secondary minimization is to bring the platform to a domain where the manipulation task can be successfully carried out—with the idea of maintaining the manipulator's configuration near a *preferred* one (Yamamoto and Yun 1994; Yoshikawa 1990). The proposed approach relies on a previous work by two of the authors on the stabilization of nonholonomic systems, and more specifically on the *transverse function* (t.f.) approach (Morin and Samson 2001, 2003). The key element is the possibility of conceptually associating a *companion omnidirectional* frame with a frame characterizing the situation of a nonholonomic vehicle. The property conveyed by the transverse function used to define this companion frame is that the vehicle's frame is bound to stay close to its companion, whatever the motion of the latter frame. Moreover, the distance between the two frames can be modified at will, and rendered arbitrarily small via the choice of the transverse function parameters. In order to control the vehicle, it then suffices to control the companion omnidirectional frame, a much easier task. This approach exploits the strong controllability properties of nonholonomic systems, and especially the fact that any (not necessarily admissible) smooth trajectory of situations can be approximately followed, with arbitrarily good precision, by a nonholonomic (controllable) mobile robot (Haynes and Hermes 1970; Sussmann and Liu 1991). Studies in this direction have already been conducted in the context of trajectory planning/motion generation. A distinctive feature of the present approach is that is purely reactive. It provides feedback control laws endowed with *stability* and *robustness* properties which are important in practice in order to cope with unmatched initial conditions, modeling errors, or perturbations (among other things). The fact that it allows, for example, the tracking of a moving object/target without any *a priori* knowledge about its future displacements (i.e., by using on-line measurements only) is also illustrative of the reactivity of the approach and its intrinsic difference from planning methods.

The paper is organized as follows. We recall in Section 2 some classical relations concerning the kinematic modeling of mobile manipulators, and a few results concerning the transverse function approach. In particular, the key concept of a companion omnidirectional frame which can be associated with a nonholonomic frame, is defined at the end of this section. The general control framework is presented in Section 3. A target tracking application example is described in Section 4, and illustrated through simulation results. To simplify the exposition, the presentation of the framework in Sections 2, 3, and 4, is done for the sub-class of planar manipulators mounted on unicycle-like platforms. However, the control approach extends *mutatis mutandis* when the end-effector of the manipulation arm evolves in the three-dimensional Cartesian space. Because of the generality of the t.f. approach, the present method also applies to many other nonholonomic platforms, with very minor modifications. This point is discussed in more detail in Section 5, and illustrated through the example of a car-like platform.

## 2. Preliminary Recalls and Notation

Before introducing some notation, it may be useful to comment upon the choice of designing controllers from kinematic models of the mechanical systems under consideration, rather than from more complete models encompassing dynamical terms. As for many other control studies, this choice is motivated primarily by the fact that kinematic models of nonholonomic mobile manipulators account for the main difficult nonlinearities encountered when attempting to control these systems, i.e., kinematic nonholonomic constraints which restrict instantaneous motion capabilities. This explains why most of the literature devoted to the motion control of wheeled mobile robots is based on kinematic models, the idea being to focus on the major difficulties associated with the control design. Once a kinematic controller $u^*$ has been designed, it is not very difficult (at least theoretically) to derive dynamic controllers which guarantee the convergence of the system's velocity vector $u$ to $u^*$, and the stability of zero for the error $(u - u^*)$. A classical method, which applies to manipulator arms as well as to nonholonomic systems (and thus nonholonomic mobile manipulators) is the so-called Computed Torque Method which linearizes the system's dynamical equations in the form $\dot{u} = w$, with $w$ a vector of control variables diffeomorphically related to the torques produced by the actuators. Then, by setting (for instance) $w = -k(u - u^*) + \dot{u}^*$, with $\dot{u}^*$ the time-derivative of the function $u^*$ along the solutions of the controlled system, one obtains the closed-loop equation $\frac{d}{dt}(u - u^*) = -k(u - u^*)$ which yields the exponential stability of $u - u^* = 0$ if $k$ is positive. This result holds whether $u^*$ is an open-loop kinematic controller, i.e., a function of time derived, for example, via a planning algorithm, or a feedback law depending on state variables measured (or estimated) on-line. In both cases one obtains a dynamic feedback controller. However, there is an important difference between these two cases because the closed-loop dynamic model inherits the stability properties of the controlled kinematic model. In particular, the system will recover from initial configuration (position/orientation) errors only if $u^*$ is itself a stabilizing feedback law. The present paper addresses the design of such feedback laws for nonholonomic mobile manipulators.

Let us now proceed with some notation. $I_n$ is the identity matrix associated with $\mathbb{R}^n$. $R(\theta)$, $S$, and $\bar{R}(\theta)$, are the rotation matrices defined by:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad S = R(\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\bar{R}(\theta) = \begin{pmatrix} R(\theta) & 0 \\ 0 & 1 \end{pmatrix}.$$

The following notation is used to describe the configuration of the planar mobile manipulator depicted on Figure 1.

- $q$ is the vector of either prismatic or revolute joint coordinates of the manipulator arm. Its dimension is $n_q$.
- $\mathcal{F}_0 = \{O, \vec{\imath_0}, \vec{\jmath_0}\}$ is a fixed frame in the plane.
- $\mathcal{F}_b = \{B, \vec{\imath_b}, \vec{\jmath_b}\}$ is a frame attached to the mobile platform in the plane. If the mobile platform is of the unicycle type (and thus nonholonomic), we impose that the instantaneous velocity of the point $B$ w.r.t. (with respect to) $\mathcal{F}_0$ is along the unit vector $\vec{\imath_b}$.
- $\mathcal{F}_e = \{E, \vec{\imath_e}, \vec{\jmath_e}\}$ is a frame attached to the manipulator's end-effector.
- Given two arbitrary frames $\mathcal{F}_a = \{A, \vec{\imath_a}, \vec{\jmath_a}\}$ and $\mathcal{F}_b = \{B, \vec{\imath_b}, \vec{\jmath_b}\}$, the "situation" of $\mathcal{F}_b$ w.r.t. $\mathcal{F}_a$ is defined as the element $r_{ab} \in SE(2) = \mathbb{R}^2 \times S^1$ (with $S^1 = \mathbb{R}/2\pi\mathbb{Z}$) given by

$$r_{ab} = \begin{pmatrix} p_{ab} \\ \theta_{ab} \end{pmatrix}$$

  with $p_{ab} = (x_{ab}, y_{ab})^T$ the components of $\overrightarrow{AB}$ in the basis of $\mathcal{F}_a$, and $\theta_{ab}$ the oriented angle between $\vec{\imath_a}$ and $\vec{\imath_b}$. When $\mathcal{F}_a$ is the fixed frame $\mathcal{F}_0$ defined above, $r_{ab} = r_{0b}$ is simply called the "situation" of $\mathcal{F}_b$, and the index 0 is omitted, i.e.,

$$r_{0b} = r_b = \begin{pmatrix} p_b \\ \theta_b \end{pmatrix}.$$

### *2.1. Group Operation in* SE(2)

The space of frames in the plane is isomorphic to the Lie group $SE(2)$. With the above notation, a possible isomorphism between these two spaces is given by the mapping $\mathcal{F} \longmapsto r$.

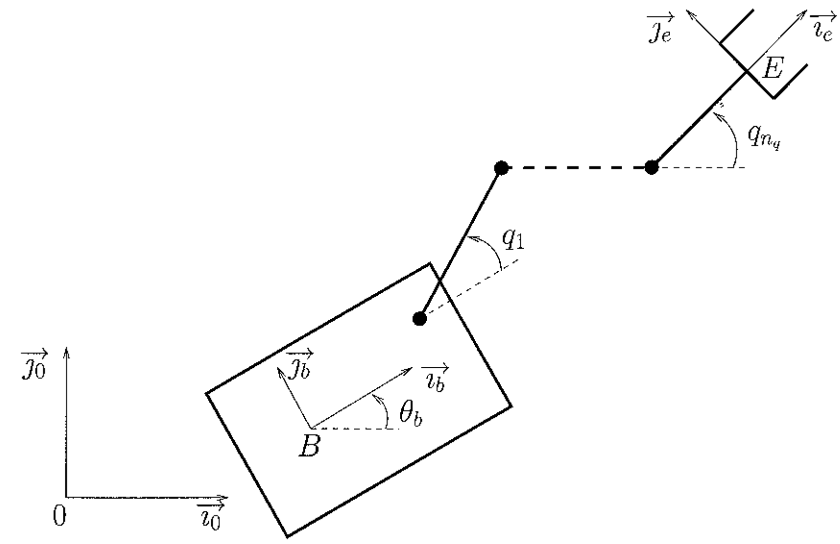



Fig. 1. Planar mobile manipulator.

Furthermore, it is also well known that the mapping

$$\begin{pmatrix} p \\ \theta \end{pmatrix} \longmapsto \begin{pmatrix} R(\theta) & p \\ 0 & 1 \end{pmatrix}$$

with $R(\theta)$ the $2 \times 2$ rotation matrix of angle $\theta$ in the plane, defines a Lie group isomorphism between $SE(2)$ and the matrix Lie group of $3 \times 3$ homogeneous matrices. On this latter group, the group product is just given by the matrix product, i.e.,

$$\begin{pmatrix} R(\theta_a) & p_a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} R(\theta_b) & p_b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R(\theta_a + \theta_b) & p_a + R(\theta_a)p_b \\ 0 & 1 \end{pmatrix}$$

and, accordingly, the group operation on $SE(2)$ is defined by

$$\begin{pmatrix} p_a \\ \theta_a \end{pmatrix}\begin{pmatrix} p_b \\ \theta_b \end{pmatrix} = \begin{pmatrix} p_a + R(\theta_a)p_b \\ \theta_a + \theta_b \end{pmatrix}. \tag{1}$$

The identity element $e$ on $SE(2)$, and the inverse $r^{-1}$ of $r = (p, \theta)^T$ (such that $rr^{-1} = r^{-1}r = e$) are therefore given by

$$e = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad r^{-1} = \begin{pmatrix} -R(-\theta)p \\ -\theta \end{pmatrix} = -\bar{R}(-\theta)r.$$

Since the group operation plays the same role as the addition in the case of $\mathbb{R}^n$, a "difference" between two elements of $SE(2)$ is given by the product of one element by the inverse of the other one (and thus not by a component-to-component subtraction). For instance, a difference between $r_a$ and $r_b$ is $r_{ba} = (r_b)^{-1}r_a = r_{b0}r_{0a}$; let us recall that $r_{ba}$ is the situation of the frame $\mathcal{F}_a$ w.r.t. the frame $\mathcal{F}_b$. Note that, due to the non-commutativity of the group operation, the product order is important, and one can form several "differences" between two elements of $SE(2)$. For example, $r_{ab} = (r_a)^{-1}r_b = r_{a0}r_{0b} = (r_{ba})^{-1}$ characterizes the situation of the frame $\mathcal{F}_b$ w.r.t. $\mathcal{F}_a$. A possible distance between two frames $\mathcal{F}_a$ and $\mathcal{F}_b$ is given by

$$\begin{aligned} d(\mathcal{F}_a, \mathcal{F}_b) &:= \mathrm{d}_\gamma(r_a, r_b) \\ &:= \sqrt{\|p_a - p_b\|^2 + 2\gamma(1 - \cos(\theta_a - \theta_b))} \\ &= \sqrt{\|p_{ab}\|^2 + 2\gamma(1 - \cos\theta_{ab})} \end{aligned} \tag{2}$$

with $\gamma > 0$, and $\|.\|$ the Euclidean norm (on $\mathbb{R}^2$ in the present case). When $\gamma = 2$ this distance coincides with the classical Fröbenius matrix norm of $(\bar{R}_{ab} - I_3)$, with $\bar{R}_{ab}$ denoting the homogeneous matrix associated with $r_{ab}$. Note that the last equality in (2) further indicates that $\mathrm{d}_\gamma$ is a left-invariant distance on $SE(2)$, i.e., $\mathrm{d}_\gamma(r_a, r_b) = \mathrm{d}_\gamma(e, r_{ab})$.

### 2.2. Kinematic Relations

The kinematic model of the manipulator arm is simply given by $\dot{q} = u_q$. Let us now consider the mobile platform. The components of the velocity of the point $B$ w.r.t. $\mathcal{F}_0$, expressed in $\mathcal{F}_b$, are denoted as $v_{b,1}$ and $v_{b,2}$, i.e.,

$$\frac{d\overrightarrow{OB}}{dt} = v_{b,1}\vec{i_b} + v_{b,2}\vec{j_b}.$$

The angular velocity of the frame $\mathcal{F}_b$ w.r.t. $\mathcal{F}_0$ is denoted as $\omega_b$, i.e.,

$$\frac{d\theta_b}{dt} = \omega_b.$$

By setting $u_b := (v_{b,1}, v_{b,2}, \omega_b)$, one easily verifies that

$$\dot{r}_b = \bar{R}(\theta_b)u_b. \tag{3}$$

Obviously, a similar relation holds not only for $\mathcal{F}_b$ but also for any other frame. Following Murray, Li, and Sastry (1994), $u_b$ is called the "body velocity of the frame $\mathcal{F}_b$."

Let $r_c$ and $r_d$ denote two smooth curves on $SE(2)$ with $\dot{r}_c = \bar{R}(\theta_c)u_c$ and $\dot{r}_d = \bar{R}(\theta_d)u_d$. The following relation, which can also be found e.g., in Murray, Li, and Sastry (1994, Proposition 2.15), is established in the Appendix:

$$\dot{\widehat{r_c r_d}} = \bar{R}(\theta_c + \theta_d)\left(u_d + \mathrm{Ad}_{r_d^{-1}} u_c\right) \tag{4}$$

with $\mathrm{Ad}_{r^{-1}}$ the matrix defined by the following relations:

$$\mathrm{Ad}_r = \begin{pmatrix} R(\theta) & -Sp \\ 0 & 1 \end{pmatrix},$$

$$\mathrm{Ad}_{r^{-1}} = (\mathrm{Ad}_r)^{-1} = \begin{pmatrix} R(-\theta) & R(-\theta)Sp \\ 0 & 1 \end{pmatrix}. \tag{5}$$

From the above relations and the fact that $r^{-1}r = e$, one deduces that for any smooth curve $r$ on $SE(2)$ with $\dot{r} = \bar{R}(\theta)u$,

$$\dot{\widehat{r^{-1}}} = \bar{R}(-\theta)(-\mathrm{Ad}_r u). \tag{6}$$

Finally, by combining (4) and (6), one also obtains:

$$\dot{\widehat{r_c r_d^{-1}}} = \bar{R}(\theta_c - \theta_d)\mathrm{Ad}_{r_d}(u_c - u_d) \tag{7}$$

and

$$\dot{\widehat{r_c^{-1} r_d}} = \bar{R}(\theta_d - \theta_c)(u_d - \mathrm{Ad}_{r_d^{-1} r_c} u_c) \tag{8}$$

which may also be written as

$$\dot{r}_{cd} = \bar{R}(\theta_{cd})u_{cd}, \quad \text{with} \quad u_{cd} = u_d - \mathrm{Ad}_{r_{dc}} u_c. \tag{9}$$

Let $h : (q, r, t) \longmapsto h(q, r, t)$ denote a differentiable function from $\mathbb{R}^{n_q} \times SE(2) \times \mathbb{R}$ to $\mathbb{R}^p$. With some abuse of notation, we denote by $\frac{\partial h}{\partial r}$ the function such that, for any smooth curves $(q, r)(.)$ on $\mathbb{R}^{n_q} \times SE(2)$,

$$\frac{d}{dt}h(q(t), r(t), t) = \frac{\partial h}{\partial q}(q(t), r(t), t)\dot{q}(t) + \frac{\partial h}{\partial r}(q(t), r(t), t)u(t) + \frac{\partial h}{\partial t}(q(t), r(t), t) \tag{10}$$

with $\dot{r} = \bar{R}(\theta)u$, and $\frac{\partial h}{\partial q}$ and $\frac{\partial h}{\partial t}$ the (standard) partial derivatives of $h$ w.r.t. $q$ and $t$.

### *2.3. Transverse Functions and Companion Omnidirectional Frame*

Consider a smooth function

$$f : \alpha \longmapsto f(\alpha) = \begin{pmatrix} p_f(\alpha) \\ \theta_f(\alpha) \end{pmatrix}$$

from $S^1$ to $SE(2)$. Then,

$$\bar{r}_b(\alpha) := r_b f(\alpha)^{-1} = \begin{pmatrix} p_b - R(\theta_b - \theta_f(\alpha))p_f(\alpha) \\ \theta_b - \theta_f(\alpha) \end{pmatrix} = \begin{pmatrix} \bar{p}_b \\ \bar{\theta}_b \end{pmatrix} \tag{11}$$

belongs to $SE(2)$ and, as such, represents the situation of some frame $\mathcal{F}_{\bar{b}}(\alpha)$, the origin of which has components

$$-R(-\theta_f(\alpha))p_f(\alpha)$$

in the frame $\mathcal{F}_b$. With the distance between $\mathcal{F}_b$ and $\mathcal{F}_{\bar{b}}(\alpha)$ defined by (2), one obtains:

$$d_\gamma(\mathcal{F}_b, \mathcal{F}_{\bar{b}}(\alpha)) = \mathrm{d}_\gamma(r_b, \bar{r}_b(\alpha)) = \mathrm{d}_\gamma(r_b, r_b(f(\alpha))^{-1})$$
$$= \mathrm{d}_\gamma(e, (f(\alpha))^{-1}) = \mathrm{d}_\gamma(f(\alpha), e).$$

Therefore,

$$\forall \alpha \in S^1, \quad d_\gamma(\mathcal{F}_b, \mathcal{F}_{\bar{b}}(\alpha)) = \sqrt{\|p_f(\alpha)\|^2 + 2\gamma(1 - \cos\theta_f(\alpha))} \leq \sup_\alpha \mathrm{d}_\gamma(f(\alpha), e).$$

By using (7), one easily obtains that the kinematics of $\mathcal{F}_{\bar{b}}(\alpha)$ is given by:

$$\dot{\bar{r}}_b = \bar{R}(\bar{\theta}_b)\bar{u}_b \tag{12}$$

with

$$\bar{u}_b := \begin{pmatrix} I_2 & -Sp_f(\alpha) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{R}(\theta_f(\alpha)) & -\frac{\partial f}{\partial \alpha}(\alpha) \end{pmatrix} \begin{pmatrix} u_b \\ \dot{\alpha} \end{pmatrix} \tag{13}$$

the body velocity of the frame $\mathcal{F}_{\bar{b}}(\alpha)$. Note that the argument $\alpha$ of $\bar{r}_b(\alpha)$ in (12) has been omitted to lighten the notation. From now on, $\dot{\alpha}$ will be viewed as a complementary control input to monitor the motion of the frame $\mathcal{F}_{\bar{b}}(\alpha)$. The important point is that the frame $\mathcal{F}_{\bar{b}}(\alpha)$ is omnidirectional provided that $\bar{u}_b$ can be rendered equal to any vector in $\mathbb{R}^3$ (i.e., provided that the mapping $(u_b, \dot{\alpha}) \longmapsto \bar{u}_b$ is onto). When the mobile platform is omnidirectional, this property is satisfied for any function $f$. For example, one can take $f = 0$, in which case $\mathcal{F}_b$ and $\mathcal{F}_{\bar{b}}(\alpha)$ coincide. The case of nonholonomic mobile platforms is more interesting. Consider a unicycle-like platform. Then, the second component of $u_b$ (i.e., $v_{b,2}$) is equal to zero, so that:

$$\bar{u}_b = \begin{pmatrix} I_2 & -Sp_f(\alpha) \\ 0 & 1 \end{pmatrix} H(\alpha) \begin{pmatrix} v_{b,1} \\ \omega_b \\ \dot{\alpha} \end{pmatrix} = \bar{H}(\alpha) \begin{pmatrix} v_{b,1} \\ \omega_b \\ \dot{\alpha} \end{pmatrix} \tag{14}$$

with

$$H(\alpha) := \begin{pmatrix} \cos\theta_f(\alpha) & 0 & -\frac{\partial x_f}{\partial \alpha}(\alpha) \\ \sin\theta_f(\alpha) & 0 & -\frac{\partial y_f}{\partial \alpha}(\alpha) \\ 0 & 1 & -\frac{\partial \theta_f}{\partial \alpha}(\alpha) \end{pmatrix} \quad \text{and}$$

$$\bar{H}(\alpha) := \begin{pmatrix} I_2 & -Sp_f(\alpha) \\ 0 & 1 \end{pmatrix} H(\alpha). \tag{15}$$

Therefore, $\mathcal{F}_{\bar{b}}(\alpha)$ is omnidirectional provided that the matrix $H(\alpha)$ is invertible. A function $f$ for which this property is satisfied for *any* $\alpha \in S^1$ is called a "transverse function" (Morin and Samson 2001). The issue of the existence of such functions has been treated in the much more general context of the transverse function approach (Morin and Samson 2001, 2003). In the present case, a family of transverse functions is as follows:

LEMMA 1. For any $\varepsilon_1, \varepsilon_2 > 0$, the function $f$ defined by

$$f(\alpha) = \begin{pmatrix} \varepsilon_1 \sin\alpha \\ \varepsilon_1\varepsilon_2 \frac{\sin 2\alpha}{4} \\ \arctan(\varepsilon_2 \cos\alpha) \end{pmatrix} \quad (16)$$

is a transverse function (i.e., $\det H(\alpha) \neq 0$ for any $\alpha \in S^1$).

The proof of this lemma (easily obtained by calculation) is given in the Appendix.

DEFINITION 1. Given a transverse function $f$ and some $\alpha \in S^1$, the corresponding omnidirectional frame $\mathcal{F}_{\bar{b}}(\alpha)$ associated with the nonholonomic frame $\mathcal{F}_b$ is called a *companion frame* of $\mathcal{F}_b$.

From what precedes, whatever the motion of $\mathcal{F}_{\bar{b}}(\alpha)$, the distance between $\mathcal{F}_b$ and $\mathcal{F}_{\bar{b}}(\alpha)$ never exceeds $\sup_\alpha d_\gamma(f(\alpha), e)$, a bound which can be made arbitrarily small (but different from zero) via the choice of $\varepsilon_1$ and $\varepsilon_2$ in (16). Therefore, a way of controlling $\mathcal{F}_b$ consists in controlling its companion frame $\mathcal{F}_{\bar{b}}(\alpha)$, a much easier task since $\mathcal{F}_{\bar{b}}(\alpha)$ is omnidirectional.

Note that the determinant of $H(\alpha)$ tends to zero when making $\varepsilon_1$ and $\varepsilon_2$ tend to zero, so that choosing very small values for these parameters will typically result in large control gains in the expressions of $v_{b,1}$, $\omega_b$, and $\dot{\alpha}$ –which, in view of (14), are obtained by premultiplying $\bar{u}_b$ by the inverse of $\bar{H}(\alpha)$. Therefore, in practice, the choice of these parameters will be made in order to satisfy some compromise between keeping the frame of the mobile platform near its companion frame, on the one hand, and minimizing (well-known) adverse effects associated with large control gains, on the other. This compromise, which also involves the frequency of the platform's maneuvers (via the value of $\dot{\alpha}$), is discussed in more detail in Artus, Morin, and Samson (2004). Another relation between the maximal size of the transverse function and the property of not colliding into the manipulator's joint limits when performing the manipulation task will be pointed out in Section 4.2.

## 3. Control Approach

Prior to addressing the control design issue itself, one must specify control objectives in relation to the tasks which the mobile manipulator is meant to perform. In this respect, it is useful to have in mind the respective roles played by the manipulator, on the one hand, and the supporting mobile platform, on the other. In a typical application, a tool or a sensor is mounted on the manipulator's end-effector in order to perform a task involving precise movements inside a limited domain of application. We will call this task the "manipulation task." In many cases, it is preferable that the mobile platform stays motionless during the execution of the manipulation task. There are several reasons for this, such as the imperfection of the filtering of perturbations resulting from moving on an uneven terrain, or energy saving. However, moving the supporting platform may also be useful, or even become necessary, when the size of the manipulator's operational space is not sufficient to cover the range required to complete the manipulation task on a large object, or when this object moves and tends to leave the region reachable by the manipulator's end-effector.

The general idea is that the prime role of the mobile platform is to displace the manipulator so as to compensate for its limited domain of operation, in such a way that the manipulation task can be carried out without interruption. Clearly, this objective is looser than the one set on the manipulator, and it also demonstrates a kind of "master/slave" natural decoupling and hierarchy between the two mechanical subsystems.

### *3.1. Control Problem Statement*

A way to translate the above considerations into general mathematical terms, reminiscent of a classical approach developed for the treatment of redundancy in the case of manipulators mounted on a fixed support (see for example Samson, Leborgne, and Espiau 1991), consists in considering the problem of minimizing a secondary cost function, via the control of the mobile platform, under the constraint of enforcing a set of "virtual linkages" which characterize the manipulation task, via the control of the manipulator. More precisely, we will assume that the overall control objective is the (local) asymptotic stabilization of an *ideal* trajectory solution to the following problem:

$$\begin{aligned} \mathcal{P}_0: \quad & \text{At each time-instant } t, \text{ minimize } h_s(r_b, t) \\ & \text{under the constraint that } e_m(q, r_b, t) = 0 \end{aligned}$$

with $e_m$ and $h_s$ being accessible to measurement (or combined measurement and estimation). The vector-valued function $e_m$ characterizes the manipulation task. It obviously depends on the manipulator's joint coordinates $q$. Its dependence upon the mobile platform's situation $r_b$ comes from the fact that this platform is also the base of the manipulator. The independent time-variable $t$ is used to parameterize the possible evolution of the manipulation objective, as well as possible variations of the environment (resulting, for instance, from the motion of an object/target tracked by the end-effector). The secondary objective concerning the minimization of $h_s$ is usually less demanding than the manipulation task because, in many cases, the "exact" minimization of this function is not a strict requirement.

When the mobile platform is omnidirectional, so that the mobile manipulator can be viewed as an extended manipulator arm with $\dim(q) + \dim(r_b)$ degrees of freedom, the control problem evoked above can be solved under relatively mild assumptions upon $h_s$ and $e_m$. For example, if $h_s(r_b, t)$ admits a unique minimum $r_*(t)$ for each $t$, and if the function $(q, t) \longmapsto e_m(q, r_*(t), t)$ is *admissible* (Samson, Leborgne, and Espiau 1991) so that there exists a solution $q_*(t)$ to the equation $e_m(q, r_*(t), t) = 0$, the problem is solved by any control law for $(u_q, u_b)$ which asymptotically stabilizes the *ideal* trajectory $(q_*(t), r_*(t))$. Since the mapping $(u_q, u_b) \longmapsto (\dot{q}, \dot{r}_b)$ is onto in this case, the existence of a smooth ideal trajectory is sufficient to ensure the existence of such a feedback control. See, for example, Samson, Leborgne, and Espiau 1991 (Chapter 4) for the determination of such control laws from the measurement of $e_m$ and $h_s$, and the calculation of their derivatives.

In the case of a nonholonomic platform, the situation is significantly more complicated. The difficulty is twofold: i) the ideal trajectory $r_*(t)$ associated with the minimization of the secondary cost $h_s$ has no reason of being *feasible* for the nonholonomic platform, and ii) even if this trajectory were feasible, the design of a feedback control law capable of ensuring its asymptotic stabilization would still be problematic, due to the non-existence of *universal* stabilizers for these systems, as pointed out in Brockett (1983) and Lizárraga (2004). For these reasons, the constrained minimization problem $\mathcal{P}_0$ used to specify the desired trajectory for the mobile manipulator is ill-adapted for control purposes. It is too restrictive in the sense that it does not underlie the existence of a control solution. In order to circumvent the abovementioned difficulties, we propose to replace this problem by the following one, which applies equally well to the omnidirectional and nonholonomic cases:

$$\mathcal{P}: \quad \text{At each time-intant } t, \text{ minimize } h_s(\bar{r}_b, t) \\ \text{under the constraint that } e_m(q, r_b, t) = 0$$

with $\bar{r}_b$, given by (11), the situation of the companion frame $\mathcal{F}_{\bar{b}}(\alpha)$ associated with $\mathcal{F}_b$ (for some transverse function). Since $\mathcal{F}_{\bar{b}}(\alpha)$ is omnidirectional, $\bar{r}_b$ is not subjected to nonholonomic constraints, and the asymptotic stabilization of the solution to Problem $\mathcal{P}$ can be addressed with the techniques developed for the control of manipulator arms. Note that the specification of an ideal trajectory for the companion frame does not uniquely determine a trajectory for the mobile platform, since the relative situation of these frames depends on the transverse function variable $\alpha$ whose value at time $t$ is obtained by the integration of $\dot{\alpha}$, and thus depends on both the chosen control law and the initial conditions. This new formulation of the control problem (asymptotic stabilization of an ideal trajectory solution to $\mathcal{P}$) yields the following two-step approach for the control design.

**Step 1:** Selection of the the manipulation task $e_m$.

Ideally, $e_m$ should be chosen such that the constraint $e_m(q, r_b, t) = 0$ is equivalent to the realization of the manipulation objective. In practice however, some precautions must be taken in order to avoid the explosion of the control at the crossing of Jacobian singularities, and also give the priority to the avoidance of joint limits over the execution of the manipulation task. We propose to select $e_m$ on the basis of the following rules:

1. $e_m$ is $n_q$-dimensional and the square "Jacobian" matrix $\frac{\partial e_m}{\partial q}(q, r_b, t)$ is invertible for any $(q, t)$, and any $r_b$ in some "large" domain $\mathcal{D}_r(t)$.
2. Keeping all components of $e_m$ bounded ensures that the manipulator's joints are away from their limits.
3. Keeping all components of $e_m$ equal to zero is equivalent to the perfect realization of the manipulation objective, provided that $(q(t), r_b(t))$ belongs to some domain $\mathcal{O}_q(t) \times \mathcal{O}_r(t)$.

Rule 1 is inspired by the property of $\rho$-admissibility introduced in the context of the task-function approach (Samson, Leborgne, and Espiau 1991). It is instrumental for the good conditionning of the control problem, and the usefulness of the invertibility of the manipulation Jacobian matrix will also appear in Section 3.4 when specifying the expression of a feedback control $u_q$ for the manipulator arm, with the view of ensuring the convergence of $e_m$ to zero. Once this convergence is granted, Rule 2 guarantees the avoidance of the joint limits. Finally, Rule 3 expresses that the regulation of $e_m$ to zero must be coherent with the realization of the manipulation objective.

**Step 2:** Selection of the secondary cost $h_s$.
In view of Rules 1 and 3, the objective is to select $h_s$ such that its minimization implies that $r_b(t)$ belongs to $\mathcal{D}_r(t)$ (so that $u_q$ is well defined), and possibly also inside $\mathcal{O}_r(t)$ (so that the manipulation objective can be achieved). For example, if $\mathcal{D}_r(t)$ is an open set which varies continuously with time, a possible choice consists in defining $h_s$ by $h_s(\bar{r}_b, t) = (\mathrm{d}_\gamma(\bar{r}_b, r_*(t)))^2$, with $r_*(t) \in \mathcal{D}_r(t)$ representing a desired situation for the mobile platform at time $t$. In this case, the "minimization" of $h_s$ is equivalent to $\bar{r}_b(t) = r_*(t)$ and, since the distance between $\bar{r}_b$ and $r_b$ is bounded by a value which can be made arbitrarily small (through the choice of the transverse function), one can guarantee in this way that the distance between $r_b(t)$ and $r_*(t)$ is always small.

We now discuss in more detail possible choices for $e_m$ and $h_s$.

### 3.2. Guidelines for the Choice of the Primary Task-Function

The objective is to define some $n_q$-dimensional task-function $e_m$ satisfying Rules 1–3 of the previous section. A possible approach consists in setting

$$e_m(q, r_b, t) = \left(\frac{\partial h}{\partial q}(q, r_b, t)\right)^T \tag{17}$$

with $h$ denoting a smooth positive cost function whose minimization w.r.t. $q$ is equivalent to zeroing its derivative $\frac{\partial h}{\partial q}$. For example, if $h$ is a strictly convex function of $q$ for any $(r_b, t)$, it has a unique minimum $q_*(r_b, t)$, and $e_m(q, r_b, t) = 0$ if and only if $q = q_*(r_b, t)$. Then, the asymptotic stabilization of $e_m$ to zero is equivalent to keeping $h$ minimal. A possibility is to choose $h$ as the sum of three cost functions

$$h(q, r_b, t) = h_p(q, r_b, t) + h_\ell(q, t) + h_r(q, t) \tag{18}$$

The function $h_p$ is associated with Rule 3. It must be equal to zero if and only if the manipulation objective is perfectly achieved. Take, for instance

$$h_p(q, r_b, t) = \frac{1}{2}\|e_p(q, r_b, t)\|^2 \tag{19}$$

with $e_p$ a smooth vector-function the zeroing of which corresponds to enforcing the virtual linkages associated with the manipulation objective.

The cost function $h_\ell$ is associated with Rule 2, i.e., it is there to ensure that the joint limits cannot be reached as long as $e_m$ remains bounded. This function should thus grow unbounded when any $q_i$ tends to a corresponding joint limit. On the other hand, in view of Rule 3, it is desirable that this cost does not affect the realization of the manipulation objective (i.e., the vanishing of $h_p$) when the $q_i$s are far from their limits. Let $q_i^-, q_i^+$ $(i = 1, \ldots, n_q)$ denote the lower and upper limits for the joint variable $q_i$, we propose

$$h_\ell(q) = \frac{1}{2}\sum_{i=1}^{n_q} \alpha_i \, f_i^2(q) \qquad (\alpha_i > 0) \tag{20}$$

with $f_i$ $(i = 1, \ldots, n_q)$ real-valued convex functions defined by:

$$f_i(q_i) = \begin{cases} \frac{(q_i^- + \delta_i - q_i)^2}{q_i - q_i^-} & \text{if } q_i \le q_i^- + \delta_i \\ & \quad (0 < \delta_i < (q_i^+ - q_i^-)/2) \\ 0 & \text{if } q_i^- + \delta_i \le q_i \le q_i^+ - \delta_i \\ \frac{(q_i^+ - \delta_i - q_i)^2}{q_i^+ - q_i} & \text{if } q_i \ge q_i^+ - \delta_i \end{cases} \tag{21}$$

Note that $h_\ell$ is globally convex, and that

$$h_\ell(q) = 0 \Longleftrightarrow \frac{\partial h_\ell}{\partial q}(q) = 0 \Longleftrightarrow q_i \in [q_i^- + \delta_i, q_i^+ - \delta_i], \forall i$$

Finally, $h_r$ is a *regularization* cost function, convex w.r.t. $q$ for any fixed $t$, whose role is to help ensuring the satisfaction of Rule 1. By choosing

$$h_r(q, t) = \frac{1}{2}(q - \hat{q}(t))^T \Gamma(t)(q - \hat{q}(t)) \tag{22}$$

with $\Gamma(t)$ a positive symmetric matrix and $\hat{q}(.)$ some function, the choice of which will be discussed shortly, one obtains, in view of (17)–(22),

$$e_m = \left(\frac{\partial e_p}{\partial q}\right)^T e_p + \left(\frac{\partial h_\ell}{\partial q}\right)^T + \Gamma(t)(q - \hat{q}(t)) \tag{23}$$

and

$$\frac{\partial e_m}{\partial q} = \left(\frac{\partial e_p}{\partial q}\right)^T \frac{\partial e_p}{\partial q} + \frac{\partial^2 h_\ell}{\partial q^2} + \Gamma(t) + \sum_{k=1}^{n_q} e_{p,k}\frac{\partial^2 e_{p,k}}{\partial q^2}. \tag{24}$$

With $h_\ell$ defined by (20)-(21), the first two matrices in the right-hand side of (24) are positive. Therefore

$$\frac{\partial e_m}{\partial q} \ge \Gamma(t) + \sum_{k=1}^{n_q} e_{p,k}\frac{\partial^2 e_{p,k}}{\partial q^2}. \tag{25}$$

Since $e_p$ is a smooth function, the invertibility of $\frac{\partial e_m}{\partial q}$ on any compact set (Rule 1) can be obtained by choosing $\Gamma(t)$ "positive enough". The choice of $\hat{q}(.)$ directly affects the satisfaction of Rule 3. In order to limit the adverse role of $h_r$ in this respect, we propose to compute $\hat{q}(t)$ by filtering the measured joint vector $q(t)$ according to

$$\begin{cases} \dot{\hat{q}} = -\sigma \frac{\hat{q} - q(t)}{\xi + \|\hat{q} - q(t)\|} & (\sigma > 0, \xi > 0) \\ \hat{q}(0) = q(0) \end{cases} \tag{26}$$

This ensures that i) $\|\dot{\hat{q}}(t)\|$ is uniformly bounded, ii) $\|\hat{q}(t) - q(t)\|$ is small when $\|\dot{q}(t)\|$ is small itself, and iii) $(\hat{q}(t) - q(t))$, and thus $h_r(q(t), t)$, tend to zero when $\dot{q}(t)$ tends to zero.

### 3.3. Guidelines for the Choice of the Secondary Cost-Function

The choice of the secondary cost-function $h_s$ will often result from the attempt to satisfy several requirements which may be, in some cases, complementary but also, in other cases, antagonistic. One of them, of particular practical relevance and importance, is to move the mobile platform so that the manipulation task can be carried out without the manipulator's joints colliding into their hardware limits. A cost-function whose minimization ensures this requirement will be denoted as $h_{s,1}$. At this level of generality, it is hardly possible to specify this function further. Indeed, there may be several ways to fulfill the abovementioned requirement, depending on the properties of the manipulator (its number of d.o.f., for instance, the existence of revolute joints not subjected to limits etc.), the nature of the manipulation task, and also the completeness of the information provided by the available sensors about the situation of the mobile platform w.r.t. its environment. Examples of functions $h_{s,1}$ will, however, be provided in the next section.

A second typical requirement is to keep the platform motionless whenever this is possible, for instance when the manipulator's joints are "far away" from their limits. The fulfillment of this requirement can be handled via the minimization of a cost-function $h_{s,2}$ of the form (reminiscent of the regularization function $h_r$ evoked previously)

$$h_{s,2}(\bar{r}_b, t) = (\mathrm{d}_\gamma(\bar{r}_b, \hat{r}(t)))^2 = (\mathrm{d}_\gamma(e, \tilde{r}_b(\bar{r}_b, t)))^2 \quad (27)$$

with $\tilde{r}_b(\bar{r}_b, t) = \bar{r}_b^{-1}\hat{r}(t)$ computed according to

$$\begin{aligned} \dot{\tilde{r}}_b &= \bar{R}(\tilde{\theta}_b)(\hat{u}(t) - \mathrm{Ad}_{\tilde{r}_b^{-1}}\bar{u}_b) \\ &= \bar{R}(\tilde{\theta}_b)\hat{u}(t) - \begin{pmatrix} I_2 & S\tilde{p}_b \\ 0 & 1 \end{pmatrix} \bar{u}_b \\ \hat{u}(t) &= -\frac{1}{\xi + d_\gamma(e, \tilde{r}_b(t))}\mathrm{Diag}\{\sigma_i\}\tilde{r}_b(t) \quad (\sigma_{1,2,3} > 0 \,, \xi > 0) \\ \tilde{r}_b(0) &= 0 \end{aligned} \quad (28)$$

In relation (27), $\hat{r}(t)$ represents the situation of a virtual frame defined by filtering the motion of the mobile platform's companion frame. The explicit calculation of its coordinates is not necessary for the control computation, as this can be observed from relation (28). The cost $h_{s,2}$ is minimal when $\tilde{r}_b = 0$. In view of (28), the satisfaction of this equality on a time-interval implies that $\bar{u}_b = 0$ on this time-interval. From (13), or (14), this in turn implies that the velocity $u_b$ of the mobile platform is equal to zero. This conveys the idea of how the minimization of $h_{s,2}$ is related to the preoccupation of preventing the mobile platform from moving. However, such a function is not, by itself, very useful, since one can rightfully argue that it suffices to set $u_b = 0$ to keep the platform motionless. The real usefulness of this function results from its combination with another cost-function, in order to end up with a control law containing a term which tends to slow down the mobile platform. For instance, its combination with the control objective represented by the function $h_{s,1}$ discussed previously yields the following secondary cost-function $h_s$:

$$h_s(\bar{r}_b, t) = \Psi(t)h_{s,1}(\bar{r}_b, t) + (1 - \Psi(t))h_{s,2}(\bar{r}_b, t) \quad (29)$$

with $\Psi(t) \in [0, 1]$ denoting a weight function whose derivative is bounded. Examples of such a function will be given. More generally, $h_s$ will be the weighted sum of several terms, each of them associated with a desirable feature concerning the situation of the mobile platform, i.e.,

$$h_s(\bar{r}_b, t) = \sum_i \Psi_i(t)h_{s,i}(\bar{r}_b, t).$$

### 3.4. Control Design

#### 3.4.1. Control of the Manipulator

A possible control law $u_q$ for the manipulator arm is defined by

$$u_q = \left(\frac{\partial e_m}{\partial q}(q, r_b, t)\right)^{-1} \cdot \left(Ge_m(q, r_b, t) - \frac{\partial e_m}{\partial r_b}(q, r_b, t)u_b - \frac{\partial e_m}{\partial t}(q, r_b, t)\right) \quad (30)$$

with $G$ denoting a Hurwitz-stable matrix, and the partial derivative $\frac{\partial e_m}{\partial r_b}(q, r_b, t)$ defined according to eq. (10). The application of this control yields the closed-loop equation $\dot{e}_m = Ge_m$ and, therefore, the exponential convergence of $e_m$ to zero.

#### 3.4.2. Control of the Mobile Platform

Given a secondary cost-function $h_s$, the question is then to determine a feedback control $\bar{u}_b$, the application of which brings and maintains the value of this function "near" its minimum. In the case where $h_s$ does not depend on time and admits a unique minimizing argument $r_*$ (as a result of the strict convexity of the function, for instance), a possible control law is

$$\bar{u}_b = -\left(\frac{\partial h_s}{\partial \bar{r}_b}(\bar{r}_b)\right)^T h_s(\bar{r}_b) \quad (31)$$

Indeed, since $\dot{h}_s(\bar{r}_b) = \frac{\partial h_s}{\partial \bar{r}_b}(\bar{r}_b)\bar{u}_b$, this control yields, in closed loop, $\dot{h}_s = -\|\frac{\partial h_s}{\partial \bar{r}_b}(\bar{r}_b)\|^2 h_s$ $(\leq 0)$, which in turn implies that $\|\frac{\partial h_s}{\partial \bar{r}_b}(\bar{r}_b)\|^2 h_s(\bar{r}_b)$ tends to zero, and that $h_s$ decreases to some limit value $(\geq 0)$. If this limit value is not to equal to zero, then $\frac{\partial h_s}{\partial \bar{r}_b}$ necessarily tends to zero, so that $h_s$ tends to its minimum. Therefore, whatever the value of the minimum of $h_s$, the above control makes $\bar{r}_b$ converge to the minimizing situation $r_*$. Note that the calculation of this control does not require the explicit determination of $r_*$. The measurement of $h_s(\bar{r}_b)$ and its gradient $\frac{\partial h_s}{\partial \bar{r}_b}(\bar{r}_b)$, plus the property of convexity of the function, are sufficient (just as for any gradient-based minimization algorithm). From there one easily infers that, provided that the task-function $e_m(q, r_*, t)$ is admissible for $t \in [0, +\infty)$, a solution $(q_*(t), r_*)$ to the constrained minimization problem $\mathcal{P}$ exists and the control (30)-(31) (locally) asymptotically stabilizes this trajectory. Note that this does not yet imply, by itself, that the manipulation objective corresponding to the minimization of $h_p$ is achieved, since the convergence of $e_m$ to zero does not necessarily imply that $e_p$ tends to zero. In this respect, the choice of $r_*$ plays a central role which the application example treated in Section 4 illustrates.

The case of a time-dependent function $h_s$ is more difficult. In this case, the time derivative of $h_s(\bar{r}_b, t)$ along the trajectories of the mobile manipulator is given by

$$\dot{h}_s = a^T(\bar{r}_b, t)\bar{u}_b + \frac{\partial h_s}{\partial t}(\bar{r}_b, t) \quad (32)$$

with

$$a^T(\bar{r}_b, t) = \frac{\partial h_s}{\partial \bar{r}_b}(\bar{r}_b, t) \tag{33}$$

As long as $\frac{\partial h_s}{\partial \bar{r}_b}(\bar{r}_b, t) \neq 0$, a control value for $\bar{u}_b$ which makes $h_s$ decrease (i.e., $\dot{h}_s < 0$) can always be found. However, when $\frac{\partial h_s}{\partial \bar{r}_b}(\bar{r}_b, t)(\bar{r}_b, t) = 0$ – which corresponds to a necessary condition for $\bar{r}_b$ to be a local minimizer (or maximizer) of $h_s$ – the control has no influence on the variation of the function any more. This implies that the minimization of $h_s$ at all time-instants cannot be guaranteed in this case. The time evolution of $h_s$ will then depend on the choice of the control law, and the properties of $h_s$ itself. We propose below an expression for $\bar{u}_b$ which, associated with a suitable decomposition of the "drift term" $\frac{\partial h_s}{\partial t}$, is "liable" to perform well in many situations.

In what follows, the arguments $\bar{r}_b$ and $t$ of all functions will be omitted for the sake of readability. Consider a decomposition of $\frac{\partial h_s}{\partial t}$ in the form

$$\frac{\partial h_s}{\partial t} = a^T b_1 + b_2 \tag{34}$$

with $b_1$ and $b_2$ some functions of $\bar{r}_b$ and $t$. Then, in view of (32),

$$\dot{h}_s = a^T(\bar{u}_b + b_1) + b_2. \tag{35}$$

This decomposition is, of course, not unique. It already appears from the above relation that a decomposition with $b_2 = 0$ must be favorable to the minimization of $h_s$. However, this choice of $b_2$ is not always possible, because $\frac{\partial h_s}{\partial t}$ may be different from zero when $\frac{\partial h_s}{\partial \bar{r}_b} = 0$. The relation (35) also shows that $b_1$ contains the part of $\frac{\partial h_s}{\partial t}$ which can be "exactly" pre-compensated by the control. Let us then consider the problem of minimizing $\frac{1}{2}(\bar{u}_b + b_1)^T K^{-1}(\bar{u}_b + b_1)$ (with $K > 0$) w.r.t. $\bar{u}_b$, under the constraint that $\dot{h}_s = -(a^T Da)$. The idea behind this problem is to i) make $h_s$ decrease with a rate (partly) specified by the matrix $D$ (this is the role of the constraint), and ii) among the controls yielding this rate of decrease of $h_s$, pick the one whose difference with the pre-compensation of $\frac{\partial h_s}{\partial t}$ through the term $b_1$ is minimal in the weighted norm associated with the positive matrix $K^{-1}$. This minimization objective is related to energy saving. It also corresponds to the preoccupation of limiting the number of maneuvers during transient phases, when the mobile platform is nonholonomic, by choosing $K$ adequately. More details about this issue can be found in Artus, Morin, and Samson (2004). For instance, by choosing

$$K(\alpha) = \bar{H}(\alpha)\bar{K}\bar{H}(\alpha)^T \tag{36}$$

with $\bar{H}(\alpha)$ the matrix defined in (15) and $\bar{K} = \text{Diag}\{\bar{k}_b, \bar{k}_b, \bar{k}_\alpha\}$ a constant positive matrix, one has $\bar{u}_b^T K(\alpha)^{-1}\bar{u}_b = (1/\bar{k}_b)\|u_b\|^2 + (1/\bar{k}_\alpha)\dot{\alpha}^2$. This latter relation in turn suggests choosing $\bar{k}_b$ smaller than $\bar{k}_\alpha$ in order to penalize the size of platform's velocity $u_b$ more than the size of the computed auxiliary control $\dot{\alpha}$. By using (35), the Lagrangian associated with the aforementioned minimization problem is

$$L(\bar{u}_b) = \frac{1}{2}(\bar{u}_b + b_1)^T K^{-1}(\bar{u}_b + b_1) + \lambda\left(a^T(\bar{u}_b + b_1) + b_2 + (a^T Da)\right)$$

with $\lambda$ the Lagrange parameter. The optimality condition

$$\frac{\partial L}{\partial \bar{u}_b}(\bar{u}_b) = 0$$

yields $\bar{u}_b = -b_1 - \lambda K a$, and by using this expression in the constraint equation it follows that

$$\lambda = \frac{a^T Da + b_2}{a^T Ka}.$$

Therefore, the solution to the above minimization problem is

$$\bar{u}_b = -b_1 - \frac{a^T Da}{a^T Ka}Ka - \frac{b_2}{a^T Ka}Ka.$$

The regularization of this expression, in order to ensure that $\bar{u}_b$ is well defined when $\frac{\partial h_s}{\partial \bar{r}_b} = 0$ (i.e., when $a = 0$), leads (among other possibilities) to the following control expression

$$\bar{u}_b = -\hat{b}_1 - \frac{a^T Da + \hat{b}_2}{a^T Ka + \mu}Ka \tag{37}$$

with $\mu$ denoting a small positive number, and $\hat{b}_i$ (with $i = 1, 2$) some *convenient* estimation of $b_i$ the choice of which is discussed below. From (35), the application of the control law (37) yields, in closed-loop

$$\dot{h}_s = -\frac{a^T Ka}{a^T Ka + \mu}a^T Da + \frac{\partial h_s}{\partial t} - (a^T\hat{b}_1 + \frac{a^T Ka}{a^T Ka + \mu}\hat{b}_2). \tag{38}$$

Let us now comment on the choice of $(\hat{b}_1, \hat{b}_2)$ in relation with possible decompositions $(b_1, b_2)$ of $\frac{\partial h_s}{\partial t}$.

**Choice 1**: full pre-compensation of $\frac{\partial h_s}{\partial t}$ by setting $(\hat{b}_1 = b_1, \hat{b}_2 = b_2)$.
In this case, the relations (34) and (38) yield

$$\dot{h}_s = -\frac{a^T Ka}{a^T Ka + \mu}a^T Da + \frac{\mu}{a^T Ka + \mu}b_2. \tag{39}$$

This equation shows that the introduction of the regularizing constant $\mu$ prevents the desired constraint from being satisfied exactly. This was expected. It also points out the fact that the pre-compensation of $\frac{\partial h_s}{\partial t}$ is all the more efficient, in terms of the minimization of $h_s$, when $\|b_2\|$ is small. Indeed, when $b_2 \equiv 0$, then $h_s$ is non-increasing, and $a$, and thus $\frac{\partial h_s}{\partial \bar{r}_b}$, converge to zero. If $h_s$ is strictly convex with respect to the variable $\bar{r}_b$ and does not depend on time, this in turn implies that $h_s$ converges to its minimum. When $b_2 \neq 0$, it is no longer possible to ensure that $h_s$ converges to a minimum. Nevertheless, the following relation (which is a direct consequence of (39))

$$(a^T Da)(a^T Ka) \geq \mu b_2 \Longrightarrow \dot{h}_s \leq 0$$

indicates that the decrease of $h_s$ is all the more likely when $\|b_2\|$ is small. When $b_2$ cannot be taken equal to zero, a possible simple decomposition $(b_1, b_2)$ of $\frac{\partial h_s}{\partial t}$, coherent with the rule of keeping $\|b_2\|$ small, is, for instance

$$b_1 = \frac{\partial h_s}{\partial t}\frac{1}{\|a\|^2 + \nu}a \quad \Rightarrow \quad b_2 = \frac{\partial h_s}{\partial t}\frac{\nu}{\|a\|^2 + \nu} \tag{40}$$

with $\nu$ denoting a small positive number.

**Choice 2**: no pre-compensation of $\frac{\partial h_s}{\partial t}$ by setting ($\hat{b}_1 = 0$, $\hat{b}_2 = 0$).
Common consequences of this choice are a less "reactive" control and a degradation of performance, in terms of minimizing $h_s$. On the other hand, a beneficial counterpart of this loss of reactivity is a reduction of the control effort. This effect can be foreseen from the constrained optimization problem considered for the control design, after remarking that setting $b_1 = 0$ yields the minimization of $\bar{u}_b^T K^{-1} \bar{u}_b$. In view of (38), the variation of $h_s$ is then given by

$$\dot{h}_s = -\frac{a^T Ka}{a^T Ka + \mu} a^T Da + \frac{\partial h_s}{\partial t}. \tag{41}$$

**Choice 3**: pre-compensation of $\frac{\partial h_s}{\partial t}$ combined with a reduction of the control effort, by setting ($\hat{b}_1 = 0, \hat{b}_2 = \frac{\partial h_s}{\partial t}$).
This choice corresponds to the decomposition ($b_1 = 0, b_2 = \frac{\partial h_s}{\partial t}$) of $\frac{\partial h_s}{\partial t}$. In this case, the control effort is reduced (for the same reasons as in the previous case) at the expense of the pre-compensation of $\frac{\partial h_s}{\partial t}$, which is not achieved exactly, and, subsequently, at the expense of the minimization of $h_s$. This control solution can be seen as intermediary between Choices 1 and 2. Obviously, there is also a continuum of possibilities involving partial pre-compensation of $\frac{\partial h_s}{\partial t}$, each of them achieving a different compromise in terms of control energy expenditure and minimization of $h_s$.

From there, a deeper analysis of the control properties requires specifying the function $h_s$ further, as illustrated by an example treated in the next section.

Let us finally recall that the control $u_b$ of the physical mobile platform is calculated from $\bar{u}_b$ by using the relation (13) (or (14)), i.e., $u_b = \bar{u}_b$, when the mobile platform is omnidirectional, and $(v_{b,1}, \omega_b, \dot{\alpha})^T = \bar{H}(\alpha)^{-1}\bar{u}_b$ with the matrix $\bar{H}(\alpha)$ defined in (15), when the platform is nonholonomic and unicycle-like. Let us also shortly comment on the existence of limitations upon the velocity components of the platform, i.e., upon the components of $u_b$. A simple way to handle these limitations is to "saturate" the control. In this case, it is useful to have in mind that the saturation should be implemented by dividing *all* the components of $(u_b, \dot{\alpha})$ by the same number (greater or equal to one), so as to augment the chances of preserving the stabilization properties of the control.

# 4. Application Example: Target Tracking in $SE(2)$

This application, depicted in Figure 2, consists in tracking in both position and orientation an object (materialized on the figure by the frame $\mathcal{F}_d$) moving in the plane. The mobile manipulator consists of a unicycle-like platform equipped with a three-joint RPR manipulator. The axis of the first joint is attached to the platform at the point of coordinates $(L, 0)$ w.r.t. the frame $\mathcal{F}_b$. The distance between the last joint's axis and the origin $E$ of the end-effector's frame $\mathcal{F}_e$ is denoted by $L_e$.

## *4.1. The Manipulation Task and Associated Control*

The manipulation objective is the stabilization of the target's situation w.r.t. the end-effector, $r_{ed}$, at a fixed desired value $r^*_{ed} = (X^*, 0, 0)^T$. The realization of this objective implies that, whatever the target's motion in the robot's environment, the target appears to be motionless with respect to the end-effector, at a distance $X^*$ from it. This is a three-dimensional virtual linkage. One can assume, for instance, that $r_{ed}$ is measured via the observation of the target by a camera mounted on the end-effector. With the notation of Section 3.2, a possible choice for the task-function $e_p$ associated with this manipulation objective is

$$e_p(q, r_b, t) = \tilde{r}_{ed} := (r^*_{ed})^{-1} r_{ed}. \tag{42}$$

Note that $e_p$ can also be expressed as a function of $q$ and $r_{db}$ (since $r_{ed} = r_{eb}r_{bd} = r_{eb}r_{db}^{-1}$ and $r_{eb}$ only depends on $q$). In order to calculate the joint control (30), one needs to determine the partial derivatives of $e_p$ w.r.t. $q$, $r_b$, and $t$. Since $\theta^*_{ed} = 0$ and $u^*_{ed} = 0$, one deduces from (8) and (42) that

$$\dot{e}_p = \dot{r}_{ed}. \tag{43}$$

Since $r_{ed} = r_{eb}r_{bd}$, one also has, by using (4) and (9),

$$\dot{r}_{ed} = \bar{R}(\theta_{ed})(u_{bd} + \mathrm{Ad}_{r_{db}} u_{eb})$$

with

$$u_{bd} = u_d - \mathrm{Ad}_{r_{db}} u_b.$$

Therefore

$$\dot{r}_{ed} = \bar{R}(\theta_{ed})\left(u_d + \mathrm{Ad}_{r_{db}}(u_{eb} - u_b)\right) \tag{44}$$

Now, $r_{eb}$ (the situation of the mobile platform w.r.t. the manipulator's end-effector) is a function of $q$ only, and it is in

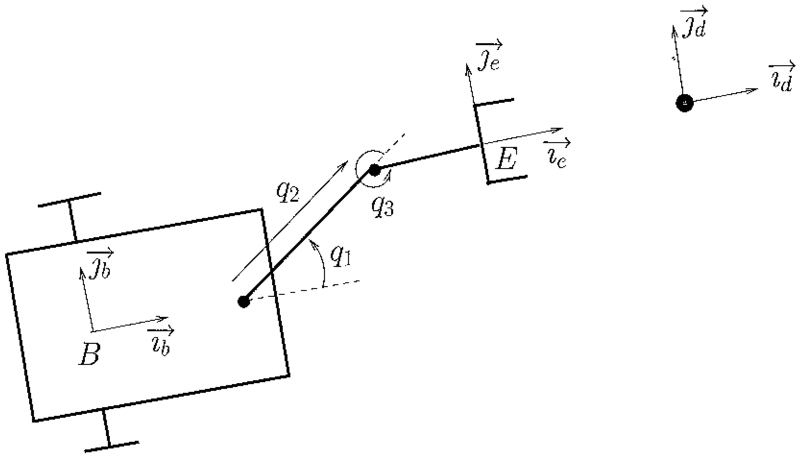



Fig. 2. Tracking in SE(2).

fact simple to verify that

$$r_{eb} = \begin{pmatrix} -R(-q_1 - q_3)\begin{pmatrix} L \\ 0 \end{pmatrix} - R(-q_3)\begin{pmatrix} q_2 \\ 0 \end{pmatrix} - \begin{pmatrix} L_e \\ 0 \end{pmatrix} \\ -q_1 - q_3 \end{pmatrix}. \tag{45}$$

Therefore

$$\begin{aligned} \dot{r}_{eb} &= \bar{R}(\theta_{eb})u_{eb} \\ &= \frac{\partial r_{eb}}{\partial q}\dot{q} \end{aligned} \tag{46}$$

and, from (43)-(44),

$$\dot{e}_p = \bar{R}(\theta_{ed})\left(u_d + \mathrm{Ad}_{r_{db}}(\bar{R}(\theta_{be})\frac{\partial r_{eb}}{\partial q}\dot{q} - u_b)\right) \tag{47}$$

With the definition specified by (10) of the partial derivatives of a function, one deduces from the above equation (by a term-by-term identification) that

$$\begin{aligned} \frac{\partial e_p}{\partial q} &= \bar{R}(\theta_{ed})\mathrm{Ad}_{r_{db}}\bar{R}(\theta_{be})\frac{\partial r_{eb}}{\partial q} = \begin{pmatrix} I_2 & R(\theta_{eb})Sp_{bd} \\ 0 & 1 \end{pmatrix}\frac{\partial r_{eb}}{\partial q} \\ \frac{\partial e_p}{\partial r_b} &= -\bar{R}(\theta_{ed})\mathrm{Ad}_{r_{db}} = -\begin{pmatrix} R(\theta_{eb}) & R(\theta_{eb})Sp_{bd} \\ 0 & 1 \end{pmatrix} \\ \frac{\partial e_p}{\partial t} &= \bar{R}(\theta_{ed})u_d \end{aligned} \tag{48}$$

with $\theta_{eb} = -\theta_{be} = -(q_1 + q_3)$. One easily verifies from (45) that $\frac{\partial r_{eb}}{\partial q}$, and thus $\frac{\partial e_p}{\partial q}$, are invertible everywhere except at $q_2 = 0$.

Having determined $e_p$, $e_m$ is given by (17)–(18), with $h_p$ given by (19), $h_l$ by (20)–(21), and $h_r$ by (22) and (26). The control $u_q$ is then given by (30). The various (constant) parameters involved in these expressions will be specified further.

### *4.2. Secondary Cost-Function and Control of the Mobile Platform*

For this application, if the mobile platform were omnidirectional, an obvious way to ensure the avoidance of the joint's limits while tracking the target would be to keep the relative situation $r_{db} = r_d^{-1} r_b$ between the platform and the target equal to some nominal value $r_*$ (i.e., any constant value such that $(q, r_{db}) = (q_*, r_*) \Longrightarrow e_p = 0$ with $q_*$ a joint value inside $\prod_i (q_i^-, q_i^+)$). Since the mobile platform is subjected to nonholonomic constraints, this objective cannot be achieved for all the target's motions. However, one can try to "nearly" obtain this result by controlling the platform's companion frame instead. This suggests considering the following secondary cost-function $h_{s,1}$:

$$h_{s,1}(\bar{r}_b, t) = \frac{1}{2}(\mathrm{d}_\gamma(r_*, (r_d(t))^{-1}\bar{r}_b))^2 = \frac{1}{2}(\mathrm{d}_\gamma(e, \tilde{r}))^2 \tag{49}$$

with $\mathrm{d}_\gamma$ denoting the distance defined by (2), and $\tilde{r} = r_*^{-1} r_d^{-1} \bar{r}_b$. One deduces from (8) and (12) that

$$\dot{\tilde{r}} = \bar{R}(\tilde{\theta})(\bar{u}_b - \mathrm{Ad}_{(r_*\tilde{r})^{-1}} u_d) \tag{50}$$

with $u_d$ defined by $\dot{r}_d = \bar{R}(\theta_d)u_d$. From (2),

$$\frac{\partial(\mathrm{d}_\gamma(e, \tilde{r}))^2}{\partial \tilde{r}} = 2(\tilde{x}, \tilde{y}, \gamma \sin\tilde{\theta})\bar{R}(\tilde{\theta})$$

so that, by (50), eq. (32) is satisfied for the function $h_s = h_{s,1}$ with

$$\begin{aligned} a^T(\bar{r}_b, t) &= (\tilde{x}, \tilde{y}, \gamma \sin\tilde{\theta})\bar{R}(\tilde{\theta}) \\ \tfrac{\partial h_s}{\partial t}(\bar{r}_b, t) &= (\tilde{x}, \tilde{y}, \gamma \sin\tilde{\theta})\bar{R}(\tilde{\theta})\left(-\mathrm{Ad}_{(r_*\tilde{r})^{-1}} u_d\right). \end{aligned}$$

These equations also indicate that, in this case, a decomposition of $\frac{\partial h_s}{\partial t}$ in the form $\frac{\partial h_s}{\partial t} = a^T b_1 + b_2$ is obtained by setting

$$b_1(\bar{r}_b, t) = -\mathrm{Ad}_{(r_*\tilde{r})^{-1}} u_d, \qquad b_2 = 0. \tag{51}$$

The application of the control (37) with $\hat{b}_1 = b_1$ and $\hat{b}_2 = b_2$ then makes $h_s$ decrease (in view of (39) when $b_2 = 0$) and, in fact, makes this cost converge to zero. In this case, one can establish the following result.

PROPOSITION 1. Let $(q_*, r_*)$ denote any pair such that $(q, r_{db}) = (q_*, r_*) \Longrightarrow e_p = 0$, with $q_* \in \prod_i (q_i^- + \delta_i, q_i^+ - \delta_i)$, and consider the control law $\bar{u}_b$ defined by (37) with $\hat{b}_1 = b_1$ and $\hat{b}_2 = b_2$, and the control law $u_q$ defined by (30) with the matrix $\Gamma$ involved in the regularization cost $h_r(q, t)$ equal to zero. There exist some constants $\bar{\varepsilon}, \eta_1, \eta_2 > 0$ such that, if

*i)* the "size" of the transverse function, $\|f\|_{max} = \sup_\alpha \mathrm{d}_\gamma(f(\alpha), e)$, is smaller than $\bar{\varepsilon}$,

*ii)* the initial norm $\|q(0) - q_*\|$ is smaller that $\eta_1$,

*iii)* the initial distance $\mathrm{d}_\gamma(r_{db}(0), r_*)$ of the mobile platform w.r.t. $r_*$ is smaller that $\eta_2$,

then the tracking error converges exponentially to zero and the manipulator's joints always stay away from their limits.

The proof of this result is given in the Appendix.

Let us comment on the maximal value of $\bar{\varepsilon}$ in the above proposition by considering, for instance, the transverse function (16). Let $\nu$ denote the function defined by $\|f\|_{max} = \nu(\varepsilon_1, \varepsilon_2)$. For any $\varepsilon_1$ and $\varepsilon_2$ one can define the set $D(\varepsilon_1, \varepsilon_2) = \{q \;:\; e_p = 0,\; \tilde{r} = 0,\; \alpha \in S^1\}$ of values that $q$ can take when the manipulation objective is perfectly realized (i.e., when $e_p = 0$), with the companion frame moving along the ideal trajectory defined by $\tilde{r} = 0$. Let $Q_l = \{q \;:\; q_i \in (q_i^-, q_i^+), i = 1, 2, 3\}$ denote the set of allowed joint variables. If $D(\varepsilon_1, \varepsilon_2) \subset Q_l$, then no collision with joint limits occurs while the task is perfectly executed. However, there exists a smallest positive number $\bar{\varepsilon}$ such that none of the sets $D(\varepsilon_1, \varepsilon_2)$, with $(\varepsilon_1, \varepsilon_2) \in \nu^{-1}(\bar{\varepsilon})$, is contained in $Q_l$. By definition, once the transverse function is given, this number depends on the manipulator's geometry, the set $Q_l$, and the choice of $r_*$. Its calculation, even in the case of the simple manipulator considered in the example, is far from being straightforward, so that one may be satisfied in practice with the numerical estimation of a close lowerbound. Since it depends on $r_*$, one may also try to determine a *preferred* situation $r_*$ which maximizes this number, in the spirit of the notion of manipulability (Yoshikawa 1990). The significance of this number comes from the fact that, as long as $\|f\|_{max} < \bar{\varepsilon}$, there exists $(\varepsilon_1, \varepsilon_2) \in \nu^{-1}(\|f\|_{max})$ such that the manipulation objective can be perfectly realized without the manipulator running into joint limits. If $\|f\|_{max} > \bar{\varepsilon}$ then, whatever the choice of $(\varepsilon_1, \varepsilon_2) \in \nu^{-1}(\|f\|_{max})$ such that $D(\varepsilon_1, \varepsilon_2)$ is not empty, there exist target motions for which a collision with at least one of the joint limits is unavoidable unless the objective of perfect manipulation is abandoned (i.e., $e_p \neq 0$).

Proposition 1 points out two important properties granted by the proposed approach. The first one is the possibility of performing the target tracking task with the guarantee—*whatever* the motion of the target—of avoiding collisions with joint limits. The second one is the convergence of the tracking error to zero for small enough initial tracking errors, with an attraction domain *independent* of the target's motion. A pending question, beyond the scope of this paper, is the estimation of the size of this domain. Note that this issue is not specific to the method presented here. Indeed, global convergence and stability is exceptional in the realm of nonlinear systems. For these systems, ensuring local stability is the prime objective and the norm. In the present case, an estimate of the attraction domain could be obtained analytically. However, as usual in this case, this estimate would be quite conservative. An alternative and probably more rewarding approach would be to evaluate the attraction domain through extensive simulations.

Now, the realization of the secondary task, as specified above, implies that the mobile platform has to move whenever the target moves. If one would prefer to keep the platform motionless until one of the manipulator's joints gets close to one of its limits (before the exact execution of the manipulation task has to be abandoned in order to avoid a collision with the joint limit), one should consider a combination of $h_{s,1}$ with another cost function, such as the function $h_{s,2}$ specified by relations (27) and (28). This yields the secondary cost-function of relation (29), i.e.,

$$h_s(\bar{r}_b, t) = \Psi(t) h_{s,1}(\bar{r}_b, t) + (1 - \Psi(t)) h_{s,2}(\bar{r}_b, t)$$

with $\Psi$ denoting a weight function, the determination of which is now discussed. The idea is to have $\Psi(t)$ equal to zero as long as the joints coordinates $q$ are far from their limits, and to have $\Psi(t)$ strictly positive otherwise. Let us, however, recall that $\Psi$ cannot be a function of $q$ (because, otherwise, $h_s$ would no longer be a function of $\bar{r}_b$ and $t$ only). It has to be a function of time whose derivative is bounded. In order to comply with this constraint, a possibility is to filter $\tilde{\Psi}(q(t))$, with $q(t)$ the measured joint vector, and $\tilde{\Psi}$ a function whose value is equal to zero when the joint coordinates are far from their limits, and strictly positive otherwise. An example of such a function is

$$\tilde{\Psi}(q) = \frac{1}{n_q} \sum_{i=1}^{n_q} \tilde{\sigma}_i^2(q_i) \,,$$

$$\tilde{\sigma}_i(q_i) = \begin{cases} 1 & \text{if } q_i \leq q_i^- + \delta_i^{--} \text{ or } q_i \geq q_i^+ - \delta_i^{++} \\ \frac{q_i^- + \delta_i^- - q_i}{\delta_i^- - \delta_i^{--}} & \text{if } q_i^- + \delta_i^{--} < q_i < q_i^- + \delta_i^- \\ 0 & \text{if } q_i^- + \delta_i^- \leq q_i \leq q_i^+ - \delta_i^+ \\ \frac{q_i - q_i^+ - \delta_i^+}{\delta_i^+ - \delta_i^{++}} & \text{if } q_i^+ - \delta_i^+ < q_i < q_i^+ - \delta_i^{++} \end{cases} \tag{52}$$

with the margins $\delta_i^+$ and $\delta_i^-$ chosen larger than the margins $\delta_i$ used in (20), in order to reduce the risk of the manipulator arm reaching a joint configuration beyond which the perfect execution of the manipulation task is abandoned because $h_l(q) \neq 0$. The margins $\delta_i^{++}$ and $\delta_i^{--}$ are then chosen inside the intervals $(\delta_i, \delta_i^+)$ and $(\delta_i, \delta_i^-)$ respectively.

A possible filtering of $\tilde{\Psi}(q(t))$ is as follows (compare with (26)):

$$\begin{cases} \dot{\Psi} &= -\lambda \dfrac{\Psi - \tilde{\Psi}(q(t))}{\xi + \|\Psi - \tilde{\Psi}(q(t))\|} \quad (\lambda > 0\,, \xi > 0) \\ \Psi(0) &= \tilde{\Psi}(q(0)) \end{cases} \tag{53}$$

Let us remark that $\Psi(t) \in [0, 1]$ for any $t$ (because $\tilde{\Psi}(q) \in [0, 1]$ for any $q$), so that $\Psi$ corresponds to a weight function. Note also that the larger $\lambda$, the weaker the filtering.

A slightly different possibility for the determination of $\Psi(t)$ consists in setting

$$\Psi(t) = \frac{1}{n_q} \sum_{i=1}^{n_q} \sigma_i^2(t) \tag{54}$$

with $\sigma_i(t)$ denoting a filtered value of $\tilde{\sigma}_i(q_i(t))$ numerically calculated as follows:

$$\begin{cases} \dot{\sigma}_i &= -\lambda \dfrac{\sigma_i - \tilde{\sigma}_i(q_i(t))}{\xi + |\sigma_i - \sigma_i(q_i(t)|} \quad (\lambda > 0\,, \xi > 0) \\ \sigma_i(0) &= \tilde{\sigma}_i(q_0) \end{cases}$$

A third possibility, which reduces the computational load, consists in setting

$$\Psi(t) = \frac{1}{n_q} \sum_{i=1}^{n_q} \sigma_i^2(\hat{q}_i(t))$$

with $\hat{q}_i$ $(i = 1, \ldots, n_q)$ the components of the filtered function $\hat{q}$ of $q$ defined by (26).

### 4.3. Simulation Results

The geometric parameters of the mobile manipulator have been chosen as follows:

- $L = 0.4$ m, $L_e = 0.15$ m,
- $(q_1^-, q_1^+) = (-\frac{\pi}{3}, \frac{\pi}{3})$ rad, $(q_2^-, q_2^+) = (0, 0.7)$ m, and $(q_3^-, q_3^+) = (-\frac{\pi}{2}, \frac{\pi}{2})$ rad.

The manipulation task-function $e_m$ is designed as described in Section 3.2, with the function $e_p$ given by (42) with $X^* = 0.5$ (m). The parameters involved in the definition of $h_\ell$ (eq. (20) and (21)) are $\alpha_i = 1, \delta_i = 0.05\ (q_i^+ - q_i^-)$. The regularization term $h_r$ is given by (22), with $\Gamma(t) = 0.2 \exp(-t) I_3$. The gain matrix in the definition of $u_q$ (eq. (30)) is $G = 3I_3$.

Concerning the calculation of the mobile platform control, the function $h_s$ is given by (29), with $h_{s,1}$ and $h_{s,2}$ given by (49) and (27)-(28) respectively. The parameters used in the definition of $h_{s,1}$ are $\gamma = 1$ (for the distance $d_\gamma$), and $r_* = -(X^* + L_e + L + 0.45, 0, 0)^T$. This value has been chosen so that $e_p = 0$ when $r_{db} = r_*$ and $q = q_* = (0, 0.45, 0)^T \in \prod_i (q_i^-, q_i^+)$. The parameters used in the definition of $h_{s,2}$ are $\gamma = 1$, $\sigma_i = 10$, and $\xi = 1$. Two choices of $\Psi$ (specified below) have been considered. The control $\bar{u}_b$ is then calculated according to (37) with $K$ given by (36) for $\bar{K} = \text{Diag}(1, 1, 100)$, $D = I_3$, and $\mu = 10^{-4}$. The functions $\hat{b}_1$ and $\hat{b}_2$ will be specified below. The transverse function $f$, from which $H(\alpha)$ and $\bar{H}(\alpha)$ can be calculated, is defined by (16) with $(\varepsilon_1, \varepsilon_2) = (0.2, 0.4)$. The control $u_b$ for the mobile platform is then calculated from (14). There remains the specification of the functions $\Psi$, $\hat{b}_1$, and $\hat{b}_2$. In order to illustrate the influence of these functions on the closed-loop system's behavior, we consider four possible choices below and present simulation results for two different trajectories of the target.

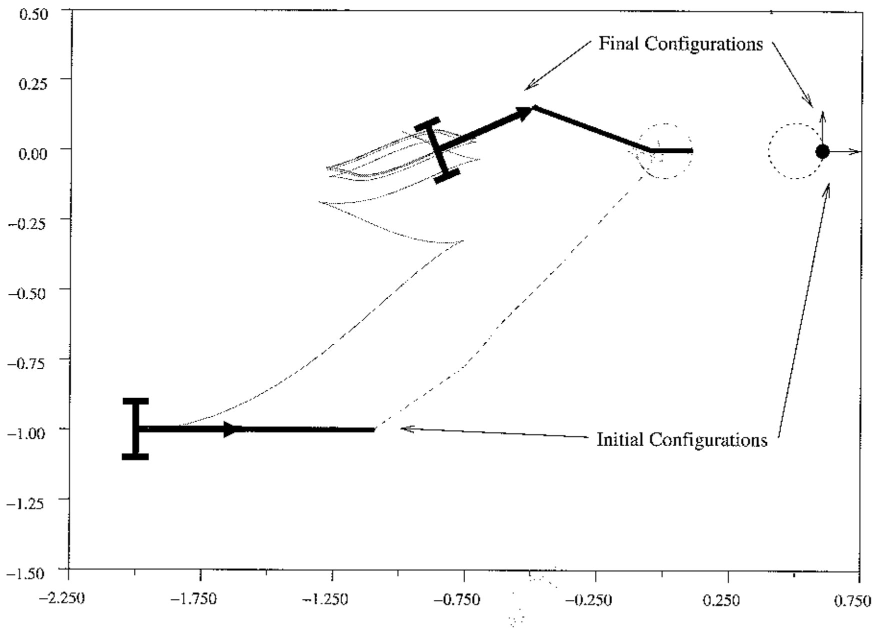
**Target's trajectory 1:** For the simulation results reported in Figures 3–6, the amplitude of the target's motion is kept small so that, once the platform has come close enough to the target, the manipulation objective can be achieved with the mobile platform at rest. The target's motion is defined by the velocity $u_d$ (i.e., $\dot{r}_d = \bar{R}(\theta_d)u_d$) with

$$u_d(t) = -\frac{\pi}{20}\left(\sin\frac{\pi}{2}t, \cos\frac{\pi}{2}t, 0\right)^T.$$

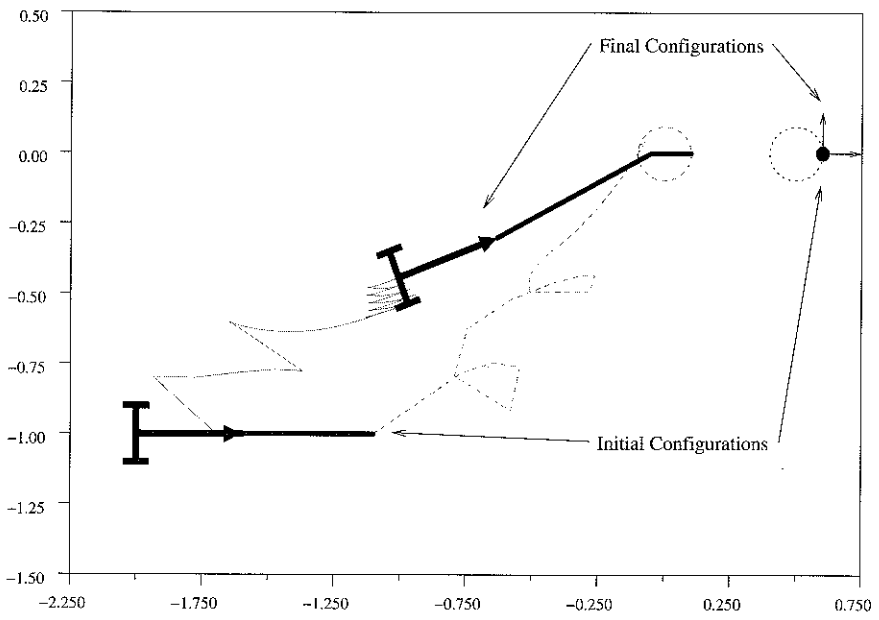
This corresponds to a circular motion of the origin of the target's frame, with a fixed orientation of this frame (see Figure 3 where the target's frame and the associated circular motion are represented on the upper-right part of the figures). The initial situation of the target w.r.t. the platform is $r_{bd}(0) = (2.6, 1, 0)^T$, and the initial configuration of the manipulator arm is $q(0) = (0, 0.35, 0)^T$ (with $q$ as specified on Figure 2). We now detail the different choices made for $\Psi$, $\hat{b}_1$, and $\hat{b}_2$.

(a) $\Psi = 1$, and $(\hat{b}_1, \hat{b}_2)$ is set according to **Choice 1** in Section 3.4.2, i.e., $(\hat{b}_1, \hat{b}_2) = (b_1, b_2)$ with $b_1, b_2$ defined by (51). Since $\Psi = 1$, $h_s$ is equal to $h_{s,1}$. Then, because of the definition of $(\hat{b}_1, \hat{b}_2)$, and since $b_2 = 0$, the convergence of $h_s$ to zero is guaranteed in this case, as this can be observed in Figure 6(a). The Cartesian motion of the mobile manipulator is shown in Figure 3(a). One can distinguish two phases. During the first part of the simulation, the platform approaches the target and both $h_s$ and $e_p$ tend to zero (Figures 6(a) and 4(a)). This transient convergence phase is followed by a steady behavior which involves periodic (non-stationary) motions of both the platform and manipulator. Although $h_s$ and $e_p$ are very well controlled to zero, a practical drawback of this solution is that it does not allow the mobile platform to remain still. This fact is also illustrated by Figure 5(a) which shows how the control inputs $v_{b,1}$ and $\omega_b$ evolve with time.

(b) $\Psi(t)$ is defined by (53), and $(\hat{b}_1, \hat{b}_2)$ is again set according to **Choice 1** in Section 3.4.2, except that $b_1$ and $b_2$ are now defined by (40) with $\nu = 10^{-4}$. The




(a)

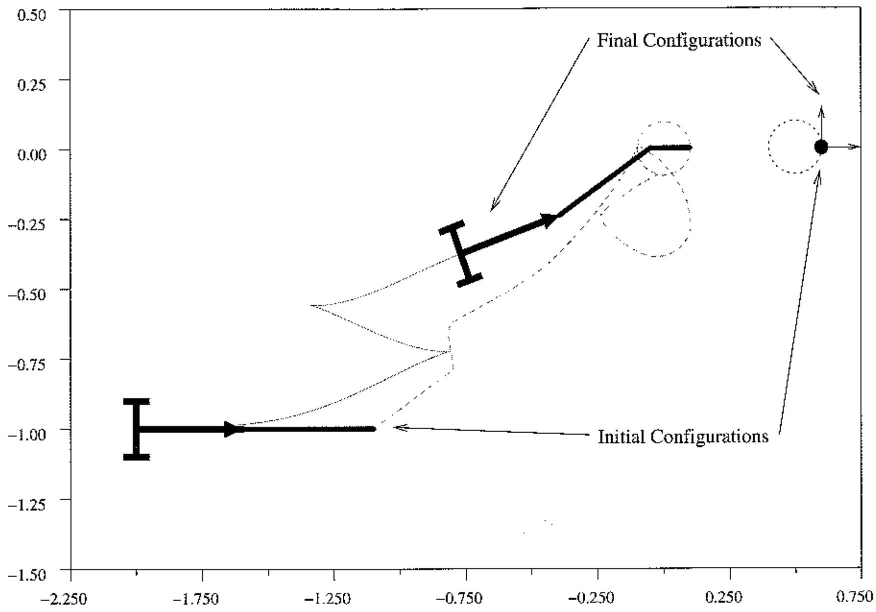



(b)

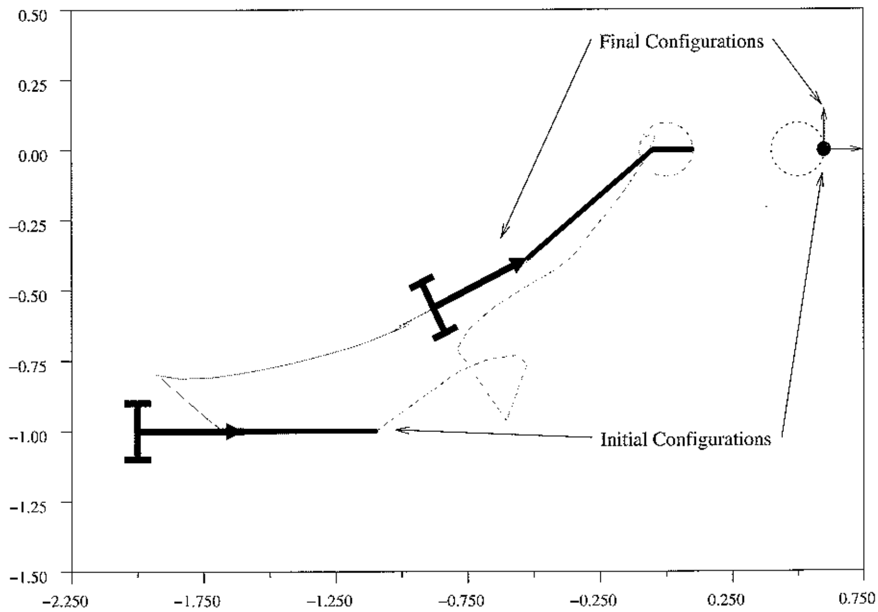
Fig. 3. Target's trajectory 1: Cartesian trajectories of the mobile manipulator and target:

- trajectory of the origin of the platform's frame $\mathcal{F}_b$ (quasi-solid gray line)
- trajectory of the origin of the end effector's frame $\mathcal{F}_e$ (dashed gray line)
- trajectory of the origin of the target's frame $\mathcal{F}_d$ (dotted gray line).

*(Continues on next page)*




(c)




(d)

Fig. 3. *(Continued from previous page)*

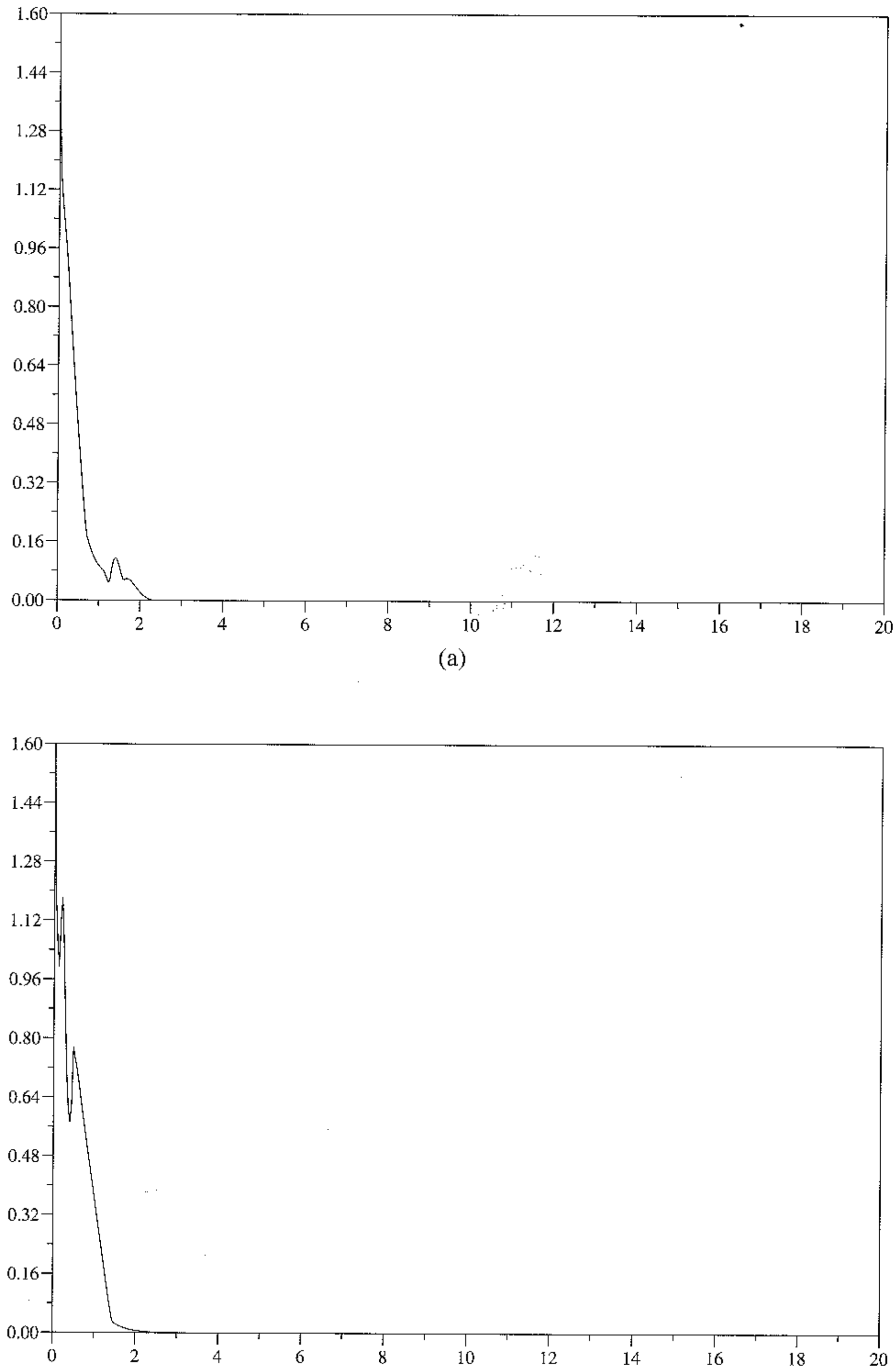



Fig. 4. Target's trajectory 1: norm of the function $e_p$ associated with the manipulation objective. *(Continues on next page)*

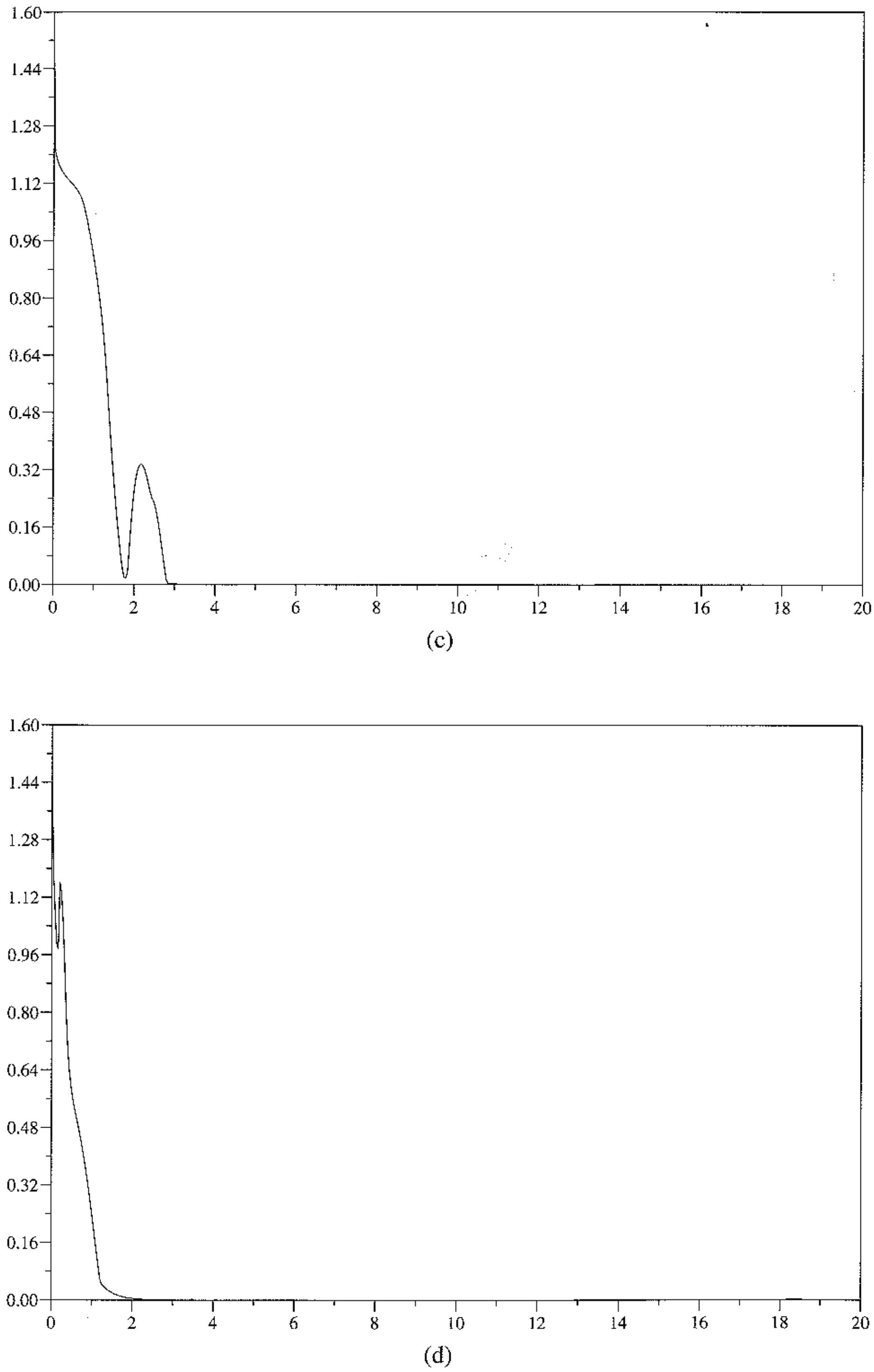



Fig. 4. *(Continued from previous page)*

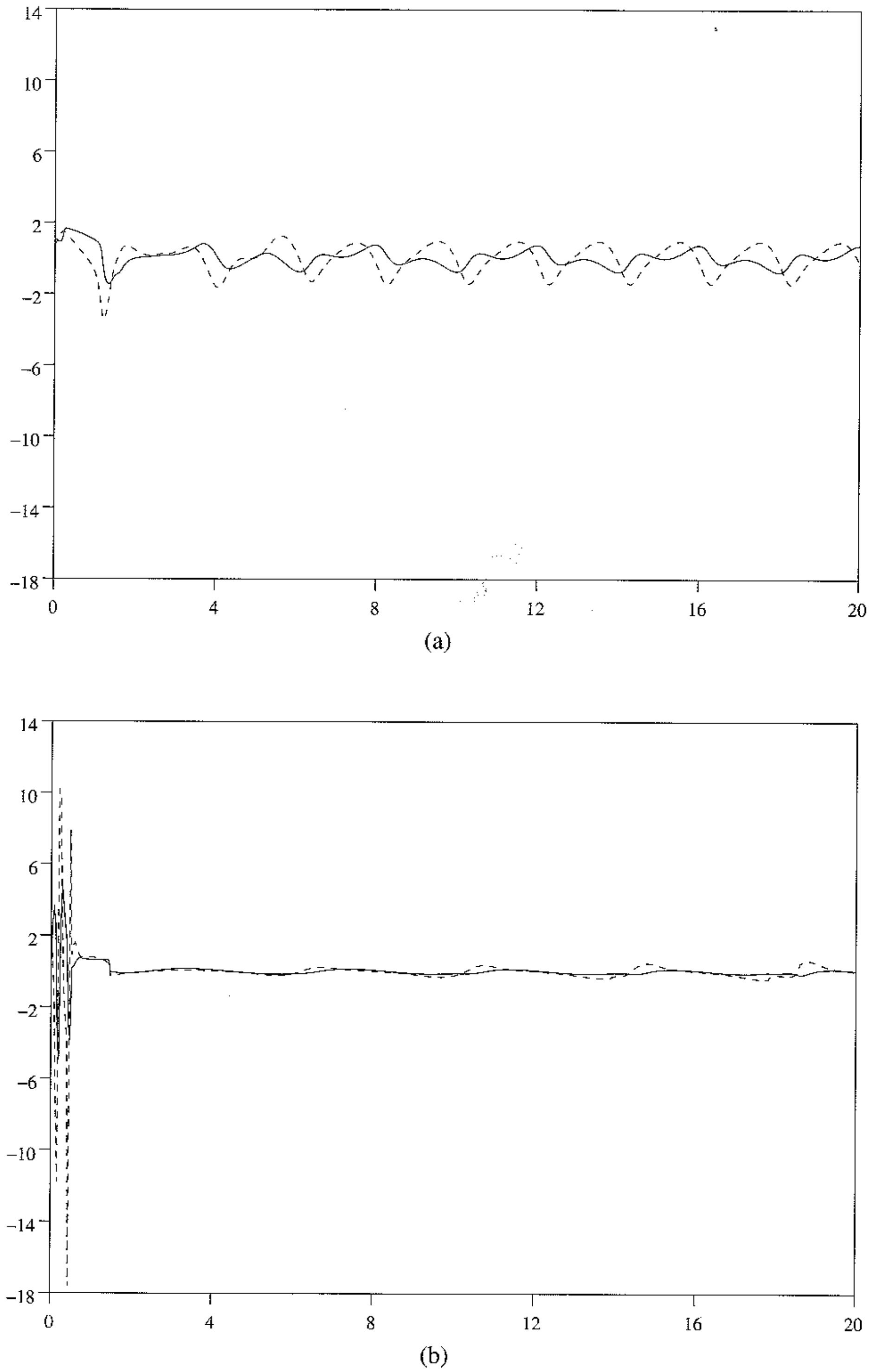



Fig. 5. Target's trajectory 1: mobile platform velocities $v_{b,1}$ (solid line) and $\omega_b$ (dashed line). *(Continues on next page)*

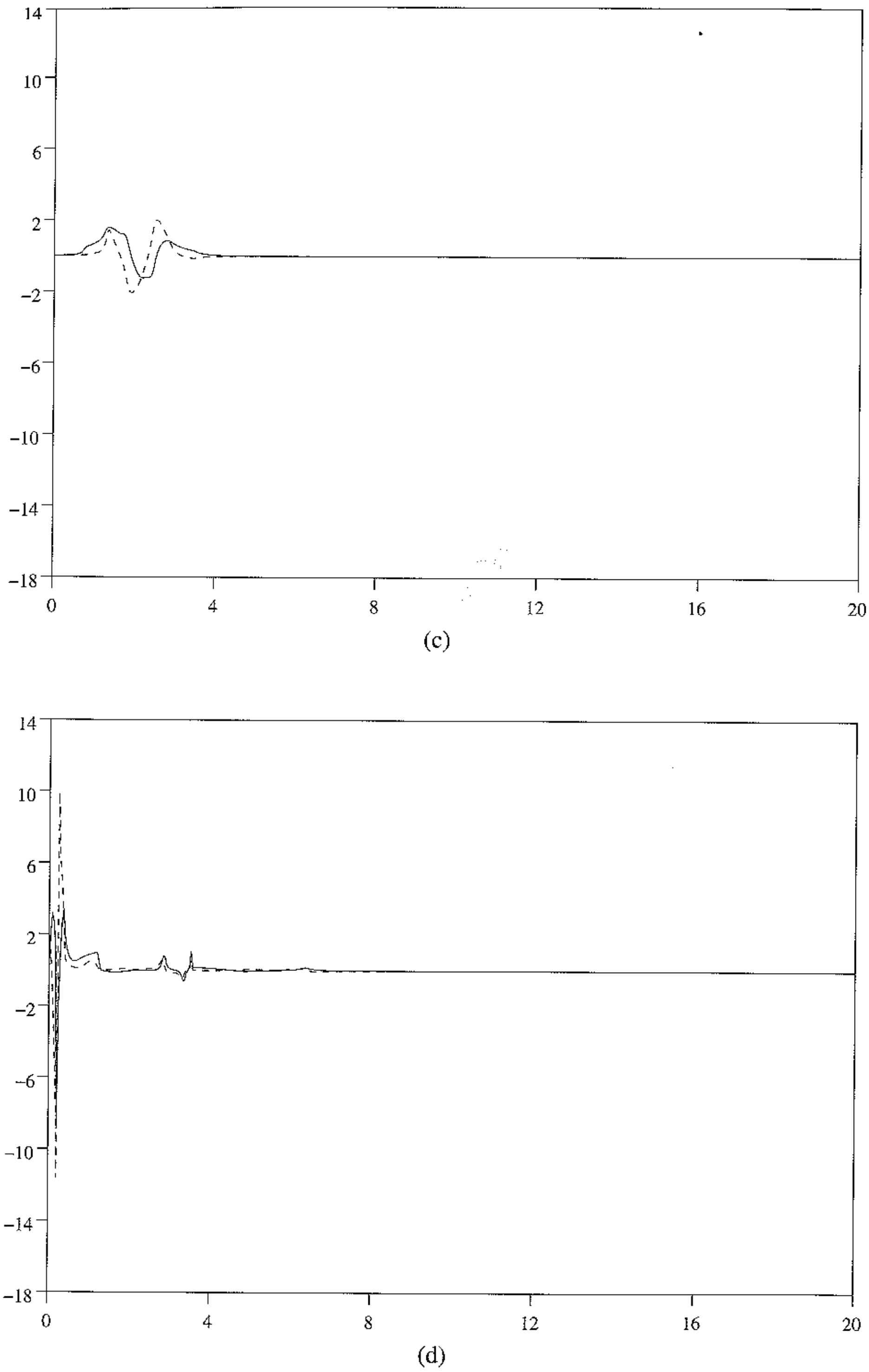
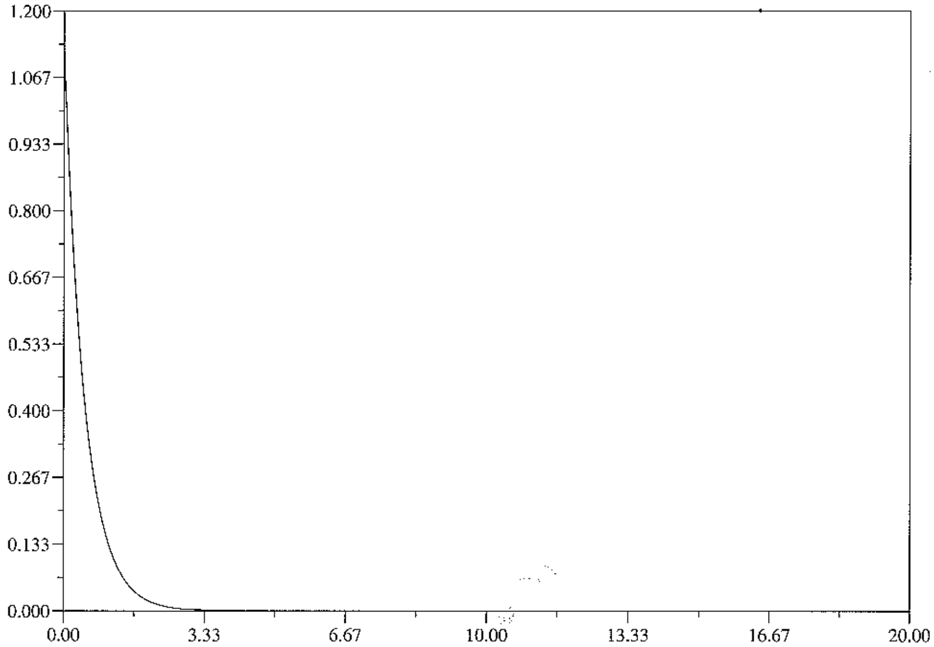
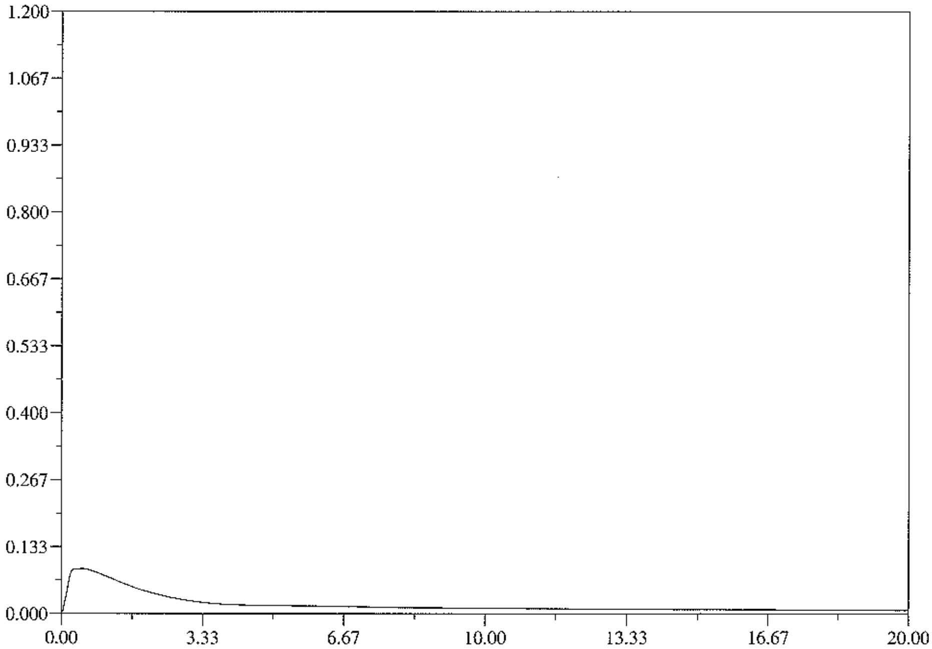



Fig. 5. *(Continued from previous page)*

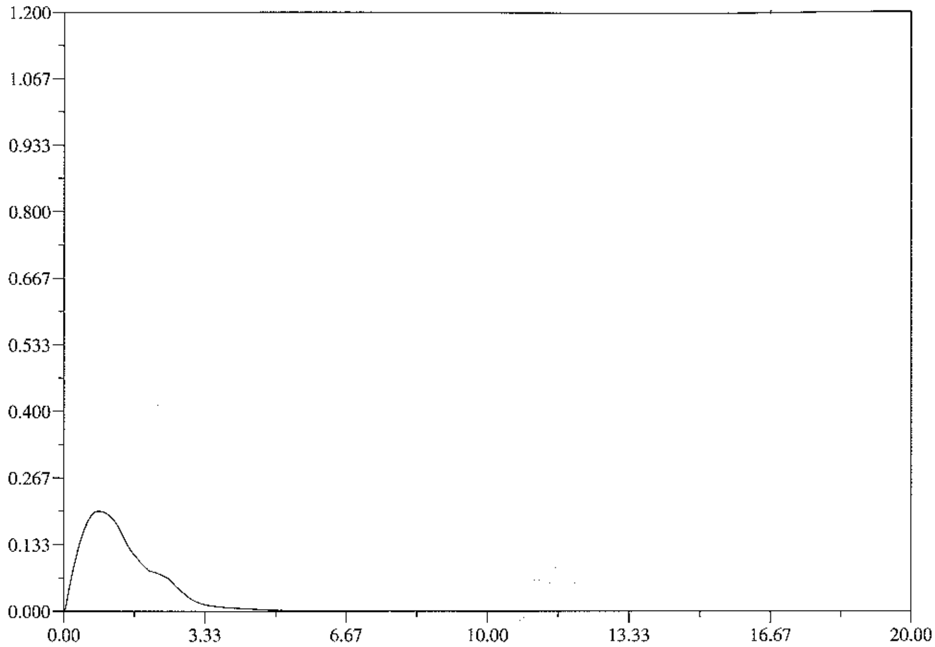



(a)

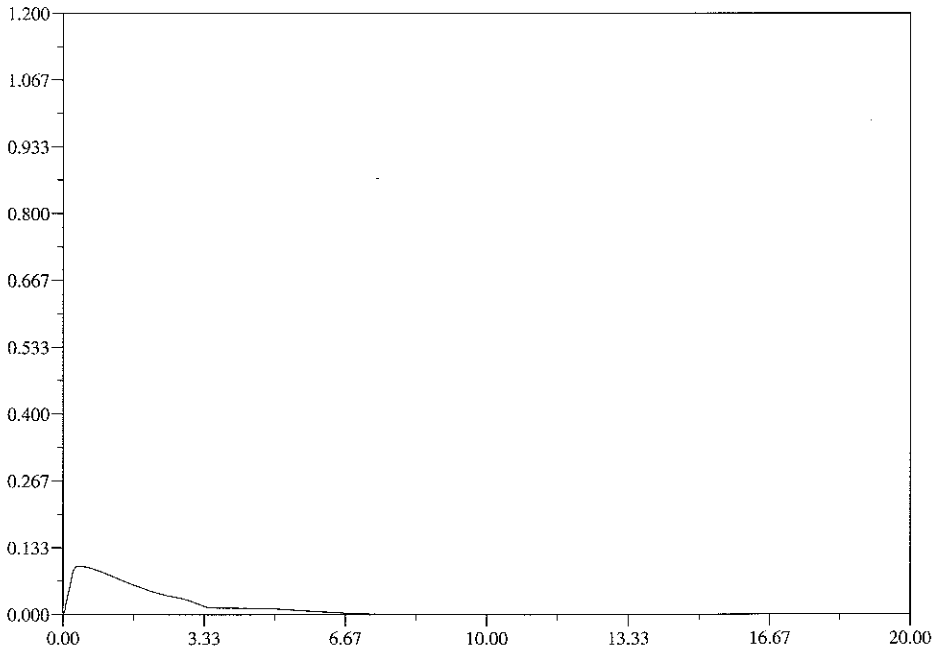



(b)

Fig. 6. Target's trajectory 1: secondary cost-function $h_s$. *(Continues on next page)*




(c)




(d)

Fig. 6. *(Continued from previous page)*

**Table 1. Target's Velocity for Trajectory 2**

| Time Interval in s | Target's Velocity (in $m.s^{-1}$, $m.s^{-1}$ and $rad.s^{-1}$) | Target's Motion |
|---|---|---|
| $t \in [0, 5)$ | $u_d(t) = (0, 0, 0)^T$ | Motionless |
| $t \in [5, 27)$ | $u_d(t) = (0, 0, \frac{\pi^2}{20} \cos \frac{\pi (t-5)}{2})^T$ | Rotation |
| $t \in [27, 37)$ | $u_d(t) = (0.2, 0, 0)^T$ | Translation |
| $t \in [37, 59)$ | $u_d(t) = (0, 0, \frac{\pi^2}{20} \cos \frac{\pi (t-37)}{2})^T$ | Rotation |
| $t \in [59, 99)$ | $u_d(t) = (0, 0.05, 0)^T$ | Translation |
| $t \in [99, 105)$ | $u_d(t) = (0, 0, 0)^T$ | Motionless |
| $t \in [105, 115)$ | $u_d(t) = (-0.2, 0, 0)^T$ | Translation |
| $t \in [115, 120)$ | $u_d(t) = (0, 0, 0)^T$ | Motionless |

parameters $(\lambda, \xi) = (1.66, 1)$, $\delta_i^- = 0.2(q_{i,*} - q_i^-)$, $\delta_i^+ = 0.2(q_i^+ - q_{i,*})$, $\delta_i^{--} = \frac{2}{3}\delta_i^-$, and $\delta_i^{++} = \frac{2}{3}\delta_i^+$ have been used in the definition of $\tilde{\Psi}$ (eq. (52)). The main difference w.r.t. the previous case concerns the definition of $h_s$, and the effect of inertia resulting from the function $h_{s,2}$. Figures 3(b) and 5(b) confirm that this strategy reduces the amplitude of the platform's displacements. This does not impede the achievement of the manipulation objective, as shown by Figure 4(b). Note that although $h_s$ no longer tends to zero, because $b_2 \neq 0$, it is ultimately small (Figure 6(b)).

(c) $\Psi(t)$ is defined as above (Simulation (b)), and $(\hat{b}_1, \hat{b}_2)$ is set according to **Choice 2** in Section 3.4.2, i.e., $(\hat{b}_1, \hat{b}_2) = (0, 0)$. By comparison with the previous simulation, this choice yields a much smoother motion of the platform and convergence to a fixed posture. The manipulation objective is again perfectly achieved after the transient phase is extinct, although this phase lasts longer (Figure 4(c)).

(d) $\Psi(t)$ is defined as above, and $(\hat{b}_1, \hat{b}_2)$ is set according to **Choice 3** in Section 3.4.2, i.e., $(\hat{b}_1, \hat{b}_2) = (0, \frac{\partial h_s}{\partial t})$. This corresponds to a compromise between the choices made in simulations (b) and (c), since it allows the partial precompensation of the drift term $\frac{\partial h_s}{\partial t}$ in (38). For this type of target's motion, the resulting behavior of the mobile manipulator is not very different from the one obtained in simulation (c).

**Target's trajectory 2:** For the simulation results reported in Figures 7–10, the same control parameters as above have been considered and tested with the four different choices of $\Psi$, $\hat{b}_1$, and $\hat{b}_2$ corresponding to simulations (a), (b), (c), and (d). However, the target's motion is now more complex and its larger amplitude is such that, during some time intervals, the platform has to move and execute maneuvers in order to allow for the tracking of the target by the manipulator's end-effector. The initial situation of the target w.r.t. the platform is $r_{bd}(0) = (2.5, 1, 0)^T$, and the initial configuration of the manipulator arm is $q(0) = (0, 0.35, 0)^T$. The target's velocity $u_d$ is specified in Table 1.

From this table, and as one can observe in Figure 7, the trajectory of the origin of the target's frame is composed of three straight lines. Contrary to the previous set of simulations, the orientation of the target is no longer constant. The simulation results (Figures 7–10) basically confirm the observations made in the previous case. In particular,

1. Choice (a), for the control design, yields an excellent tracking of the target by the manipulator's end-effector. In accordance with Proposition 1, one can indeed verify on Figure 8 that the tracking error converges to zero. This is the only control scheme for which this property is satisfied. However, this performance is obtained at the expense of the steady motion of the mobile platform and numerous maneuvers.

2. Choice (b) reduces the number of the platform's maneuvers, but the platform's velocities are still large during some of these maneuvers, as can be observed by comparing the platform's velocities in Figures 9(a) and 9(b). In addition, this control strategy no longer ensures perfect tracking of the target by the manipulator's end-effector all the time. The problem occurs at times when the delayed maneuvering of the mobile platform, in order to catch-up with the evading target, prevents the manipulation objective from being achieved without a manipulator's joint colliding into one of its limits.

3. In comparison with Choice (b), choices (c) and (d) yield even fewer maneuvers and smaller platform's velocities, as can be observed by comparing Figures 9(c)-9(d) with Figures 9(a)-9(b). The amount of time during which the tracking of the target by the manipulator's end-effector is perfectly executed is not very different, even though the frequency of the time intervals during which perfect tracking is lost does not seem to be favorable to choice (b) (see Figures 8(b), 8(c), and 8(d)). In

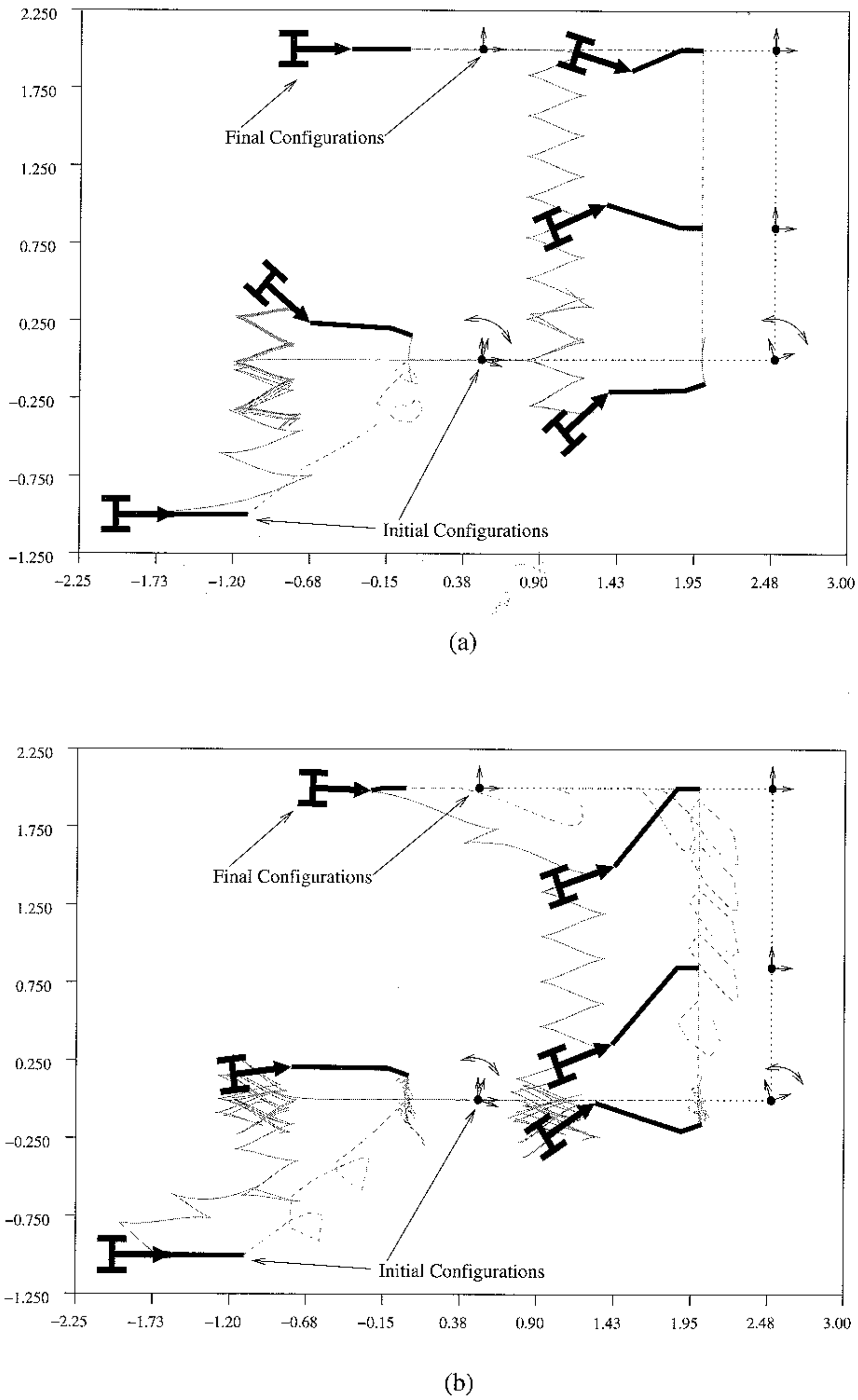



Fig. 7. Target's trajectory 2: Cartesian trajectories of the mobile manipulator and target:

- trajectory of the origin of the platform's frame $\mathcal{F}_b$ (quasi-solid gray line)
- trajectory of the origin of the end effector's frame $\mathcal{F}_e$ (dashed gray line)
- trajectory of the origin of the target's frame $\mathcal{F}_d$ (dotted gray line).

*(Continues on next page)*

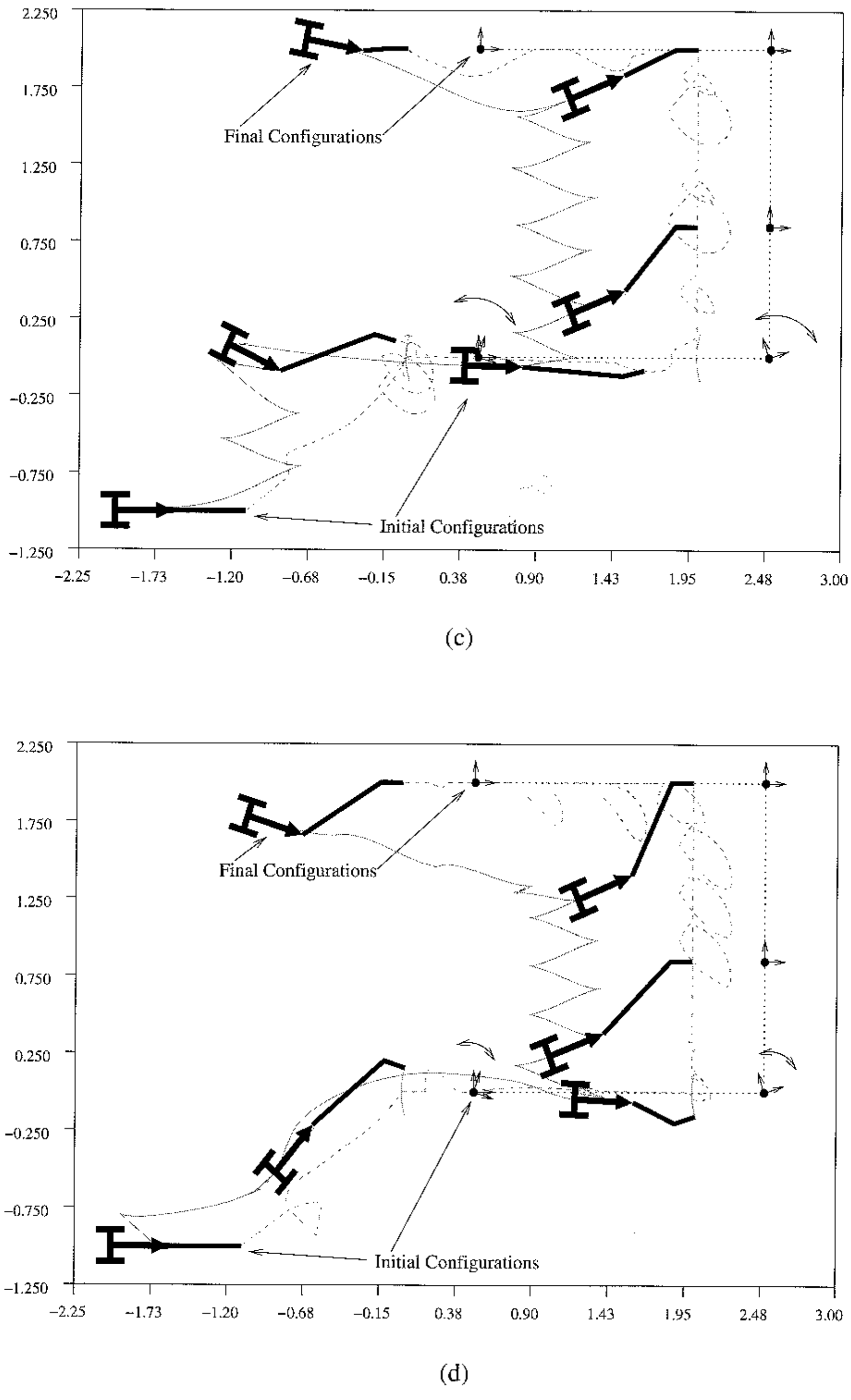
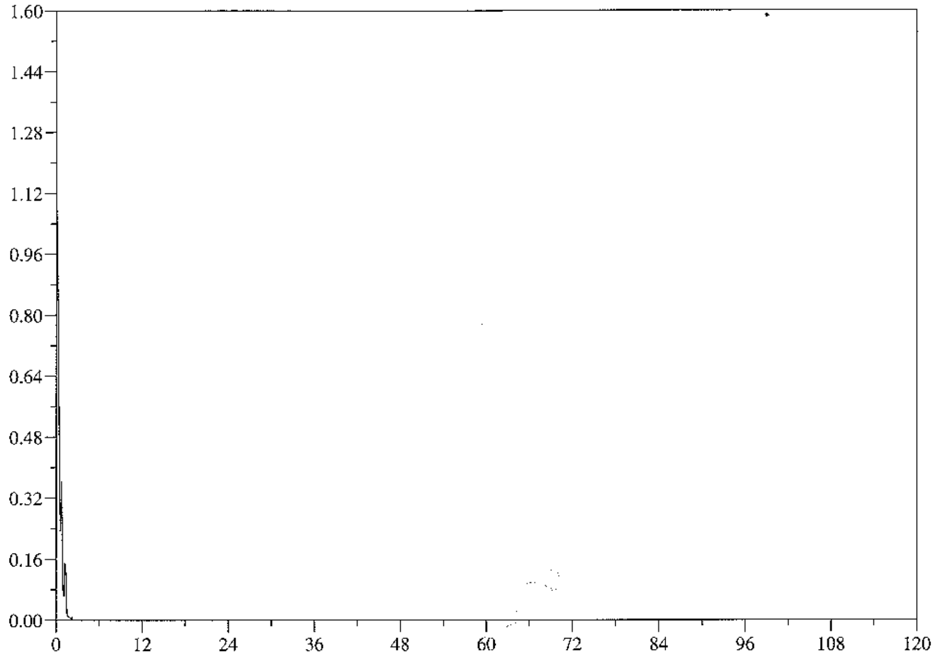
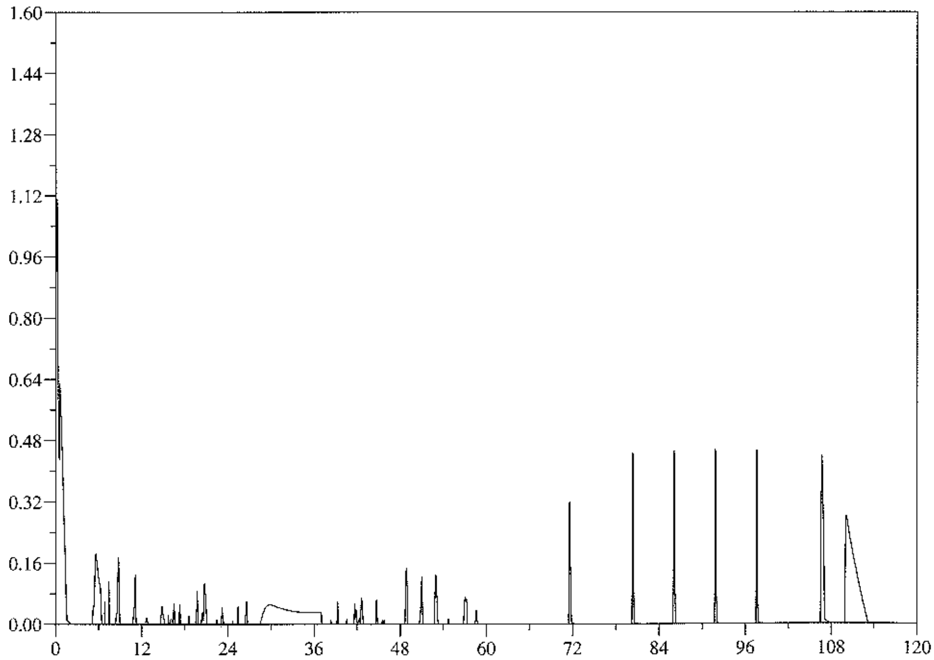



Fig. 7. *(Continued from previous page)*




(a)




(b)

Fig. 8. Target's trajectory 2: norm of the function $e_p$ associated with the manipulation objective. *(Continues on next page)*

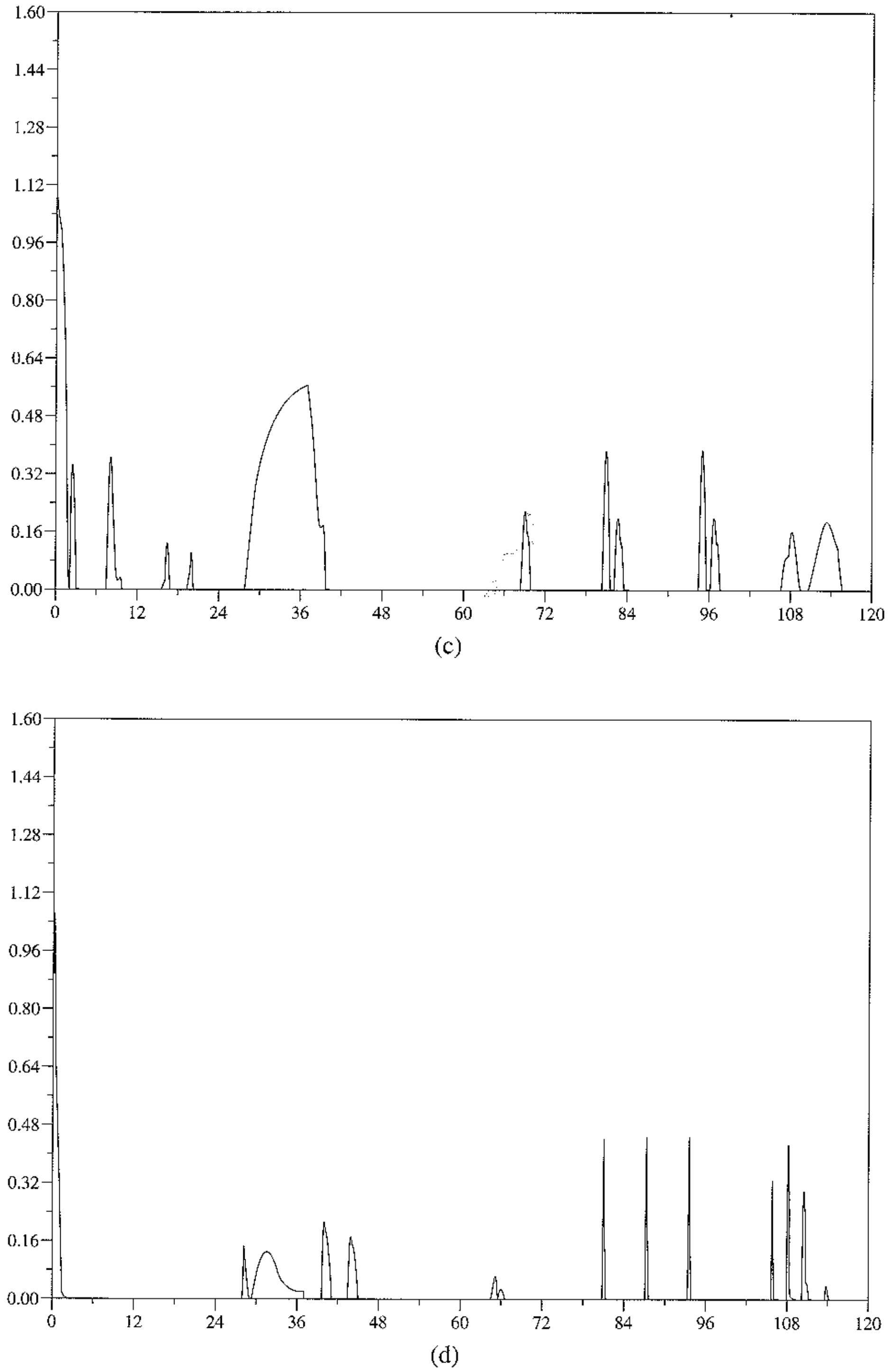
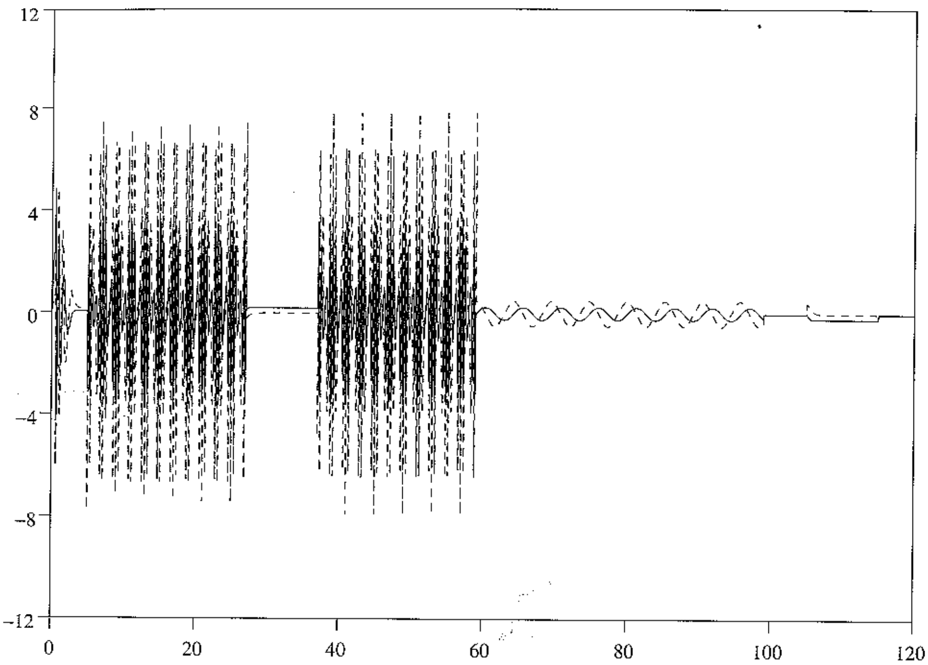
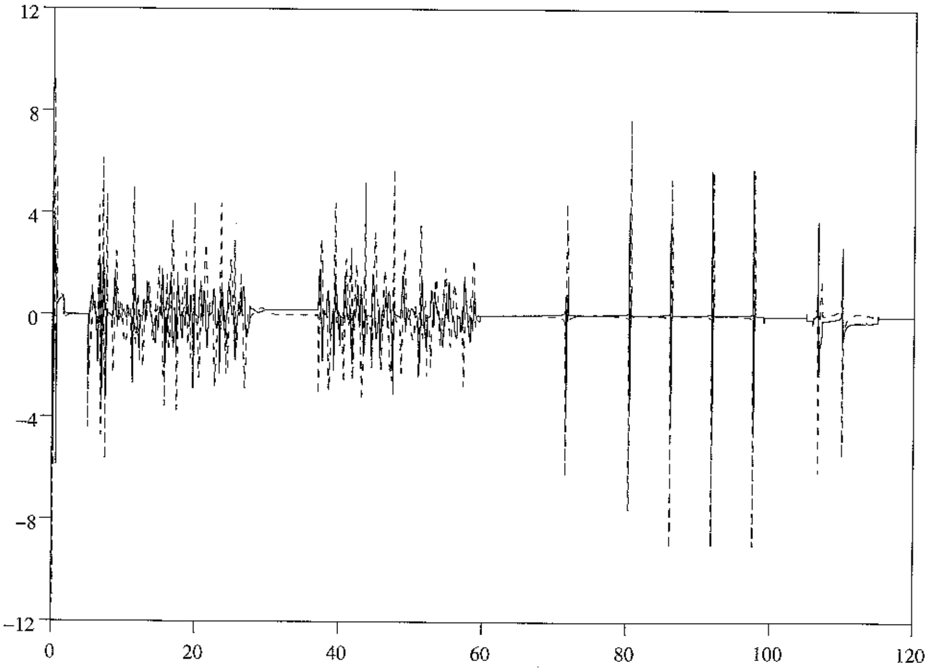



Fig. 8. *(Continued from previous page)*

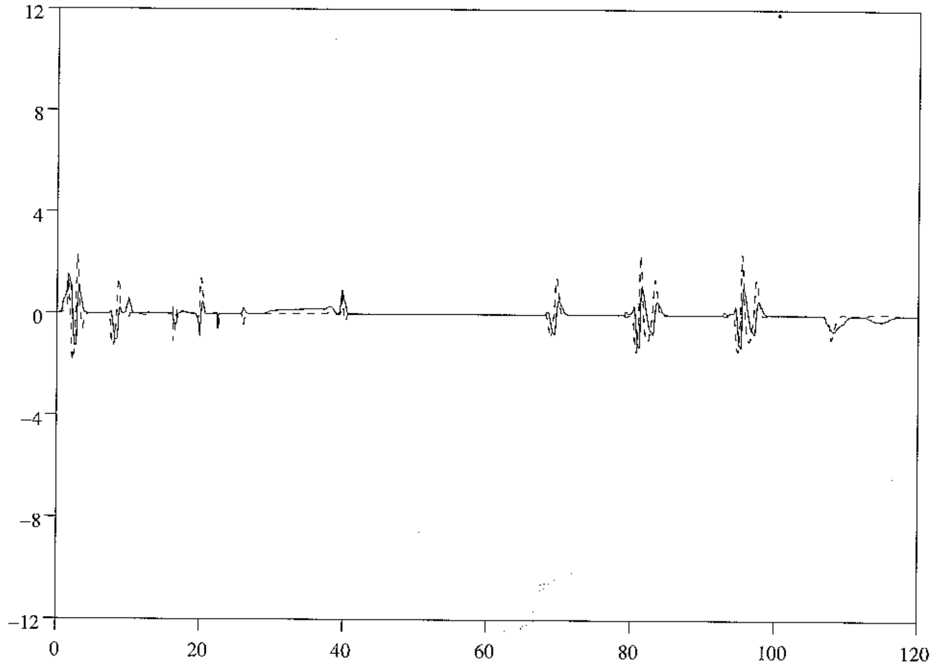



(a)

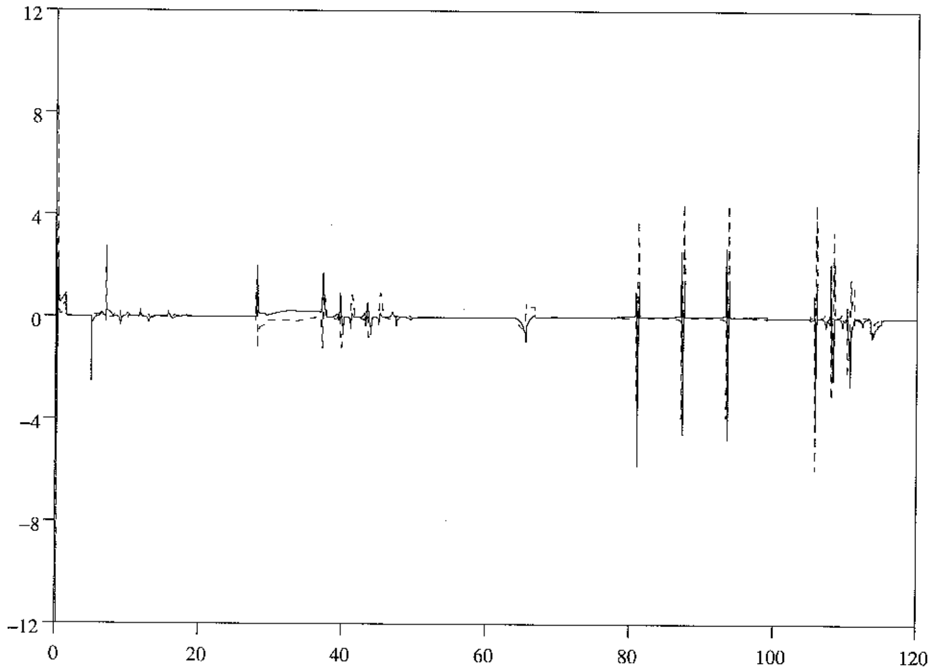



(b)

Fig. 9. Target's trajectory 2: mobile platform velocities $v_{b,1}$ (solid line) and $\omega_b$ (dashed line). *(Continues on next page)*




(c)




(d)

Fig. 9. *(Continued from previous page)*

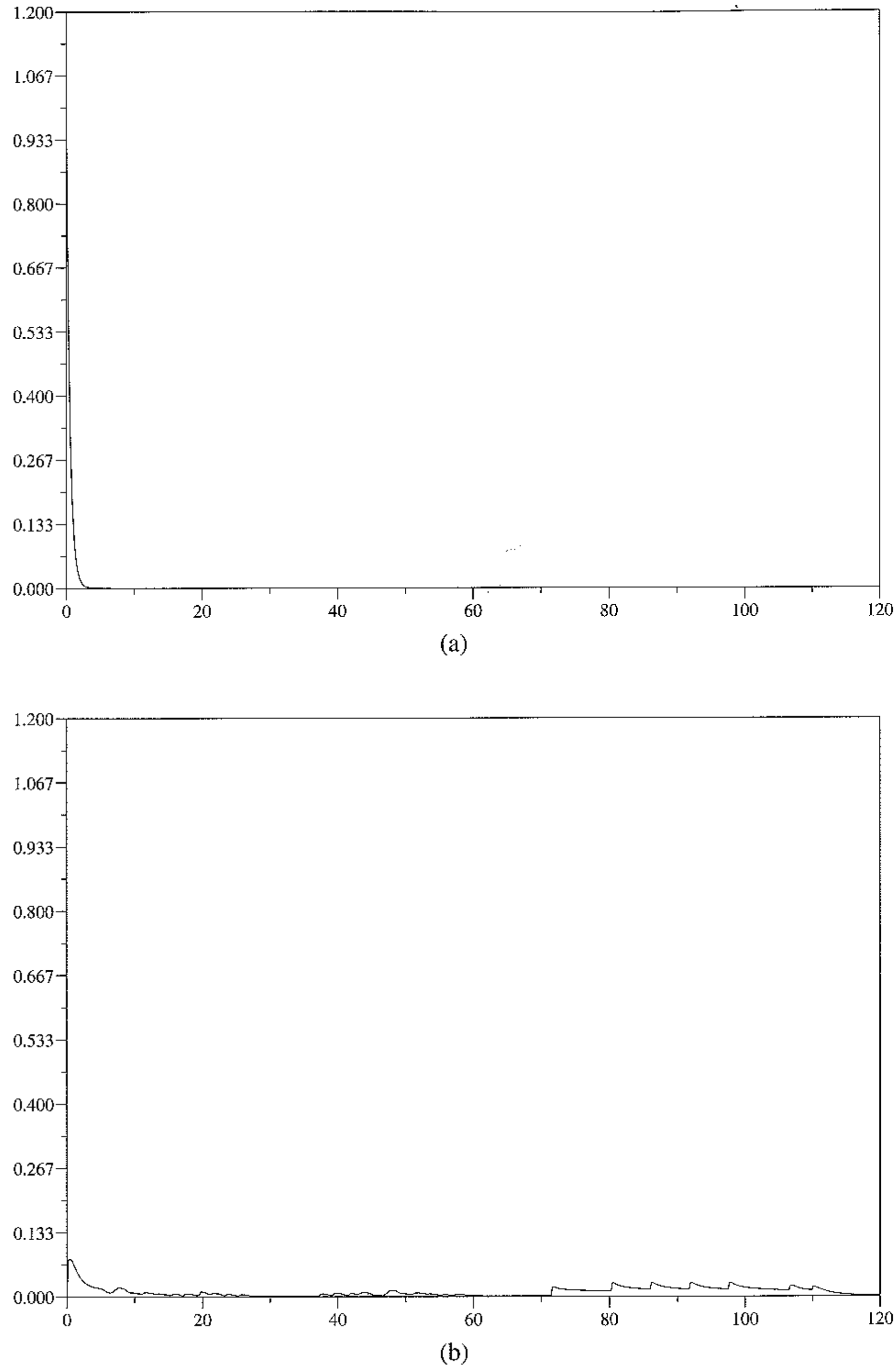



Fig. 10. Target's trajectory 2: secondary cost-function $h_s$. *(Continues on next page)*

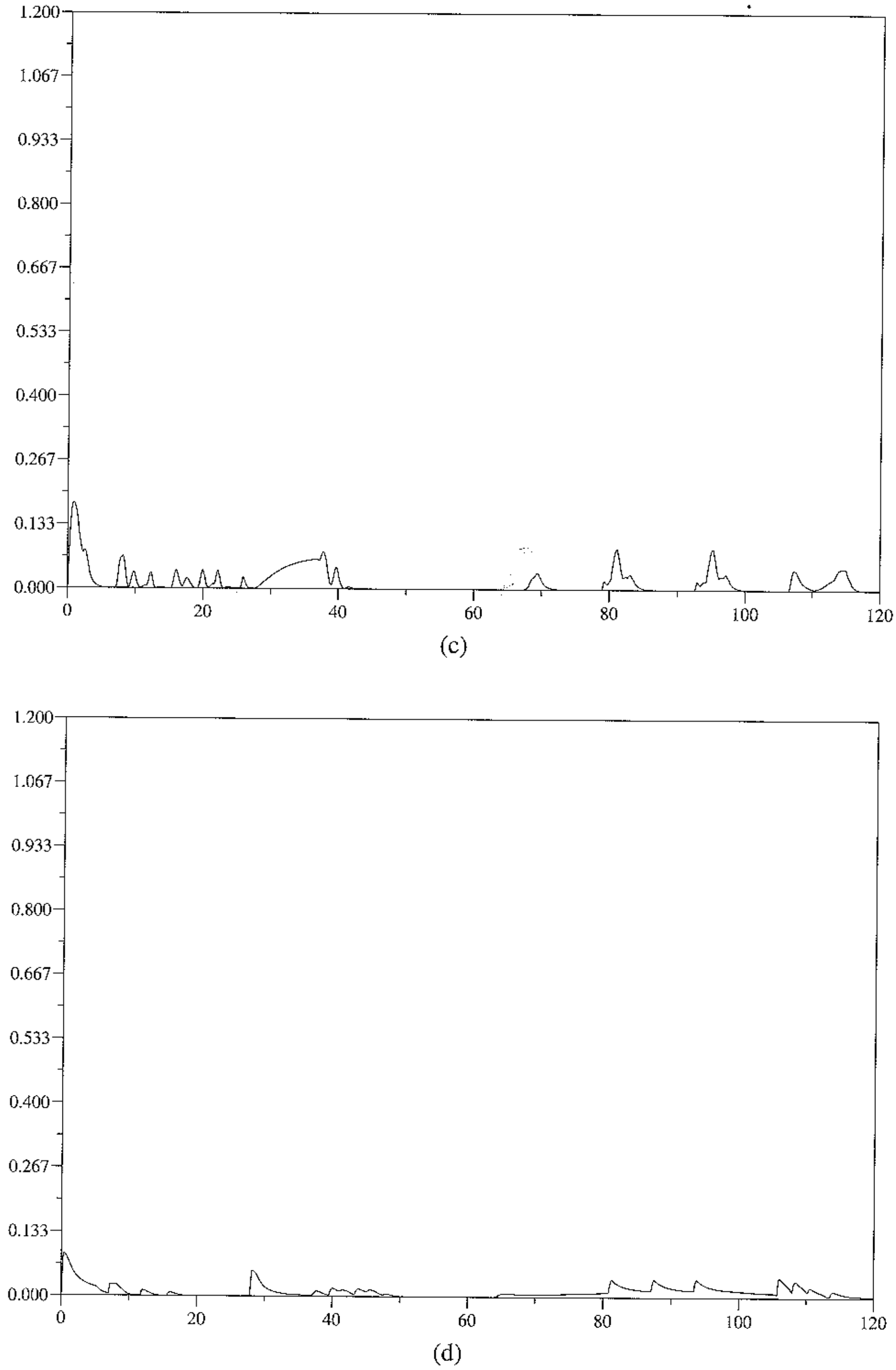



Fig. 10. *(Continued from previous page)*

this respect, and as expected from the analysis, choice (d) produces a slightly more reactive and effective control than choice (c). This is coherent with sharper variations of the platform velocities, as can be observed from Figures 9(c) and 9(d).

## 5. Extension to More General Mobile Manipulators

Some extensions of the present approach, beyond the framework of planar mobile manipulators and unicycle-like platforms, are now discussed.

First, it is important to note that the approach directly applies to general manipulators. For instance, target tracking in $SE(3)$ with conventional six d.o.f. manipulator arms can be addressed in the same way as in Section 4 which dealt with applications in $SE(2)$. To be more specific, the task function $e_p$ can then be defined (compare with 42) as $e_p(q, r_b, t) = \tilde{\mathbf{r}}_{ed} = (\mathbf{r}^*_{ed})^{-1}\mathbf{r}_{ed}$, with $\mathbf{r}_{ed}$ the situation in $SE(3)$ of the target's frame with respect to the end-effector's frame, and $\mathbf{r}^*_{ed}$ the associated desired situation. This is a six-dimensional vector-valued function. The secondary cost-function can still be defined by (49), with $r_d \in SE(2)$ the "projection" of the situation $\mathbf{r_d} \in SE(3)$ of the target's frame on the vehicle's plane (i.e., $r_d = (p_d, \theta_d)^T$ with $p_d$ the position vector associated with the projection on the plane of the target's frame origin, and $\theta_d$ the yaw Euler-angle associated with this frame). From there, one can proceed as in Section 4 for the control design, and establish a convergence result similar to the one of Proposition 1. But of course, the convergence of the tracking error to zero necessarily involves some assumptions concerning the compatibility of the target's motion (along the vertical axis, in particular) and the domain reachable by the end-effector.

The present approach can also be extended to other types of mobile platforms. Unicycle-like platforms are specific because their kinematic equations can be described by a left-invariant system on the Lie group $SE(2)$. Although the transverse function approach is best suited for systems on Lie groups, it is not restricted to this framework. For example, the case of car-like platforms (and, more generally, the case of $N$-trailers systems) can also be addressed via minor modifications of the control design presented in Section 3, as we will now demonstrate.

The control design methodology described in Section 3 relies on the fact that $\bar{r}_b$ is the configuration of an omnidirectional *companion* frame $\mathcal{F}_{\bar{b}}(\alpha)$, close to $\mathcal{F}_b$. In the case of a unicycle-like platform, we have shown in Section 2.3 how $\bar{r}_b$ can be defined from the transverse function approach. We explain next how the results of Section 2.3 (and, therefore, the whole control approach) can be extended to a car-like platform.

With the notation of Section 2, the kinematic equations of a car-like platform w.r.t. the fixed frame $\mathcal{F}_0$ are given by

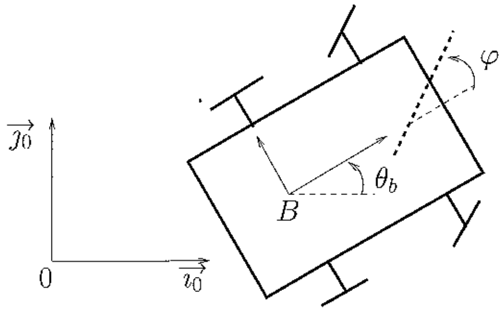



Fig. 11. Car-like mobile platform.

$$\begin{cases} \dot{x}_b &= v_{b,1}\cos\theta_b \\ \dot{y}_b &= v_{b,1}\sin\theta_b \\ \dot{\theta}_b &= v_{b,1}\xi \\ \dot{\xi} &= v_\xi \end{cases} \tag{55}$$

with $\xi = (\tan\varphi)/\ell$, $\varphi$ the steering angle of the vehicle (see Figure 11 below), and $\ell$ the distance between the front and rear wheels' axles.

System (55) can also be written as (compare with (3))

$$\begin{cases} \dot{r}_b &= \bar{R}(\theta_b)u_b \\ \dot{\xi} &= v_\xi \end{cases} \tag{56}$$

with $u_b = (v_{b,1}, 0, v_{b,1}\xi)$. Let us now consider a function

$$f : \alpha \longmapsto f(\alpha) = \begin{pmatrix} r_f(\alpha) \\ \xi_f(\alpha) \end{pmatrix} := \begin{pmatrix} p_f(\alpha) \\ \theta_f(\alpha) \\ \xi_f(\alpha) \end{pmatrix}$$

from $S^1 \times S^1$ to $SE(2) \times \mathbb{R}$ (i.e., $\alpha = (\alpha_1, \alpha_2)$). We still define $\bar{r}_b(\alpha)$ as in (11), i.e.,

$$\bar{r}_b(\alpha) := \begin{pmatrix} p_b - R(\theta_b - \theta_f(\alpha))p_f(\alpha) \\ \theta_b - \theta_f(\alpha) \end{pmatrix} = \begin{pmatrix} \bar{p}_b \\ \bar{\theta}_b \end{pmatrix}$$

so that eq. (12) is still valid, i.e.,

$$\dot{\bar{r}}_b = \bar{R}(\bar{\theta}_b)\bar{u}_b$$

with

$$\bar{u}_b := \begin{pmatrix} I_2 & -Sp_f(\alpha) \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \bar{R}(\theta_f(\alpha)) & -\frac{\partial r_f}{\partial \alpha_1}(\alpha) & -\frac{\partial r_f}{\partial \alpha_2}(\alpha) \end{pmatrix} \begin{pmatrix} u_b \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix}.$$

Since $u_b = (v_{b,1}, 0, v_{b,1}\xi)$, the previous equation can be written as

$$\bar{u}_b := \begin{pmatrix} I_2 & -Sp_f(\alpha) \\ 0 & 1 \end{pmatrix} H(\alpha) \begin{pmatrix} v_{b,1} \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix} + \begin{pmatrix} I_2 & -Sp_f(\alpha) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v_{b,1}(\xi - \xi_f(\alpha)) \end{pmatrix} \quad (57)$$

with

$$H(\alpha) := \begin{pmatrix} \cos\theta_f(\alpha) & -\frac{\partial x_f}{\partial \alpha_1}(\alpha) & -\frac{\partial x_f}{\partial \alpha_2}(\alpha) \\ \sin\theta_f(\alpha) & -\frac{\partial y_f}{\partial \alpha_1}(\alpha) & -\frac{\partial y_f}{\partial \alpha_2}(\alpha) \\ \xi_f(\alpha) & -\frac{\partial \theta_f}{\partial \alpha_1}(\alpha) & -\frac{\partial \theta_f}{\partial \alpha_2}(\alpha) \end{pmatrix}. \quad (58)$$

Let us recall that $\xi_f$ denotes the last component of the function $f$. Since $\dot{\xi} = v_\xi$, the following choice of the steering wheel angular velocity:

$$v_\xi := -k(\xi - \xi_f(\alpha)) + \dot{\xi}_f(\alpha) \qquad (k > 0) \quad (59)$$

yields the exponential stabilization of $\xi - \xi_f(\alpha)$ to zero. Let us therefore assume from now on that $\xi - \xi_f(\alpha)$ is exactly equal to zero. We deduce from (57) that (compare with (14)):

$$\bar{u}_b = \begin{pmatrix} I_2 & -Sp_f(\alpha) \\ 0 & 1 \end{pmatrix} H(\alpha) \begin{pmatrix} v_{b,1} \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix} = \bar{H}(\alpha) \begin{pmatrix} v_{b,1} \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \end{pmatrix} \quad (60)$$

so that the frame $\mathcal{F}_{\bar{b}}(\alpha)$ associated with $\bar{r}_b$ can be viewed as an omnidirectional frame provided that the matrix $H(\alpha)$ given by (58) is invertible. Let us momentarily assume that this property holds. Then, since the control approach presented in Section 3 only relies on the fact that the frame $\mathcal{F}_{\bar{b}}(\alpha)$ is omnidirectional (and not on the characteristics of the mobile platform), the feedback laws $u_q$ and $\bar{u}_b$ given by (30) and (37) can again be used to yield the *same closed-loop dynamics* for the manipulation task $e_m$ and the secondary cost $h_s$ as those derived for the unicycle-like platform[1]. The control inputs for the platform (i.e., $v_{b,1}$ and $v_\xi$) are then computed according to (59) and (60), instead of (14) in the case of a unicycle-like platform.

There remains the issue of the invertibility of $H(\alpha)$ (for any $\alpha$). This corresponds to the transversality property of the function $f$, as defined in Morin and Samson (2001). A family of transverse functions for the car has been proposed in Morin and Samson (2004):

1. Note however that, when $K$ in (37) is chosen according to (36), the dynamics of $h_s$ can be slightly different for the two types of platforms since the associated matrices $\bar{H}(\alpha)$ are not the same.

LEMMA 2. (Morin and Samson 2004) For any $\varepsilon > 0$, and $\eta_1, \eta_2, \eta_3$ such that

$$\eta_1, \eta_2, \eta_3 > 0, \quad \text{and} \quad 6\eta_2\eta_3 > 8\eta_3 + \eta_1\eta_2$$

the function $f$ defined by

$$f(\alpha) = \begin{pmatrix} \bar{f}_1(\alpha) \\ \bar{f}_4(\alpha) \\ \arctan(\bar{f}_3(\alpha)) \\ \bar{f}_2(\alpha)\cos^3 f_3(\alpha) \end{pmatrix}$$

with $\bar{f} : S^1 \times S^1 \longrightarrow \mathbb{R}^4$ given by

$$\bar{f}(\alpha) = \begin{pmatrix} \varepsilon(\sin\alpha_1 + \eta_2 \sin\alpha_2) \\ \varepsilon\eta_1 \cos\alpha_1 \\ \varepsilon^2(\frac{\eta_1 \sin 2\alpha_1}{4} - \eta_3 \cos\alpha_2) \\ \varepsilon^3(\eta_1 \frac{\sin^2\alpha_1 \cos\alpha_1}{6} - \frac{\eta_2\eta_3 \sin 2\alpha_2}{4} - \eta_3 \sin\alpha_1 \cos\alpha_2) \end{pmatrix}$$

satisfies the transversality condition $\det H(\alpha) \neq 0 \; \forall\alpha$, where $H(\alpha)$ is defined by (58).

Note that for the family of functions defined in this lemma, the norm of $\bar{f}$ (like the distance from $r_b$ to $\bar{r}_b$) tends to zero as $\varepsilon$ tends to zero. Therefore, the distance from $\mathcal{F}_b$ to the companion frame $\mathcal{F}_{\bar{b}}(\alpha)$ can also be rendered arbitrarily small in this case, via the choice of the parameter $\varepsilon$. However, as in the unicycle case, $\varepsilon$ should not be chosen too small in practice in order to limit a certain number of adverse effects induced by large control gains (see comments at the end of Section 2.3). Again, some extra work is needed to determine values of $\varepsilon$ and $\eta_{1,2,3}$ which, for a given application, achieve a satisfactory compromise between several, often antagonistic, requirements. Bearing in mind that there are an infinite number of transverse functions, this aspect of the problem could more generally be termed as transverse function *shaping*.

## 6. Conclusion

A framework for the feedback control of mobile manipulators, with a focus on motion coordination between the manipulator and the supporting mobile platform, has been proposed. It applies to omnidirectional platforms as well as to nonholonomic ones. In this respect, the concept of a companion omnidirectional frame associated with the body of the mobile platform, and derived from the transverse function approach (Morin and Samson 2003), is instrumental. It basically allows the control of any nonholonomic platform as if it were omnidirectional.

From there, one is left with the problem of using the redundancy provided by the mobile platform's d.o.f. as effectively as possible. At this level, the proposed control design methodology resembles closely the approach classically used for redundant manipulators consisting in the minimization of a cost (potential) function associated with a secondary objective constrained by the realization of a set of equalities representing the primary objective. In the case of mobile manipulators the choice of the equality constraints is further influenced by the complementarity of the roles played by the manipulator and the mobile platform. The idea is that a manipulation task, which the manipulator is in charge of performing independently of the mobile platform's motion, is first specified in the form of a vector-valued task-function which has to be regulated to zero. Setting this function equal to zero is equivalent to specifying a set of equality constraints. This may also be interpreted as the enforcement of a set of virtual linkages. For safety reasons, this function must also incorporate terms which allow the unconditional avoidance of collisions with the manipulator's joint limits. This is an absolute requirement which precludes its treatment as a secondary objective. Then, the problem of stabilizing the resulting function at zero (this is the role devoted to the control) must be well posed in the sense of the "task-admissibility" defined in Samson, Leborgne, and Espiau (1991). A way of obtaining such a function is by considering the gradient, w.r.t. the manipulator's joint variables, of a multicriteria penalty function. The platform's motion is determined via the secondary cost-function that the control attempts to minimize. This function, which depends on the mobile platform's situation—it may also depend on time—can be a weighted sum of several elementary functions, each of them corresponding to a specific, sometimes antagonistic, objective. The fact that the exact minimization of this function is often unachievable is not necessarily penalizing because the precise positioning of the mobile platform is usually not required in practice. From these general design rules, many motion coordination strategies can be worked out. We have tried to illustrate some of them via a target tracking example and simulation results. For instance, these simulations validate the possibility of synthesizing (smooth) feedback controllers capable of making the nonholonomic platform maneuver automatically in order to perform the manipulation task while avoiding collisions with joint-limits. To our knowledge, this has not been demonstrated before. Finally, we wish to mention that the present study has been conducted with the hope of attracting the attention of fellow researchers in Robotics and Automatic Control to a broad, rich and still largely open subject.

## Appendix

### *Proof of Equation 4*

From (1), the product $r_c r_d$ can also be written as

$$r_c r_d = r_c + \bar{R}(\theta_c) r_d$$

Therefore, we deduce from (3) that

$$\begin{aligned}\dot{\widehat{r_c r_d}} &= \dot{r}_c + \bar{R}(\theta_c)\dot{r}_d + \dot{\bar{R}}(\theta_c) r_d \\ &= \bar{R}(\theta_c)u_c + \bar{R}(\theta_c)\bar{R}(\theta_d)u_d + \dot{\bar{R}}(\theta_c) r_d \\ &= \bar{R}(\theta_c + \theta_d)\left(u_d + \bar{R}(-\theta_d)u_c + \bar{R}(-\theta_c - \theta_d)\dot{\bar{R}}(\theta_c) r_d\right)\end{aligned}$$

By using (5) and the fact that $\dot{\theta}_c$ is equal to the last component of $u_c$, one can verify that

$$\bar{R}(-\theta_d)u_c + \bar{R}(-\theta_c - \theta_d)\dot{\bar{R}}(\theta_c) r_d = \mathrm{Ad}_{r_d^{-1}} u_c$$

and eq. (4) follows.

### *Proof of Lemma 1*

It follows from (15) and (16) that

$$\det H(\alpha) = \varepsilon_1\left(\frac{\varepsilon_2}{2}\cos(2\alpha)\cos(\arctan(\varepsilon_2\cos\alpha)) - \cos\alpha\sin(\arctan(\varepsilon_2\cos\alpha))\right)$$

Since $\sin(\arctan y) = y\cos(\arctan y)$,

$$\begin{aligned}\det H(\alpha) &= \varepsilon_1\left(\frac{\varepsilon_2}{2}(\cos^2\alpha - \sin^2\alpha)\cos(\arctan(\varepsilon_2\cos\alpha)) - \varepsilon_2\cos^2\alpha\cos(\arctan(\varepsilon_2\cos\alpha))\right) \\ &= -\frac{\varepsilon_1\varepsilon_2}{2}\cos(\arctan(\varepsilon_2\cos\alpha))\end{aligned}$$

Therefore, $\det H(\alpha) \neq 0$ for any $\alpha$ if $\varepsilon_1, \varepsilon_2 > 0$.

### *Proof of Proposition 1*

Since $h_s$ is decreasing along the solutions of the closed-loop system, we have (for all $t \geq 0$),

$$\begin{aligned}\mathrm{d}_\gamma(r_*, r_{db}(t)f^{-1}(\alpha(t))) &\leq \mathrm{d}_\gamma(r_*, r_{db}(0)f^{-1}(\alpha(0))) \\ &\leq \mathrm{d}_\gamma(r_*, r_{db}(0)) + \mathrm{d}_\gamma(r_{db}(0), r_{db}(0)f^{-1}(\alpha(0))) \\ &\leq \mathrm{d}_\gamma(r_*, r_{db}(0)) + \mathrm{d}_\gamma(e, f^{-1}(\alpha(0))) \\ &\leq \eta_2 + \bar{\varepsilon}\end{aligned}$$

so that

$$\begin{aligned}\mathrm{d}_\gamma(r_*, r_{db}(t)) &\leq \mathrm{d}_\gamma(r_*, r_{db}(t)f^{-1}(\alpha(t))) + \mathrm{d}_\gamma(r_{db}(t)f^{-1}(\alpha(t)), r_{db}(t)) \\ &\leq \mathrm{d}_\gamma(r_*, r_{db}(t)f^{-1}(\alpha(t))) + \mathrm{d}_\gamma(f^{-1}(\alpha(t)), e) \\ &\leq \eta_2 + 2\bar{\varepsilon}\end{aligned} \tag{61}$$

It follows from (42) that $e_p$ can be expressed as a function of $q$ and $r_{db}$ (since $r_{ed} = r_{eb}r_{bd} = r_{eb}(q)r_{db}^{-1}$), i.e., $e_p(q, r_b, t) = \bar{e}_p(q, r_{db})$. As a consequence, since $\Gamma = 0$, it follows from (23) that the manipulation task $e_m$ is also a function of $q$ and $r_{db}$, i.e., $e_m(q, r_b, t) = \bar{e}_m(q, r_{db})$. Now, by definition of $q_*$ and $r_*$, we deduce from (23) that

$$\bar{e}_m(q_*, r_*) = \left(\frac{\partial \bar{e}_p}{\partial q}\right)^T \bar{e}_p(q_*, r_*) = 0$$

and it follows from (24) and the fact that $\frac{\partial e_p}{\partial q}$ is invertible for any $q \in \prod_i (q_i^-, q_i^+)$ that

$$\frac{\partial \bar{e}_m}{\partial q}(q_*, r_*) > 0$$

Therefore, by application of the implicit function theorem, there exist $\tau_1, \tau_2 > 0$ such that, for $(r_{db}, \bar{e}) \in E := \{(r_{db}, \bar{e}) : \mathrm{d}_\gamma(r_*, r_{db}) < \tau_1 , \|\bar{e}\| < \tau_2\}$, the equation

$$\bar{e}_m(q, r_{db}) = \bar{e} \tag{62}$$

admits a solution $q = \beta(r_{db}, \bar{e})$, with $\beta(r_*, 0) = q_*$ and $\beta$ a continuous function uniquely defined on $E$. It also follows from the implicit function theorem that $\beta(E)$ contains a neighborhood $\mathcal{V}$ of $q_*$ (because $\frac{\partial \beta}{\partial \bar{e}}$ is an invertible matrix in a neighborhood of $(r_*, 0)$). Since $q_* \in \prod_i (q_i^- + \delta_i, q_i^+ - \delta_i)$ and $\beta$ is continuous, one can further assume (by possibly reducing $\tau_1$ and $\tau_2$) that $\beta(E) \subset \prod_i (q_i^- + \delta_i, q_i^+ - \delta_i)$. Let us choose the constants $\bar{\varepsilon}$, $\eta_1$, $\eta_2$ such that:

*i)* $\eta_2 + 2\bar{\varepsilon} \leq \tau_1$ (so that, by (61), $\mathrm{d}_\gamma(r_*, r_{db}(t)) < \tau_1$ for any $t$),

*ii)* $\{q : \|q - q_*\| \leq \eta_1\} \subset \mathcal{V}$,

*iii)* $\max\{\|\bar{e}_m(q, r_{db})\| : \|q - q_*\| \leq \eta_1, \mathrm{d}_\gamma(r_*, r_{db}) \leq \eta_2\} < \tau_2$ (so that $\|\bar{e}_m(q(0), r_{db}(0))\| < \tau_2$).

We deduce from these inequalities that $q(0) = \beta(r_{db}(0), \bar{e}_m(q(0), r_{db}(0)))$. Furthermore, since $e_m$ is decreasing along the solutions of the closed-loop system, $\|\bar{e}_m(q(t), r_{db}(t))\| < \tau_2 , \forall t$, and by Point *i)* above, $\mathrm{d}_\gamma(r_*, r_{db}(t)) < \tau_1 , \forall t$, so that by the local unicity of the implicit solution to (62),

$$q(t) = \beta(r_{db}(t), \bar{e}_m(q(t), r_{db}(t))) \in \prod_i (q_i^- + \delta_i, q_i^+ - \delta_i)$$

for all $t$. Therefore the manipulator's joint coordinates are always away from their limits and the value $h_\ell(q(t))$ of the corresponding penalty function is always equal to zero. This in turn implies that

$$\bar{e}_m(q(t), r_{db}(t)) = \left(\frac{\partial \bar{e}_p}{\partial q}\right)^T \bar{e}_p(q(t), r_{db}(t))$$

The exponential convergence to zero of $\bar{e}_p$ then follows from the one of $\bar{e}_m$ and the invertibility of $\frac{\partial \bar{e}_p}{\partial q}$.

## Acknowledgments

This work has been supported by the French program ROBEA.

## References

Artus, G., Morin, P., and Samson, C. 2004. Control of a maneuvering mobile robot by transverse functions. *Symposium on Advances in Robot Kinematics (ARK)*, pp. 459–468.

Bayle, B., Fourquet, J.-Y., Lamiraux, F., and Renaud, M. 2002. Kinematic control of wheeled mobile manipulators. *International Conference on Intelligent Robots and Systems (IROS)*, pp. 1572–1577.

Bayle, B., Fourquet, J.-Y., and Renaud, M. 2001. Génération de mouvements des manipulateurs mobiles. *Journal Européen des Systèmes Automatisés (JESA)* 35:809–845.

Bayle, B., Fourquet, J.-Y., and Renaud, M. 2003. Manipulability of wheeled mobile manipulators: application to motion generation. *International Journal of Robotics Research* 22:565–581.

Brockett, R. 1983. Asymptotic stability and feedback stabilization. Brockett, R., Millman, R., and Sussmann, H., editors, *Differential Geometric Control Theory*. Boston: Birkauser.

Chung, J. and Velinsky, S. 1998. Modeling and control of a mobile manipulator. *Robotica* 16:607–613.

Desai, J., Wang, C., Zefran, M., and Kumar, V. 1996. Motion planning for multiple mobile manipulators. *IEEE Conference on Robotics and Automation (ICRA)*, pp. 2073–2078.

Galicki, M. 2003. Inverse kinematics solution to mobile manipulators. *International Journal of Robotics Research* 22:1041–1064.

Haynes, G. and Hermes, H. 1970. Nonlinear controllability via Lie theory. *SIAM Journal on Control* 8:450–460.

Lee, J.-K. and Cho, H. S. 1997. Mobile manipulator motion planning for multiple tasks using global optimization approach. *Journal of Intelligent and Robotic Systems* 18:169–190.

Lizárraga, D. 2004. Obstructions to the existence of universal stabilizers for smooth control systems. *Mathematics of Control, Signals, and Systems* 16:255–277.

Morin, P. and Samson, C. 2001. A characterization of the Lie algebra rank condition by transverse periodic functions. *SIAM Journal on Control and Optimization* 40(4):1227–1249.

Morin, P. and Samson, C. 2003. Practical stabilization of driftless systems on Lie groups: the transverse function approach. *IEEE Transactions on Automatic Control* 48:1496–1508.

Morin, P. and Samson, C. 2004. Trajectory tracking for nonholonomic vehicles: overview and case study. Kozlowski, K., editor, *4th International Workshop on Robot Motion Control (RoMoCo)*, pp. 139–153.

Murray, R., Li, Z., and Sastry, S. 1994. *A mathematical introduction to robotic manipulation*. Boca Raton, FL: CRC Press.

Perrier, C., Dauchez, P., and Pierrot, F. 1998. A global approach for motion generation of non-holonomic mobile manipulators. *IEEE Conference on Robotics and Automation (ICRA)*, pp. 2971–2976.

Samson, C., Leborgne, M., and Espiau, B. 1991. *Robot Control: The Task-function Approach*. Number 22 in Oxford Engineering. Oxford: Oxford University Press.

Seraji, H. 1998. A unified approach to motion control of mobile manipulators. *The International Journal of Robotics Research* 17:107–118.

Sussmann, H. and Liu, W. 1991. Limits of highly oscillatory controls and approximation of general paths by admissible trajectories. *IEEE Conference on Decision and Control (CDC)*, pp 437–442.

Tchon, K. 2002. Repeatability of inverse kinematics algorithms for mobile manipulators. *IEEE Transactions on Automatic Control* 47:1376–1380.

Tchon, K. and Jakubiak, J. 2003. Endogeneous configuration space approach to mobile manipulators: a derivation and performance assessment of jacobian inverse kynematics algorithms. *International Journal of Control* 76:1387–1419.

Wang, C.-C. and Kumar, V. 1993. Velocity control of mobile manipulators. *IEEE Conference on Robotics and Automation (ICRA)*, volume 2, pp. 713–718.

Yamamoto, Y. and Yun, X. 1994. Coordinating locomotion and manipulation of a mobile manipulator. *IEEE Transactions on Automatic Control* 39(6):1326–1332.

Yoshikawa, T. 1990. *Foundations of Robotics: Analysis and Control*. Cambridge, MA: MIT Press.