# Robust stabilization of the angular velocity of a rigid body with two controls

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**Abstract**

The problem of asymptotic stabilization of the angular velocity of a rigid body with two actuators (i.e. two controls) has long been solved. However, robustness properties of the feedback laws were little investigated. We show here that most proposed controls in the past are not robust with respect to errors in actuators location, and we derive robust asymptotically stabilizing feedbacks.

**Keywords:** Robust control, asymptotic stabilization, rigid body, homogeneous systems, center manifold theory.

## 1 Introduction

The angular velocity control of a rigid body with only one or two controls has been much studied in the literature. In particular, many strategies have been proposed for the stabilization and asymptotic stabilization problems (see e.g. [1], [2], [3], [4], [5], [6]). For the case where a single control is available (aligned with an axis having components along all three principal axes), an asymptotically stabilizing feedback was proposed in [3], and in [4] this control was shown to be non-robust with respect to errors on the principal moments of inertia. The problem of asymptotic stabilization with two controls had previously been treated (see e.g. [1], [2]) and robustness properties with respect to errors on the principal moments of inertia were established in [4]. However, robustness with respect to more general model errors (such as errors on the location of the actuators) was not investigated.

In this paper, we prove that many asymptotically stabilizing feedbacks (particularly those proposed in [1] or [2]-[4]) are not robust with respect to errors on the location of the actuators, and we derive robust control laws. The techniques used here combine center manifold theory with properties of homogeneous systems and Lyapunov-like techniques.

The paper is organized as follows. Section 2 recalls Euler's angular velocity equations and states the problem of robust asymptotic stabilization. In section 3, we show the non-robustness of a class of asymptotically stabilizing feedbacks. Finally, in section 4, robust controls are derived.

Throughout the paper, we use the following notations:

- $I_3$ denotes the identity matrix in $\mathbb{R}^3$.

- $|\cdot|$ denotes the Euclidean norm, and for any $n \times m$ matrix $C$, $\|C\|$ denotes the associated matrix norm.
- $diag(\alpha, \beta, \gamma)$ denotes the $3 \times 3$ matrix with $\alpha, \beta$ and $\gamma$ on its diagonal and 0 outside.
- For any frames $F_i, F_j$ and $F_k$ in $\mathbb{R}^3$, $R_{F_iF_j}$ denotes the rotation matrix from the frame $F_j$ to the frame $F_i$ and $\Omega_{F_i}(F_j/F_k)$ denotes the angular velocity vector of the frame $F_j$ with respect to the frame $F_k$ expressed in the basis of the frame $F_i$.
- $O(|x|^q)$ stands for any continuous function $f : \mathbb{R}^n \longmapsto \mathbb{R}$ such that:

$$\exists \delta > 0,\ \exists K \ : \ |x| \leq \delta \Longrightarrow |f(x)| \leq K|x|^q \quad .$$

## 2 Problem Statement

Let us consider:

- a frame $F_0$ attached to the rigid body and whose axes correspond to the principal inertia axes of the body,
- a fixed frame $F_1$,

and denote:

- $\omega = \Omega_{F_0}(F_0/F_1)$,
- $J = diag(j_1, j_2, j_3)$: the inertia matrix of the body,
- $S(\omega) = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$: the matrix representation of the cross product.

Euler's angular velocity equations (with two controls) are then given by:

$$\dot{\omega} \ = \ J^{-1}S(\omega)J\omega + J^{-1}B\,(\tau_1, \tau_2)^T \tag{1}$$

where the $\tau_i$ are the torques applied to the rigid body and the $3 \times 2$ matrix $B$ represents the direction in which these torques are applied.
Feedback laws which make the origin $\omega = 0$ of the closed loop system (1) asymptotically stable have long been proposed ( see e.g. [1] or [2]). However in practice, in view of model errors, the control system (1) may not be perfectly known. In fact, model errors can occur at many levels. First, the principal inertia axes are not, a priori, exactly localized. In this case, only an estimation $\hat{F}_0$ of the frame $F_0$ is known. As a consequence, the angular velocity vector $\omega$ expressed in $F_0$ can not be known either, and the measure of the angular velocity is not $\omega$ but $\hat{\omega}$ given by:

$$\hat{\omega} \ \stackrel{\Delta}{=} \ \Omega_{\hat{F}_0}(\hat{F}_0/F_1) \ = \ R_{\hat{F}_0F_0}\omega. \tag{2}$$

Then, the assumed values $\hat{j}_1, \hat{j}_2$ and $\hat{j}_3$ of the principal moments of inertia may also differ from the actual ones $j_1, j_2$ and $j_3$. Thus, instead of the matrix $J$, only $\hat{J} = diag(\hat{j}_1, \hat{j}_2, \hat{j}_3)$

is known. Finally, the torques applied to a rigid body are generated by mechanical systems (such as gaz jets actuators) and in practice, model errors also occur at this level. More precisely, the matrix $B$ depends on the position and the attitude of the actuators. Therefore, any misalignment of these actuators contributes to an error between the matrix $B$ and its assumed value $\hat{B}$.
To sum up, the control system which is used for the control design is given by:

$$\dot{\hat{\omega}} = \hat{J}^{-1} S(\hat{\omega}) \hat{J} \hat{\omega} + \hat{J}^{-1} \hat{B} \, (\hat{\tau}_1, \hat{\tau}_2)^T. \tag{3}$$

In order to simplify the exposition, we make the assumption that

$$\hat{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{4}$$

which corresponds to the case where the actuators are, supposedly, located on principal inertia axes. As a consequence, the system (3) can be rewritten as:

$$\left\{ \begin{array}{rcl} \dot{\hat{\omega}}_1 & = & \hat{c}_1 \, \hat{\omega}_2 \hat{\omega}_3 + \dfrac{\hat{\tau}_1}{\hat{j}_1} \\ \dot{\hat{\omega}}_2 & = & \hat{c}_2 \, \hat{\omega}_1 \hat{\omega}_3 + \dfrac{\hat{\tau}_2}{\hat{j}_2} \\ \dot{\hat{\omega}}_3 & = & \hat{c}_3 \, \hat{\omega}_1 \hat{\omega}_2 \end{array} \right. \tag{5}$$

with

$$\hat{c}_1 = \frac{\hat{j}_2 - \hat{j}_3}{\hat{j}_1}, \hat{c}_2 = \frac{\hat{j}_3 - \hat{j}_1}{\hat{j}_2} \text{ and } \hat{c}_3 = \frac{\hat{j}_1 - \hat{j}_2}{\hat{j}_3}. \tag{6}$$

It is of course assumed that $\hat{c}_3 \neq 0$, since otherwise this system would not be asymptotically stabilizable.
Our objective is to derive, from the system (5), feedback laws

$$\hat{\Gamma}(\hat{\omega}) = (\hat{\tau}_1(\hat{\omega}), \hat{\tau}_2(\hat{\omega})) \tag{7}$$

"*robust with respect to general model errors*" in the sense that these feedback laws expressed, from (2), in the variable $\omega$ i.e.

$$\Gamma(\omega) = (\tau_1(\omega), \tau_2(\omega)) = (\hat{\tau}_1(R_{\hat{F}_0 F_0} \omega), \hat{\tau}_2(R_{\hat{F}_0 F_0} \omega)) \tag{8}$$

make the origin $\omega = 0$ of the system (1) asymptotically stable whenever

$$\|J - \hat{J}\| \leq \epsilon, \quad \|B - \hat{B}\| \leq \epsilon, \quad \text{and } \|R_{\hat{F}_0 F_0} - I_3\| \leq \epsilon \tag{9}$$

where $\epsilon$ is a small enough strictly positive value.
It must be noted that feedback laws with certain robustness properties have already been proposed. One may find in [4] for instance feedback laws which are robust in the particular case where small errors are made on the principal moments of inertia (i.e. $(\hat{j}_1, \hat{j}_2, \hat{j}_3) \neq (j_1, j_2, j_3)$) but no error is made on the inertial frame or on the location of the actuators (i.e. $\hat{F}_0 = F_0$ and $\hat{B} = B$). This kind of robustness will be referred to, in the sequel, as "*robustness with respect to errors on the principal moments of inertia*". As we will see in the next section many feedback laws, which are robust in this sense, are not *robust with respect to general model errors.*

## 3 A non-robustness result

The aim of this section is to revisit robustness properties of some control laws, such as those proposed in [1] or [2]-[4] for instance. To this purpose, we here restrict ourselves to the case where no error is made on the frame of inertia axes (i.e. $\hat{F}_0 = F_0$). As a consequence, it follows from (2) and (8) that $\hat{\omega} = \omega$ and $\hat{\tau}_i = \tau_i$, $(i = 1, 2)$. The system (5) can then be rewritten as

$$\left\{ \begin{array}{lcl} \dot{\omega}_1 & = & \hat{c}_1\, \omega_2\omega_3 + \dfrac{\tau_1}{\hat{j}_1} \\ \dot{\omega}_2 & = & \hat{c}_2\, \omega_1\omega_3 + \dfrac{\tau_2}{\hat{j}_2} \\ \dot{\omega}_3 & = & \hat{c}_3\, \omega_1\omega_2 \end{array} \right. \tag{10}$$

The control laws proposed in [1] or [2]-[4] are of the form:

$$\left\{ \begin{array}{lcl} \tau_1(\omega) & = & \hat{j}_1\, (-\hat{c}_1\, \omega_2\omega_3 - \omega_1 + \alpha\, \omega_3^k) \\ \tau_2(\omega) & = & \hat{j}_2\, (-\hat{c}_2\, \omega_1\omega_3 - \omega_2 + \beta\, \omega_3^{k+1}) \end{array} \right. \tag{11}$$

The following proposition, which is the main result of this section, describes robustness properties of these feedback laws.

**Proposition 1** *For any real parameters $\alpha$, $\beta$ and any integer $k$ such that:*

$$\hat{c}_3\, \alpha\beta < 0, \quad \textit{and} \quad k \geq 1, \tag{12}$$

*the control law (11) satisfies the following properties:*

*i) the controlled system (10)-(11) is locally asymptotically stable,*

*ii) this control is robust with respect to errors on the principal moments of inertia,*

*iii) this control is not robust with respect to general model errors.*

**Proof:**
For the proof of part *i)*, we proceed (as in [2]-[4]) by application of center manifold theory (see [7] for a classical introduction on the subject).
The closed loop system (10)-(11) is given by

$$\left\{ \begin{array}{lcl} \dot{\omega}_1 & = & -\omega_1 + \alpha\, \omega_3^k \\ \dot{\omega}_2 & = & -\omega_2 + \beta\, \omega_3^{k+1} \\ \dot{\omega}_3 & = & \hat{c}_3\, \omega_1\omega_2 \end{array} \right. \tag{13}$$

and can be rewritten as:

$$\left\{ \begin{array}{lcl} \dot{\xi}_1 & = & -\xi_1 + g_1(\xi_1, \xi_2, \omega_3) \\ \dot{\xi}_2 & = & -\xi_2 + g_2(\xi_1, \xi_2, \omega_3) \\ \dot{\omega}_3 & = & \hat{c}_3\, (\xi_1 + \alpha\, \omega_3^k)(\xi_2 + \beta\, \omega_3^{k+1}) \end{array} \right. \tag{14}$$

with $\xi_1 = \omega_1 - \alpha\, \omega_3^k$, $\xi_2 = \omega_2 - \beta\, \omega_3^{k+1}$ and,

$$\left\{ \begin{array}{lcl} g_1(\xi_1, \xi_2, \omega_3) & = & -\hat{c}_3\, \alpha k(\xi_1 + \alpha\, \omega_3^k)(\xi_2 + \beta\, \omega_3^{k+1})\omega_3^{k-1} \\ g_2(\xi_1, \xi_2, \omega_3) & = & -\hat{c}_3\, \beta(k+1)(\xi_1 + \alpha\, \omega_3^k)(\xi_2 + \beta\, \omega_3^{k+1})\omega_3^k \end{array} \right. \tag{15}$$

From center manifold theory, there exists for the system (14) a center manifold given by :

$$\{\,(\xi_1,\xi_2,\omega_3)\,/\quad \xi_1 = h_1(\omega_3),\ \xi_2 = h_2(\omega_3),\quad |\omega_3| < \delta\}$$

and the asymptotic stability of (14) is equivalent to the asymptotic stability on the center manifold. As a consequence, it is sufficient to study the stability of the differential equation:

$$\dot\omega_3 \;=\; \hat c_3\,(h_1(\omega_3) + \alpha\,\omega_3^k)(h_2(\omega_3) + \beta\,\omega_3^{k+1}). \tag{16}$$

But, by application of the approximation theorem of center manifold theory, and since from (15), $g_1(0,0,\omega_3) = O(|\omega_3|^{3k})$ and $g_2(0,0,\omega_3) = O(|\omega_3|^{3k+1})$, one deduces that:

$$|(h_1(\omega_3), h_2(\omega_3))| = O(|\omega_3|^{3k}).$$

Hence, the equation (16) can be rewritten as:

$$\dot\omega_3 \;=\; \hat c_3\,\alpha\beta\,\omega_3^{2k+1} + O(|\omega_3|^{4k}) \tag{17}$$

which is locally asymptotically stable under the condition (12).

The proof of part *ii)* is similar to the proof given in [4].

Let us now proceed with the proof of part *iii)*.
We have to show that, for any control law $(\tau_1(\omega), \tau_2(\omega))$ given by (11)-(12), and any $\epsilon > 0$, there exists $J$ and $B$ satisfying (9) such that the origin $\omega = 0$ of the controlled system (1)-(11) is not asymptotically stable. To this purpose, let us define the matrix $J$ and $B$ as follows:

$$J = \hat J + \begin{pmatrix} 0 & 0 & 0\\ 0 & \epsilon_1 & 0\\ 0 & 0 & \epsilon_1 \end{pmatrix}, \quad B = \hat B + \frac{1}{\hat\jmath_2}\begin{pmatrix} 0 & 0\\ 0 & \epsilon_1\\ 0 & \epsilon_2\,(\hat\jmath_3 + \epsilon_1) \end{pmatrix} \tag{18}$$

with

$$0 < \epsilon_1 \;\le\; Min\{\epsilon,\ \hat\jmath_2\epsilon\} \quad \text{and} \quad |\epsilon_2| \;\le\; \epsilon\frac{\hat\jmath_2}{\hat\jmath_3 + \epsilon_1}. \tag{19}$$

It directly follows from (18) and (19) that $J$ and $B$ satisfy (9). Moreover, one easily verifies that the system (1) can be rewritten as:

$$\left\{\begin{array}{lcl} \dot\omega_1 &=& \hat c_1\,\omega_2\omega_3 + \dfrac{\tau_1}{\hat\jmath_1}\\ \dot\omega_2 &=& (\hat c_2 + \delta_1(\epsilon_1))\,\omega_1\omega_3 + \dfrac{\tau_2}{\hat\jmath_2}\\ \dot\omega_3 &=& (\hat c_3 + \delta_2(\epsilon_1))\,\omega_1\omega_2 + \epsilon_2\,\dfrac{\tau_2}{\hat\jmath_2} \end{array}\right. \tag{20}$$

with

$$\delta_1(\epsilon_1) = \epsilon_1\frac{\hat\jmath_2 - \hat\jmath_3 + \hat\jmath_1}{\hat\jmath_2(\hat\jmath_2 + \epsilon_1)} \quad \text{and} \quad \delta_2(\epsilon_1) = \epsilon_1\frac{-\hat\jmath_3 - \hat\jmath_1 + \hat\jmath_2}{\hat\jmath_3(\hat\jmath_3 + \epsilon_1)}. \tag{21}$$

After a first change of variables, $\xi_3 = \omega_3 - \epsilon_2\,\omega_2$, and using the expression (11) of the control law, the system (20) can be rewritten as:

$$\left\{\begin{array}{lcl} \dot\omega_1 &=& -\omega_1 + \alpha\,(\xi_3 + \epsilon_2\,\omega_2)^k\\ \dot\omega_2 &=& \delta_1(\epsilon_1)\,\omega_1(\xi_3 + \epsilon_2\,\omega_2) - \omega_2 + \beta\,(\xi_3 + \epsilon_2\,\omega_2)^{k+1}\\ \dot\xi_3 &=& (\hat c_3 + \delta_2(\epsilon_1))\,\omega_1\omega_2 - \epsilon_2\,(\hat c_2 + \delta_1(\epsilon_1))\,\omega_1(\xi_3 + \epsilon_2\,\omega_2) \end{array}\right. \tag{22}$$

and with the following change of variables:

$$\left\{ \begin{array}{lcl} \xi_1 & = & \omega_1 - \alpha\,\xi_3^k \\ \xi_2 & = & \omega_2 - (\beta + \delta_1(\epsilon_1)\alpha)\,\xi_3^{k+1} \end{array} \right. \tag{23}$$

one obtains after some calculations that (22) is transformed into the system:

$$\left\{ \begin{array}{lcl} \dot{\xi}_1 & = & -\xi_1 + c(k)\,\xi_2 + g_1(\xi_1,\xi_2,\xi_3) \\ \dot{\xi}_2 & = & -\xi_2 + g_2(\xi_1,\xi_2,\xi_3) \\ \dot{\xi}_3 & = & (\hat{c}_3 + \delta_2(\epsilon_1) - \epsilon_2^2\,(\hat{c}_2 + \delta_1(\epsilon_1)))\,(\xi_1 + \alpha\,\xi_3^k)(\xi_2 + (\beta + \delta_1(\epsilon_1)\alpha)\,\xi_3^{k+1}) \\ & & -\epsilon_2(\hat{c}_2 + \delta_1(\epsilon_1))\,\xi_3(\xi_1 + \alpha\,\xi_3^k) \end{array} \right. \tag{24}$$

where $c(k)$ is a coefficient depending only on $k$, and $g_1$ and $g_2$ are smooth functions which vanish at $(0,0,0)$, whose first order derivatives also vanish at this point, and such that:

$$\left\{ \begin{array}{lcl} g_1(0,0,\xi_3) & = & O(|\xi_3|^{2k}) \\ g_2(0,0,\xi_3) & = & O(|\xi_3|^{2k+1}). \end{array} \right. \tag{25}$$

The system (24) is now in the right form to apply center manifold theory. In particular, there exists a invariant manifold for (24) given by :

$$\{\,(\xi_1,\xi_2,\xi_3)\,/ \quad \xi_1 = l_1(\xi_3),\ \xi_2 = l_2(\xi_3), \quad |\xi_3| < \delta\}$$

and we deduce from (25), by application of the approximation theorem of center manifold theory, that $|(l_1(\xi_3), l_2(\xi_3))| = O(|\xi_3|^{2k})$. As a consequence, on this invariant manifold, we have from (24):

$$\dot{\xi}_3 \ = \ -\epsilon_2\,(\hat{c}_2 + \delta_1(\epsilon_1))\alpha\,\xi_3^{k+1} + O(|\xi_3|^{2k+1}). \tag{26}$$

Since, in view of (6), (19) and (21), $\delta_1(\epsilon_1)$ is different from zero when $\hat{c}_2 = 0$ and tends to zero as $\epsilon_1$ tends to zero, it is always possible, for any $\alpha \neq 0$, any $\hat{c}_2$ and any $k \geq 1$, to choose $\epsilon_1$ and $\epsilon_2$, satisfying (19), such that the origin $\xi_3 = 0$ of the system (26) is unstable. This ends the proof of the proposition. □

This proof shows the reason why the feedback laws (11) are not robust with respect to general model errors. In fact, due to model errors, the vector field associated with the actual closed loop system differs from the one associated with the assumed closed loop system by some extra terms (compare (10) and (20)), and these extra terms are large (i.e. of low order) with respect to the terms of the assumed closed loop system (compare (17) and (26)). As we will see in the next section, this problem no longer occurs when the stabilizing feedbacks are homogeneous of degree 2.

## 4 A robust control law

We first recall some properties on the homogeneous systems. For a more comprehensive exposition, the reader is referred to [8].

A function $f : \mathbb{R}^n \longmapsto \mathbb{R}^m$ of class $C^1$ is homogeneous of degree $k \geq 1$ if

$$f(cx) \ = \ c^k f(x), \quad \forall c \geq 0.$$

The following proposition states a robustness property of differential systems with homogeneous right hand side. This proposition is a particular case of a result due to Massera [9] (see also [8, Th. 57.3, Sec. 56], and the generalisations by Hermes [10] and Rosier[11]).

**Proposition 2** *Let $f : \mathbb{R}^n \longmapsto \mathbb{R}^n$ a function of class $C^1$ homogeneous of degree $k \geq 1$, and such that the origin $x = 0$ of the system $\dot{x} = f(x)$ is asymptotically stable. Then, there exists $M > 0$ such that for any function $g : \mathbb{R}^n \longmapsto \mathbb{R}^n$ of class $C^1$, homogeneous of degree $k$ and satisfying*

$$|g(x)| \leq M\,|x|^k, \tag{27}$$

*the origin $y = 0$ of the system $\dot{y} = f(y) + g(y)$ is also asymptotically stable.*

**Remark 1** *: For homogeneous systems, the local asymptotic stability is equivalent to the global asymptotic stability (see [8, Th. 17.1]).*

Let us now return to the control system (1).

**Lemma 1** *The control system (1) can be rewritten as:*

$$\left\{\begin{array}{lcl} \dot{\omega}_1 & = & (\hat{c}_1 + \eta_1)\,\omega_2\omega_3 + (1+\eta_4)\dfrac{\tau_1}{\hat{j}_1} + \eta_5\dfrac{\tau_2}{\hat{j}_2} \\ \dot{\omega}_2 & = & (\hat{c}_2 + \eta_2)\,\omega_1\omega_3 + \eta_6\dfrac{\tau_1}{\hat{j}_1} + (1+\eta_7)\dfrac{\tau_2}{\hat{j}_2} \\ \dot{\omega}_3 & = & (\hat{c}_3 + \eta_3)\,\omega_1\omega_2 + \eta_8\dfrac{\tau_1}{\hat{j}_1} + \eta_9\dfrac{\tau_2}{\hat{j}_2} \end{array}\right. \tag{28}$$

*where the $\eta_i$ depend on $J$ and $\hat{J}$ for $i = 1, \ldots, 3$, and on $J, B, \hat{J}$ and $\hat{B}$ for $i = 4, \ldots, 9$. Moreover, there exists $\eta_0(\hat{J}, \hat{B}, \epsilon)$ such that:*

- *i) for any matrices $J$ and $B$ satisfying (9), $|\eta_i(J, \hat{J})| \leq \eta_0(\hat{J}, \hat{B}, \epsilon)$ for $i = 1, \ldots, 3$ and $|\eta_i(J, B, \hat{J}, \hat{B})| \leq \eta_0(\hat{J}, \hat{B}, \epsilon)$ for $i = 4, \ldots, 9$,*
- *ii) $\eta_0(\hat{J}, \hat{B}, \epsilon)$ tends to zero as $\epsilon$ tends to zero.*

The proof of this lemma, which directly follows from (1) and (9), is left to the reader.

In view of this lemma, for any feedback law $\hat{\Gamma}$ of class $C^1$ and homogeneous of degree 2, the actual closed loop system (1)-(8) can be rewritten as:

$$\dot{\omega} = f(\omega) + g(\omega) \tag{29}$$

where

$$f(\omega) = \begin{pmatrix} \hat{c}_1\,\omega_2\omega_3 + \dfrac{\hat{\tau}_1(\omega)}{\hat{j}_1} \\ \hat{c}_2\,\omega_1\omega_3 + \dfrac{\hat{\tau}_2(\omega)}{\hat{j}_2} \\ \hat{c}_3\,\omega_1\omega_2 \end{pmatrix} \tag{30}$$

is the vector field associated with the assumed closed loop system (5), which is of class $C^1$ and homogeneous of degree 2, and

$$g(\omega) = \begin{pmatrix} \eta_1\omega_2\omega_3 + \eta_4 \dfrac{\hat{\tau}_1(R_{\hat{F}_0F_0}\omega)}{\hat{j}_1} + \eta_5 \dfrac{\hat{\tau}_2(R_{\hat{F}_0F_0}\omega)}{\hat{j}_2} + \dfrac{\hat{\tau}_1(R_{\hat{F}_0F_0}\omega) - \hat{\tau}_1(\omega)}{\hat{j}_1} \\ \eta_2\omega_1\omega_3 + \eta_6 \dfrac{\hat{\tau}_1(R_{\hat{F}_0F_0}\omega)}{\hat{j}_1} + \eta_7 \dfrac{\hat{\tau}_2(R_{\hat{F}_0F_0}\omega)}{\hat{j}_2} + \dfrac{\hat{\tau}_2(R_{\hat{F}_0F_0}\omega) - \hat{\tau}_2(\omega)}{\hat{j}_1} \\ \eta_3\omega_1\omega_2 + \eta_8 \dfrac{\hat{\tau}_1(R_{\hat{F}_0F_0}\omega)}{\hat{j}_1} + \eta_9 \dfrac{\hat{\tau}_2(R_{\hat{F}_0F_0}\omega)}{\hat{j}_2} \end{pmatrix} \quad (31)$$

is the vector field generated by model errors, which is also of class $C^1$ and homogeneous of degree 2. Thus, as previously announced, the vector field $g$ is now of the same order as the vector field $f$ associated with the closed loop system (5). Moreover, $g$ satisfies the following property:

**Lemma 2** *For any $C^1$ feedback law $\hat{\Gamma}$ homogeneous of degree 2, there exists $M(\hat{J}, \hat{B}, \epsilon)$ such that:*

*i) for any matrices $J$, $B$ and $F_0$ satisfying (9), $|g(\omega)| \leq M(\hat{J}, \hat{B}, \epsilon)\,|\omega|^2$,*

*ii) $M(\hat{J}, \hat{B}, \epsilon)$ tends to zero as $\epsilon$ tends to zero.*

(Proof in the Appendix).

According to this lemma, it follows from Proposition 2 applied to the system (29), that the design of a robust feedback law can be reduced to the design of a feedback $\hat{\Gamma}$ of class $C^1$ and homogeneous of degree 2, which makes the origin of the closed loop system (5) asymptotically stable. The following proposition provides feedback laws of this type.

**Proposition 3** *Let us consider the following control:*

$$\left\{ \begin{array}{rcl} \hat{\tau}_1(\hat{\omega}) &=& \hat{j}_1\,(-\hat{c}_1\,\hat{\omega}_2\hat{\omega}_3 + \lambda\hat{c}_3\,\hat{\omega}_1\hat{\omega}_2 - k_1\,(\hat{\omega}_1 - \lambda\hat{\omega}_3)|\hat{\omega}_1 - \lambda\hat{\omega}_3|) \\ \hat{\tau}_2(\hat{\omega}) &=& \hat{j}_2\,(-(\hat{c}_2 + \mu\hat{c}_3)\,\hat{\omega}_1\hat{\omega}_3 - k_2\,\hat{\omega}_2|\hat{\omega}_2|) \end{array} \right. \quad (32)$$

*then, for any real parameters $\lambda$, $\mu$, $k_1$ and $k_2$ such that:*

$$\lambda \neq 0, \quad \mu > 0, \quad k_1 > 0, \quad \textit{and} \quad k_2 > 0, \quad (33)$$

*i) the controlled system (5)-(32) is globally asymptotically stable,*

*ii) this control is robust with respect to general model errors.*

**Proof:**
We first proceed with the proof of part *i)*.
The closed loop system (5)- (32) is given by:

$$\left\{ \begin{array}{rcl} \dot{\hat{\omega}}_1 &=& \lambda\hat{c}_3\,\hat{\omega}_1\hat{\omega}_2 - k_1\,(\hat{\omega}_1 - \lambda\hat{\omega}_3)|\hat{\omega}_1 - \lambda\hat{\omega}_3| \\ \dot{\hat{\omega}}_2 &=& -\mu\hat{c}_3\,\hat{\omega}_1\hat{\omega}_3 - k_2\,\hat{\omega}_2|\hat{\omega}_2| \\ \dot{\hat{\omega}}_3 &=& \hat{c}_3\,\hat{\omega}_1\hat{\omega}_2 \end{array} \right. \quad (34)$$

Let $V(\hat\omega)=\frac{1}{2}(\hat\omega_1-\lambda\hat\omega_3)^2+\frac{1}{2}\hat\omega_2^2+\frac{\mu}{2}\hat\omega_3^2$. The derivative of $V$ along the trajectories of the controlled system (34) is then given by:

$$\dot V \;=\; -k_1\,|\hat\omega_1-\lambda\hat\omega_3|^3-k_2\,|\hat\omega_2|^3 \tag{35}$$

As a consequence, any trajectory of the system (34) is bounded and tends, by application of LaSalle's invariance principle [12], to the largest invariant set $S$ for the system (34), contained in $\{\hat\omega\ /\ \hat\omega_1=\lambda\hat\omega_3 \text{ and } \hat\omega_2=0\}$. For any trajectory $\hat\omega(t)$ contained in $S$, $\dot{\hat\omega}_2(t)\equiv\hat\omega_2(t)\equiv 0$, which implies, from (34), that $-\lambda\mu\hat c_3\,\hat\omega_3^2(t)\equiv 0$. Thus $\hat\omega_3(t)\equiv 0$, which in turn implies that $\hat\omega_1(t)\equiv 0$. This ends the proof of part *i)*.
Finally, as previously explained, the proof of part *ii)* directly follows, from part *i)* and Lemma 2, by application of Proposition 2 to the system (29). □

## 5 Conclusion

We have shown that some feedbacks, which asymptotically stabilize Euler's angular velocity equations, are not robust with respect to model errors such that misalignment of the actuators. However, robust stabilization can be obtained by using feedback laws which make the closed loop system homogeneous, and such that the homogeneity is not destroyed by the model errors. This remark also raises the problem of attitude's control of a rigid body. In fact, asymptotic stabilization of the attitude of a rigid body with two controls has recently been obtained in [13], by using center manifold theory, and in [14], by using homogeneous feedbacks (see also [15] more recently). However, even in the latter case, robustness does not seem ensured because the homogeneity of the closed loop system is destroyed by the model errors. Robust feedback stabilization for the attitude control problem remains an open question.

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**Appendix: Proof of Lemma 2**
Using the homogeneity of the fonctions $\hat\tau_i$, $i=1,2$, one obtains that:

$$\sup_{\omega\neq 0}\frac{|\hat\tau_i(R_{\hat F_0F_0}\omega)|}{|\omega|^2} \;=\; \sup_{\omega\neq 0}\frac{|\hat\tau_i(R_{\hat F_0F_0}\omega)|}{|R_{\hat F_0F_0}\omega|^2}\frac{|R_{\hat F_0F_0}\omega|^2}{|\omega|^2} \;\le\; \|R_{\hat F_0F_0}\|^2\sup_{|\omega|=1}|\hat\tau_i(\omega)| \tag{36}$$

which implies, in view of (9) and the continuity of the $\hat\tau_i$ that:

$$|\hat\tau_i(R_{\hat F_0F_0}\omega)| \;\le\; (1+\epsilon)^2K_1\,|\omega|^2 \tag{37}$$

where $K_1=\sup\limits_{|\omega|=1}|\hat\tau_i(\omega)|$ is independent of $J,B$ and $F_0$.
Similarly,

$$\begin{aligned}
&\sup_{\omega\neq 0}\frac{|\hat\tau_i(R_{\hat F_0F_0}\omega)-\hat\tau_i(\omega)|}{|\omega|^2} \;=\; \sup_{|\omega|=1}|\hat\tau_i(R_{\hat F_0F_0}\omega)-\hat\tau_i(\omega)|\\
&=\;\sup_{|\omega|=1}|\hat\tau_i(\omega+(R_{\hat F_0F_0}-I_3)\omega)-\hat\tau_i(\omega)|\\
&\le\;\sup_{|\omega|=1,\,|y|\le\epsilon}|\hat\tau_i(\omega+y)-\hat\tau_i(\omega)| \;\le\; \sup_{|\omega|=1,\,|y|\le\epsilon}|y|\sup_{|z|\le\epsilon}|\frac{\partial\hat\tau_i}{\partial\omega}(\omega+z)| \;\le\; \epsilon K_2
\end{aligned} \tag{38}$$

where $K_2 = \sup_{|\omega|=1,\,|z|\leq\epsilon} |\frac{\partial \hat{\tau}_i}{\partial \omega}(\omega + z)|$ is also independent of $J, B$ and $F_0$. It directly follows from (38) that:

$$|\hat{\tau}_i(R_{\hat{F}_0 F_0}\omega) - \hat{\tau}_i(\omega)| \;\leq\; \epsilon K_2 \, |\omega|^2 \tag{39}$$

In view of (31), the proof of Lemma 2 follows from (37), (39) and Lemma 1. □

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