12. Robust point-stabilization of nonlinear affine control systems

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Summary.

Exponential stabilization of nonlinear driftless affine control systems is addressed with the concern of achieving robustness with respect to imperfect knowledge of the system's control vector fields. The present paper gives an overview of the results developed by the authors in [11], and provides new results on the robustness with respect to sampling of the control laws. Control design for a dynamic extension of the original system is also considered. This study is inspired by [1], where the same robustness issue was first addressed. It is further motivated by the fact, proven in [7], according to which no *continuous homogeneous* time-periodic state-feedback can be a robust exponential stabilizer in the sense considered here. *Hybrid* open-loop/feedback controllers, more precisely described as continuous time-periodic feedbacks associated with a specific dynamic extension of the original system, are considered instead.

12.1 Introduction

We consider an analytic driftless system on $\mathbf{R}^n$

$$(S_0) \;:\qquad \dot{x} = \sum_{i=1}^{m} f_i(x)u_i\,, \qquad (m < n), \tag{12.1}$$

locally controllable around the origin, i.e.

$$\text{Span}\{f(0) : f \in \text{Lie}(f_1, \dots, f_m)\} = \mathbf{R}^n\,, \tag{12.2}$$

and address the problem of constructing explicit feedback laws which (locally) exponentially stabilize, in some sense specified later, the origin $x = 0$ of the controlled system. A further requirement is that these feedbacks should also be exponential stabilizers for any "perturbed" system in the form

$$(S_\varepsilon) \ : \qquad \dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\varepsilon, x))u_i \, , \tag{12.3}$$

with h_i analytic in $\mathbf{R} \times \mathbf{R}^n$ and $h_i(0, x) = 0$, when $|\varepsilon|$ is small enough. In other words, given a *nominal* control system (S_0), we would like to find *nominal* feedback controls, derived on the basis of this nominal system, that preserve the property of exponential stability when they are applied to "neighboring" systems (S_ε).

Explicit *homogeneous* exponential (time-periodic) stabilizers $u(x, t)$ for systems (S_0) have been derived in various previous studies (see [8, 10], for example). However, as demonstrated in [7], none of these controls solves the robustness problem stated above in the sense that there always exists some $h_i(\varepsilon, .)$ for which the origin of the associated controlled system is not stable when $\varepsilon \neq 0$. This negative result strongly suggests that no continuous feedback $u(x, t)$, not necessarily homogeneous, can be a robust exponential stabilizer. However, it does not imply that the problem cannot be handled via an adequate dynamic extension of the original nominal system. As a matter of fact, and as explained below, the present study may already be seen as a step in this direction.

An alternative to continuous state feedback control consists in considering *hybrid* open-loop/feedback controls such as open-loop controls which are periodically updated from the measurement $x(kT)$, $k \in \mathbf{N}$, of the state at discrete time-instants. The idea of using this type of control to achieve asymptotic stabilization of the origin of the class of nonlinear driftless systems considered here is not new. This possibility has sometimes been presented as an extension of solutions obtained when addressing the open-loop steering problem, i.e. the problem of finding an open-loop control which steers the system from an initial state to another desired one (see [9, 12], for example). Hybrid continuous/discrete time exponential stabilizers for chained systems, which do not specifically rely on open-loop steering control, have also been proposed in [14]. However, [1] is to our knowledge the first study where the robustness problem stated above has been formulated in a similar fashion and where it has been shown that this problem can be solved by using a hybrid open-loop/feedback control. In fact, although this is not specified in the abovementioned reference, the proposed control does not "strictly" ensure asymptotic stability, in the usual sense of Lyapunov, of the origin of the perturbed systems (S_ε). In order to be more specific about this technical point, and also clarify the meaning of "periodically updated open-loop control applied to a time-continuous system $\dot{x} = f(x, u)$", it is useful to introduce the following *extended* control system:

$$
\begin{cases}
\dot{x} = f(x,u) \\
\dot{y} = (\sum_{k \in \mathbf{N}} \delta_{kT})(x - y_{-\alpha}) \qquad 0 < \alpha < T \,,
\end{cases}
\tag{12.4}
$$

with T denoting the updating time-period of the control part which depends upon y, δ_{kT} the classical Dirac impulse at the time-instant kT, and $y_{-\alpha}$ the delay operator such that $y_{-\alpha}(t) = y(t-\alpha)$. The extra equation in y just indicates that $y(t)$ is constant and equal to $x(kT)$ on the time-interval $[kT,(k+1)T)$. Therefore, any control the expression of which, on the time-interval $[kT,(k+1)T)$, is a function of $x(kT)$ and t, may just be interpreted as a feedback control $u(y,t)$ for the corresponding extended system. From now on, we will adopt this point of view whenever referring to this type of control. As commonly done elsewhere, we will also say that a feedback control $u(x,y,t)$ is a (uniform) *exponential stabilizer* for the extended system (12.4) if there exist an open set $U \in \mathbf{R}^n \times \mathbf{R}^n$ containing the point $(0,0)$, a positive real number γ, and a function β of class $\mathcal{K}$ such that:

$$
\begin{aligned}
&\forall t \geq t_0 \geq 0 \,, \; \forall (x(t_0), y(t_0)) \in U \,, \\
|(x(t),y(t))| &\leq \beta(|(x(t_0),y(t_0))|) exp(-\gamma(t-t_0))
\end{aligned}
$$

with $(x(t),y(t))$ denoting any solution of the controlled system. In our opinion, the importance of the contribution in [1] comes from that it convincingly demonstrates the possibility of achieving *robust* (with respect to unmodeled dynamics, as defined earlier) *exponential stabilization* (stability being now taken in the *strict* sense of Lyapunov) of an extended control system $(\bar{S}_0)$, defined as the "nominal" system within the set of systems

$$
(\bar{S}_\varepsilon) \;\; : \qquad
\begin{cases}
\dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\varepsilon,x)) u_i \\
\dot{y} = (\sum_{k \in \mathbf{N}} \delta_{kT})(x - y_{-\alpha}) \qquad 0 < \alpha < T \,,
\end{cases}
\tag{12.5}
$$

via the use of a *continuous time-periodic feedback* $u(y,t)$. The exploration of this possibility has been carried further on in [11], and a large part of the present paper is devoted to recalling the main results proven in this reference. These include i) a theorem stating sufficient conditions under which a continuous time-periodic feedback $u(y,t)$ is a robust stabilizer (Section 12.2), ii) a general control design algorithm which applies to any controllable analytic (differentiability up to a certain order is in fact sufficient) driftless control system affine in the control (Section 12.3.1), and iii) a set of simpler stabilizers for the subclass of nilpotent chained systems, obtained by further exploiting the internal structure of these systems (Section 12.3.3). We also complement the aforementioned study with two new results. First, we prove a robustness result with respect to sampling of the control law (Section 12.3.1).

Then, we show how to derive new stabilizing control laws for a dynamic extension of the system (consisting in adding an integrator at each input level) (Section 12.4).

The following notation is used.

The identity function on $\mathbf{R}^n$ is denoted id, $|.|$ is the Euclidean norm, and the transpose of a row-vector $(x_1, \dots, x_n)$ is denoted as $(x_1, \dots, x_n)'$.

For any vector field X and smooth function f on $\mathbf{R}^n$, Xf denotes the Lie derivative of f along the vector field X. When $f = (f_1, \dots, f_n)'$ is a smooth map from $\mathbf{R}^n$ to itself, Xf denotes the map $(Xf_1, \dots, Xf_n)'$.

A square matrix A is called *discrete-stable* if all its eigenvalues are strictly inside the complex unit circle.

Given a continuous functions g, defined on some neighborhood of the origin in $\mathbf{R}^n$, we denote $o(g)$ (resp. $O(g)$) any function or map such that $\frac{|o(g)(x)|}{|g(x)|} \longrightarrow 0$ as $|x| \longrightarrow 0$ (resp. such that $\frac{|O(g)(x)|}{|g(x)|} \leq K$ in some neighborhood of the origin). When $g = |.|$, we write $o(x)$ (resp. $O(x)$) instead of $o(g)(x)$ (resp. $O(g)(x)$).

12.2 Sufficient conditions for exponential and robust stabilization

Prior to stating the main result of this section, we review some properties of Chen-Fliess series that will be used in the sequel. The exposition is based on [4, 17], and limited here to driftless systems.

A *m-valued multi-index* I is a vector $I = (i_1, \dots, i_k)$ with k denoting a strictly positive integer, and $i_1, \dots, i_k$, integers taken in the set $\{1, \dots, m\}$. We denote the length of I as $|I|$, i.e. $I = (i_1, \dots, i_k) \Longrightarrow |I| = k$.

Given piecewise continuous functions $u_1, \dots, u_m$ defined on some time-interval $[0, T]$, and a m-valued multi-index $I = (i_1, \dots, i_k)$, we define

$$\int_0^t u_I = \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} u_{i_k}(t_k) u_{i_{k-1}}(t_{k-1}) \cdots u_{i_1}(t_1)\, dt_1 \cdots dt_k \,. \qquad (12.6)$$

Given smooth vector fields $f_1, \dots, f_m$ on $\mathbf{R}^n$, and a m-valued multi-index $I = (i_1, \dots, i_k)$, we define the k-th order differential operator $f_I : \mathcal{C}^\infty(\mathbf{R}^n; \mathbf{R}) \longrightarrow \mathcal{C}^\infty(\mathbf{R}^n; \mathbf{R})$ by

$$f_I\, g = f_{i_1} f_{i_2} \cdots f_{i_k}\, g \,. \qquad (12.7)$$

The following proposition is a classical result (see e.g. [17] for the proof).

Proposition 1 *[17] Consider the analytic system (S_0) and a compact set $K \subset \mathbf{R}^n$. There exists $\mu > 0$ such that for $M, T \geq 0$ verifying*

$$MT \leq \mu\,, \tag{12.8}$$

and for any control $u = (u_1, \ldots, u_m)$ piecewise continuous on $[0,T]$ and verifying

$$|u(t)| \leq M\,, \quad \forall t \in [0,T]\,, \tag{12.9}$$

the solution $x(.)$ of (S_0), with $x_0 \stackrel{\Delta}{=} x(0) \in K$, satisfies

$$x(t) = x_0 + \sum_I (f_I\, id)(x_0) \int_0^t u_I\,, \quad \forall t \in [0,T]\,. \tag{12.10}$$

Furthermore, the series in the right-hand side of (12.10) is uniformly absolutely convergent w.r.t. $t \in [0,T]$ and $x_0 \in K$.

Note that the sum in the right-hand side of equality (12.10) can be developed as

$$\sum_{k=1}^{\infty} \sum_{i_1,\ldots,i_k=1}^{m} (f_{i_1} \cdots f_{i_k}\, id)(x_0) \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} u_{i_k}(t_k) u_{i_{k-1}}(t_{k-1}) \cdots u_{i_1}(t_1)\, dt_1 \cdots dt_k\,.$$

Let us also remark that the condition (12.8), which relates the integration time-interval to the control size, is specific to driftless systems. For a system which contains a drift term, it is *a priori* not true that decreasing the size of the control inputs allows to increase the time-interval on which the expansion (12.10) is valid. The fact that this property holds for driftless systems can be viewed as a consequence of time-scaling invariance properties.

Our first result points out sufficient conditions under which exponential stabilization robust to unmodeled dynamics is granted.

Theorem 12.2.1. *[11] Consider an analytic locally controllable system (S_0), a neighborhood U of the origin in $\mathbf{R}^n$, and a function $u : U \times \mathbf{R}^+ \longrightarrow \mathbf{R}^m$, $(x,t) \longmapsto u(x,t)$, periodic of period T w.r.t. t, continuous w.r.t. x and piecewise continuous[1] w.r.t. t. Assume that*

1. there exist $\alpha, K > 0$ such that $|u(x,t)| \leq K|x|^\alpha$ for all $(x,t) \in U \times [0,T]$,

2. the solution $x(.)$ of

[1] In [11], u is assumed continuous w.r.t. t, but the proof is unchanged if u is only piecewise continuous.

$$\dot{x} = \sum_{i=1}^{m} f_i(x)u_i(x_0, t)\,, \quad x(0) = x_0 \in U\,, \tag{12.11}$$

satisfies $x(T) = Ax_0 + o(x_0)$ *with* A *a discrete-stable matrix,*

3. *for any multi-index* I *of length* $|I| \leq 1/\alpha$ *(this assumption is only needed when* $\alpha < 1$*),*

$$\int_0^T u_I(x) = O(x)\,. \tag{12.12}$$

Then, given a family of perturbed systems (S_ε)*, there exists* $\varepsilon_0 > 0$ *such that the origin of* $(\bar{S}_\varepsilon)$ *controlled by* $u(y,t)$ *is locally exponentially stable for any* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.

The conditions imposed in the theorem upon the control law can be satisfied in many ways. For instance, when the system (S_0) is known to be differentially flat [2], adequate control functions can be obtained by considering specifically tailored flatness-based solutions to the open-loop steering problem, as done for example in [1] in the case of chained systems. Although the control design approach and robustness analysis in [1] are very different from the ones developed in [11], the set of specific conditions derived in this reference imply that the assumptions of Theorem 12.2.1 are verified. This suggests that these assumptions are not unduly strong and also illustrates the fact that the domain of application of Theorem 12.2.1 extends to different control design techniques.

12.3 Control design

This section addresses the problem of constructing explicit controllers that meet the conditions of Theorem 12.2.1. Such controllers have to be exponential stabilizers for the extended system $(\bar{S}_0)$. A general design algorithm is first proposed. It takes advantage of known techniques based on the use of oscillatory open-loop controls in order to achieve net motion in any direction of the state space. Unfortunately (and unavoidably), the procedure also inherits the complexity of the abovementioned techniques, itself directly related to the process of selecting the "right" frequencies which facilitate motion monitoring in the state space. Unsurprisingly, the selection of these frequencies gets all the more involved that controllability of the system relies on high-order Lie brackets of the control vector fields. The control design can in fact be carried out from the expression of either the original system (S_0) or any locally controllable homogeneous approximation of (S_0). Indeed, working with an homogeneous approximation preserves the robustness of the feedback law provided that an extra condition is satisfied by the control law. This is stated more precisely further in the paper after recalling basic definitions and facts about homogeneous systems.

12.3.1 A general algorithm

We present in this section a general algorithm to construct robust and exponential stabilizers for (S_0). The algorithm uses previous results by Sussmann and Liu [16], and Liu [6]. It is also much related to the one developed in [10] for the construction of continuous time-periodic feedbacks $u(x,t)$ which exponentially stabilize the origin of a driftless system (S_0), but present the shortcoming of not being endowed with the type of robustness here considered.

In order to give a complete exposition of the algorithm, it is first useful to recall some notations from [18]. With the set of control vector fields $\{f_1, \dots, f_m\}$ we associate a set of *indeterminates* $X = \{X_1, \dots, X_m\}$. Brackets in $L(X)$, the **free** Lie algebra in the indeterminates $X_1, \dots, X_m$, will be denoted with the letter $\mathcal{B}$. To any such bracket, one can associate a *length* and a *set of indeterminates*. For instance, $\mathcal{B} = [X_1, [X_2, X_1]]$ has length three, and his set of indeterminates is $\{X_1, X_2, X_1\}$. To each element A in $L(X)$, one can also associate an element in the **control** Lie algebra $Lie(f)$ by means of the *evaluation operator* Ev. More precisely, $Ev(f)(A)$ is the vector field obtained by plugging in the f_j's for the X_j's in A. For instance, if $\mathcal{B} = [X_1, X_2]$, then $Ev(f)(\mathcal{B})$ is the vector field $[f_1, f_2]$.

Finally, we recall some definitions on subsets of $\mathbf{R}$ [16, 6].

Definition 1 *Let Ω be a finite subset of $\mathbf{R}$ and $|\Omega|$ denote the number of elements of Ω. The set Ω is said to be "Minimally Canceling" (in short, MC) if and only if :*

i) $\displaystyle\sum_{\omega\in\Omega} \omega = 0$

ii)this is the only zero sum with at most $|\Omega|$ terms taken in Ω with possible repetitions:

$$\left.\begin{array}{c} \displaystyle\sum_{\omega\in\Omega} \lambda_\omega \omega = 0 \\ \displaystyle\sum_{\omega\in\Omega} |\lambda_\omega| \leq |\Omega| \\ (\lambda_\omega)_{\omega\in\Omega} \in \mathbf{Z}^{|\Omega|} \end{array}\right\} \Longrightarrow \left\{\begin{array}{rl} (\lambda_\omega)_{\omega\in\Omega} = & (0, \dots, 0) \\ & \mathit{or}\ (1, \dots, 1) \\ & \mathit{or}\ (-1, \dots, -1) \end{array}\right. \tag{12.13}$$

Definition 2 *Let $(\Omega^\xi)_{\xi\in E}$ be a finite family of finite subsets Ω^ξ of $\mathbf{R}$. The family $(\Omega^\xi)_{\xi\in E}$ is said to be "independent with respect to p" if and only if :*

$$\left.\begin{array}{l} \displaystyle\sum_{\xi \in E} \sum_{\omega \in \Omega^\xi} \lambda_\omega \omega = 0 \\ \displaystyle\sum_{\xi \in E} \sum_{\omega \in \Omega^\xi} |\lambda_\omega| \leq p \\ (\lambda_\omega)_{\omega \in \Omega^\xi, \xi \in E} \in \mathbf{Z}^{\Sigma|\Omega^\xi|} \end{array}\right\} \Longrightarrow \sum_{\omega \in \Omega^\xi} \lambda_\omega \omega = 0 \quad \forall \xi \in E \tag{12.14}$$

Algorithm

Step 1. Determine n vector fields $\tilde{f}_j$ $(j = 1, \ldots, n)$, obtained as Lie brackets of length $\ell(j)$ of the control vector fields f_i, and such that the matrix

$$\tilde{F}(x) \triangleq \left(\tilde{f}_1(x), \ldots, \tilde{f}_n(x)\right) \tag{12.15}$$

is nonsingular at $x = 0$.

Step 2. Determine a matrix G such that the matrix $(I_n + \tilde{F}(0)G)$ is discrete-stable (with I_n denoting the n-dimensional identity matrix), and define the linear feedback

$$a(x) = \frac{1}{T} Gx\,. \tag{12.16}$$

Step 3. By Step 1, there exists, for each $j = 1, \ldots, n$, a bracket $\mathcal{B}_j$ such that $\tilde{f}_j = Ev(f)(\mathcal{B}_j)$. Partition the set $\{\mathcal{B}_1, \ldots, \mathcal{B}_n\}$ in *homogeneous components* $P_1, \ldots, P_K$, i.e.

i) all brackets in a homogeneous component P_k have the same length $l(k)$, and the same set of indeterminates $\{X_{\tau_1^k}, \ldots, X_{\tau_{l(k)}^k}\}$.

ii) given two homogeneous components P_k and $P_{k'}$ (with $k \neq k'$), either $l(k) \neq l(k')$, or $\{X_{\tau_1^k}, \ldots, X_{\tau_{l(k)}^k}\} \neq \{X_{\tau_1^{k'}}, \ldots, X_{\tau_{l(k')}^{k'}}\}$.

Step 4. The last four steps can be conducted either in the control Lie algebra (c.l.a.) framework or in the framework of free Lie algebras (f.l.a.)[2].
c.l.a.: For every $k = 1, \ldots, K$, find permutations $\sigma_1, \ldots, \sigma_{\underline{C}(k)}$ in $\mathcal{S}(l(k))$ such that the vector fields

$$[f_{\tau_{\sigma(1)}^k}, [f_{\tau_{\sigma(2)}^k}, [\ldots, f_{\tau_{\sigma(l(k))}^k}] \ldots]] \qquad (\sigma \in \{\sigma_1, \ldots, \sigma_{\underline{C}(k)}\})$$

form a basis of the linear sub-space (over $\mathbf{R}$) of $Lie(f)$ spanned by the vector fields

[2] Respective advantages and drawbacks will be pointed out later.

$$[f_{\tau^k_{\sigma(1)}}, [f_{\tau^k_{\sigma(2)}}, [\ldots, f_{\tau^k_{\sigma(l(k))}}]\ldots]] \qquad (\sigma \in \mathcal{S}(l(k)))\,.$$

f.l.a.: For every $k = 1, \ldots, K$, find permutations $\sigma_1, \ldots, \sigma_{\overline{C}(k)}$ in $\mathcal{S}(l(k))$ such that the brackets

$$[X_{\tau^k_{\sigma(1)}}, [X_{\tau^k_{\sigma(2)}}, [\ldots, X_{\tau^k_{\sigma(l(k))}}]\ldots]] \qquad (\sigma \in \{\sigma_1, \ldots, \sigma_{\overline{C}(k)}\})$$

form a basis of the linear sub-space (over $\mathbf{R}$) of $L(X)$ spanned by the brackets

$$[X_{\tau^k_{\sigma(1)}}, [X_{\tau^k_{\sigma(2)}}, [\ldots, X_{\tau^k_{\sigma(l(k))}}]\ldots]] \qquad (\sigma \in \mathcal{S}(l(k)))\,.$$

Step 5.
c.l.a.: For every $k \in \{1, \ldots, K\}$ such that $l(k) \geq 2$, determine $C(k) \stackrel{\Delta}{=} \underline{C}(k)$ MC sets $\Omega^{k,c} = \{\omega_1^{k,c}, \ldots, \omega_{l(k)}^{k,c}\}$, with $c = 1, \ldots, C(k)$, such that

i) the family of sets $(\Omega^{k,c})_{c=1,\ldots,C(k)}^{k=1,\ldots,K}$ is independent w.r.t. $\max_{k\in\{1,\ldots,K\}} l(k)$
ii) all elements in these sets have a common divisor $\bar{\omega}$ $(= 2\pi/T)$, i.e.

$$\omega_i^{k,c}/\bar{\omega} \in \mathbf{Z}, \qquad \forall (k, c, i),$$

iii) the $C(k)$ elements $g^{k,c}$ $(c = 1, \ldots, C(k))$ of $Lie(f)$ defined by

$$g^{k,c} = \sum_{\sigma \in \mathcal{S}(l(k))} \frac{[f_{\tau^k_{\sigma(1)}}, [f_{\tau^k_{\sigma(2)}}, [\ldots, f_{\tau^k_{\sigma(l(k))}}]\ldots]]}{\omega^{k,c}_{\sigma(1)}(\omega^{k,c}_{\sigma(1)} + \omega^{k,c}_{\sigma(2)})\cdots(\omega^{k,c}_{\sigma(1)} + \ldots + \omega^{k,c}_{\sigma(l(k)-1)})}$$

are independent (over $\mathbf{R}$).

For every $k \in \{1, \ldots, K\}$ such that $l(k) = 1$, just set $\omega_1^{k,1} = 0$.

Each family of sets $\{\Omega^{k,c}\}_{c=1,\ldots,C(k)}$ is used to associate the following sine and cosine functions with P_k

$$\alpha^{k,c}_{\tau^k_i}(t) = \begin{cases} \cos \omega_i^{k,c} t \ (i = 1) \\ \sin \omega_i^{k,c} t \ \ (i = 2, \ldots, l(k))\,. \end{cases} \tag{12.17}$$

f.l.a.: Same as above, with $C(k) \stackrel{\Delta}{=} \overline{C}(k)$ instead of $\underline{C}(k)$, each f_i replaced by X_i, and $Lie(f)$ replaced by $L(X)$.

Step 6.
c.l.a.: For each $k \in \{1, \ldots, K\}$ and j such that $\mathcal{B}_j \in P_k$, determine coefficients $\mu_j^{k,c}$ $(c = 1, \ldots, C(k))$ such that

$$\tilde{f}_j = \frac{(-1)^{l(k)-1}}{l(k)2^{l(k)-1}} \sum_{c=1}^{C(k)} \mu_j^{k,c} g^{k,c} . \tag{12.18}$$

f.l.a.: Same as above, with $\tilde{f}_j$ replaced by $\mathcal{B}_j$.

Step 7.
c.l.a. and **f.l.a.**: For each $k \in \{1, \ldots, K\}$, determine $l(k)C(k)$ state dependent functions $v_{\tau_i^k}^{k,c}$ which are $O(|x|^{\frac{1}{l(k)}})$, and such that

$$\prod_{i=1}^{l(k)} v_{\tau_i^k}^{k,c}(x) = \sum_{j:\mathcal{B}_j \in P_k} \mu_j^{k,c} a_j(x) \tag{12.19}$$

(a_j is the j-th component of a defined by (12.16)).

The following result concludes the description of the algorithm and points out the robustness properties associated with the resulting control in connection with Theorem 12.2.1.

Theorem 12.3.1. *Let*

$$u_i(x,t) = \begin{cases} \displaystyle\sum_{k=1}^{K} \sum_{c=1}^{C(k)} \sum_{p:\tau_p^k = i} \alpha_{\tau_p^k}^{k,c}(t) v_{\tau_p^k}^{k,c}(x) & if\ \exists (k,p) : \tau_p^k = i \\ 0 & otherwise\,. \end{cases} \tag{12.20}$$

with $C(k)$ equal to $\underline{C}(k)$ in the c.l.a. case, and to $\overline{C}(k)$ in the f.l.a. case. Then,

i) in both cases, u defined by (12.20) belongs to $\mathcal{C}^0(\mathbf{R}^n \times \mathbf{R}^+; \mathbf{R}^m)$, is T-periodic w.r.t. t, and satisfies the three assumptions of Theorem 12.2.1.

ii) in the f.l.a. case, local asymptotic stability of the origin of the perturbed system $(\bar{S}_\varepsilon)$ is guaranteed for any ε such that $I_n + \tilde{F}_\varepsilon(0)G$ is discrete-stable, where $\tilde{F}_\varepsilon$ denotes the matrix-valued function obtained from (12.15) by replacing each $\tilde{f}_j = Ev(f)(\mathcal{B}_j)$ by $\tilde{f}_{j,\varepsilon} = Ev(f + h(\varepsilon, .))(\mathcal{B}_j)$.

Property *ii*) above summarizes the main advantage of working in the f.l.a. framework. In this case, *asymptotic stability of the origin of the controlled perturbed system is just equivalent to discrete-stability of the matrix $I_n + \tilde{F}_\varepsilon(0)G$.* This result is conceptually interesting because it is reminiscent of a well known robustness result associated with linear systems. On the other hand, the fact that the number $\underline{C}(k)$ is usually smaller than $\overline{C}(k)$ characterizes the main advantage of the c.l.a. framework over the f.l.a. one in terms of complexity of the control expression (12.20), as measured by the number of terms and time-periodic functions involved in this expression. Further explanations and comments about the algorithm are given in [11].

Now we show that robustness of the hybrid law (12.20) is conserved when sampling the control function at a large enough frequency.

Proposition 2 *Let u be defined by (12.20), and denote u_N (with $N \in \mathbf{N}$) the sampled function defined by*

$$\begin{gathered}\forall k \in \mathbf{N}\,,\ \forall n = 0, \ldots, N-1\,,\ \forall t \in \left[kT + \tfrac{nT}{N}, kT + \tfrac{(n+1)T}{N}\right)\,, \\ u_N(x,t) = u(x, kT + nT/N)\,.\end{gathered} \tag{12.21}$$

Then, there exists $N_0 \in \mathbf{N}$ such that, for $N \geq N_0$, u_N is also a robust exponential stabilizer for (S_0).

Proof: The proof consists in showing that u_N satisfies the three assumptions of Theorem 12.2.1. It is clear from (12.21) that Assumption 1 is satisfied for u_N since, from Theorem 12.3.1, u satisfies Assumption 1. Let us now consider Assumption 2. Using the Chen-Fliess series, the solution $x(.)$ of (12.11) with u_N as control satisfies

$$\begin{aligned} x(T) &= x_0 + \sum_{|I| \leq 1/\alpha} (f_I\, id)(x_0) \int_0^T u_{N,I}(x_0) + o(x_0) \\ &= x_0 + \sum_{|I| \leq 1/\alpha} (f_I\, id)(x_0) \int_0^T u_I(x_0) + o(x_0) \\ &+ \sum_{|I| \leq 1/\alpha} (f_I\, id)(x_0) \left(\int_0^T u_{N,I}(x_0) - \int_0^T u_I(x_0) \right) \\ &= Ax_0 + o(x_0) \\ &+ \sum_{|I| \leq 1/\alpha} (f_I\, id)(x_0) \left(\int_0^T u_{N,I}(x_0) - \int_0^T u_I(x_0) \right)\,, \end{aligned} \tag{12.22}$$

where we have used the fact that u and u_N satisfy Assumption 1 with the same value of α, and the fact that u satisfies Assumption 2. Let us now consider each term

$$\int_0^T u_{N,I}(x_0) - \int_0^T u_I(x_0)$$

in (12.22). Using (12.20), we can rewrite this term as

$$\int_0^T u_{N,I}(x_0) - \int_0^T u_I(x_0) = \sum_q v_I^q(x_0) \left(\int_0^T \alpha_{N,I}^q - \int_0^T \alpha_I^q \right)\,. \tag{12.23}$$

This expression reads as follows. Each q denotes a family $(q_1, \ldots, q_{|I|})$ with $q_i = (k_i, c_i)$, and

$$v_I^q(x_0) = v_{i_1}^{q_1}(x_0) \cdots v_{i_{|I|}}^{q_{|I|}}(x_0)\,,$$

$$\int_0^T \alpha_{N,I}^q = \int_0^T \alpha_{N,i_I}^{q_{|I|}}(t_{|I|}) \int_0^{t_{|I|}} \cdots \int_0^{t_2} \alpha_{N,i_1}^{q_1}(t_1)dt_1 \ldots dt_{|I|}\,,$$

and

$$\int_0^T \alpha_I^q = \int_0^T \alpha_{i_I}^{q_{|I|}}(t_{|I|}) \int_0^{t_{|I|}} \cdots \int_0^{t_2} \alpha_{i_1}^{q_1}(t_1)dt_1 \ldots dt_{|I|}\,.$$

Specifying further the (finite) set on which the sum in (12.23) is taken is not important. Note that the integrals in the right-hand side of (12.23) are iterated integrals of sine or cosine functions, and sampled sine or cosine functions, which are independent of x_0. To proceed with the proof, we need the following lemma.

Lemma 1 *Each term*

$$v_I^q(x_0)\left(\int_0^T \alpha_{N,I}^q - \int_0^T \alpha_I^q\right) \tag{12.24}$$

in (12.23), viewed as a function of x_0, satisfies one of the following properties

a) it is a $o(x_0)$,

b) it is a linear function of x_0,

c) it is identically zero for N large enough.

(Proof given farther)

Since each term

$$\int_0^T \alpha_{N,I}^q - \int_0^T \alpha_I^q$$

obviously tends to zero as N tends to infinity, we deduce from Lemma 1, that the term (12.24) is either a $o(x_0)$, or a term $A_N(I,q)x_0$ with $A_N(I,q)$ a matrix which tends to zero as N tends to infinity, or zero for N large enough. Therefore, from (12.22), (12.23), and using the facts that the f_i's are smooth, and that the number of multi-indices I such that $|I| \leq 1/\alpha$ is finite, there exists a matrix $B(N)$ which tends to zero as N tends to infinity, and such that

$$x(T) = Ax_0 + B(N)x_0 + o(x_0)\,.$$

This clearly implies that Assumption 2 is satisfied for N large enough. Finally the satisfaction Assumption 3 is a direct consequence of (12.23), Lemma 1, and the fact that u satisfies this assumption. There remains to prove Lemma 1.

Proof of Lemma 1: Assuming that neither a) nor b) hold, we show that c) must be satisfied. The proof consists in expanding each sampled sine or cosine function as a Fourier series, in order to evaluate each term

$$\int_0^T \alpha_{N,I}^q \,. \tag{12.25}$$

First, we establish the following

Claim 1 *Let* $\{\omega_1, \ldots, \omega_{|I|}\}$ *denote the set of frequencies associated with the functions* $\alpha_1, \ldots, \alpha_{|I|}$ *in*

$$\int_0^T \alpha_I^q \,.$$

Then, for each M.C. set $\Omega^{k,c}$, $\{\omega_1, \ldots, \omega_{|I|}\}$ *contains at most* $l(k)$ *elements which belong to* $\Omega^{k,c}$, *and does not contain* $\Omega^{k,c}$ *itself.*

We prove the claim by contradiction, and first assume that $\{\omega_1, \ldots, \omega_{|I|}\}$ contains more than $l(k)$ elements of some $\Omega^{k,c}$. Then, in view of Step 7 of the design algorithm, we deduce that $v_I^q(x_0) = o(x_0)$. This contradicts our initial assumption according to which Property a) in Lemma 1 is not satisfied. On the other hand, if the set $\{\omega_1, \ldots, \omega_{|I|}\}$ contains some set $\Omega^{k,c}$ then, either these two sets are equal and, from (12.16) and (12.19), v_I^q is a linear function (in contradiction with the assumption that Property b) of Lemma 1 is not satisfied), or $\{\omega_1, \ldots, \omega_{|I|}\}$ contains $\Omega^{k,c}$ plus extra terms, in which case $v_I^q(x_0) = o(x_0)$ (again in contradiction with our initial assumption).

Having proved Claim 1, we return to the proof of the lemma. In order to simplify the notation, we assume from now on that $T = 2\pi$. For different values of T, the proof follows by a simple change of time variable. Let $\alpha_{N,i}$ denote any sampled sine or cosine function. Away from points of discontinuity,

$$\alpha_{N,i}(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jnt} \,, \tag{12.26}$$

with

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \alpha_{N,i}(t) e^{-jnt} \, dt \,.$$

First, consider the case when $\alpha_i(t) = \cos \omega_i t$. Then, denoting $\Delta \triangleq T/N = 2\pi/N$,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\sum_{k=0}^{N-1}\int_{k\Delta}^{(k+1)\Delta}\frac{e^{j\omega_i k\Delta}+e^{-j\omega_i k\Delta}}{2}e^{-jnt}\,dt \\ &= -\frac{1}{4jn\pi}\sum_{k=0}^{N-1}\left(e^{j\omega_i k\Delta}+e^{-j\omega_i k\Delta}\right)\left(e^{-jn(k+1)\Delta}-e^{-jnk\Delta}\right) \\ &= -\frac{1}{4jn\pi}\left(e^{-jn\Delta}-1\right)\sum_{k=0}^{N-1}e^{j(\omega_i-n)k\Delta}+e^{-j(\omega_i+n)k\Delta}\,. \end{aligned}$$

If $n-\omega_i\notin N\mathbf{Z}$, then

$$\sum_{k=0}^{N-1}e^{j(\omega_i-n)k\Delta}=\frac{1-e^{j(\omega_i-n)N\Delta}}{1-e^{j(\omega_i-n)\Delta}}=\frac{1-e^{j(\omega_i-n)2\pi}}{1-e^{j(\omega_i-n)\Delta}}=0\,,$$

where the last equality comes from the fact that, from Step 5, $\omega_i\in\mathbf{Z}$. Similarly, if $n+\omega_i\notin N\mathbf{Z}$

$$\sum_{k=0}^{N-1}e^{-j(\omega_i+n)k\Delta}=0\,.$$

Therefore, c_n is possibly different from zero only if $n=\pm\omega_i(modN)$, so that (12.26) may be rewritten as

$$\begin{aligned} \cos_N\omega_i t &= \sum_{k=-\infty}^{+\infty}\eta_{i,k}^{1}e^{j(\omega_i+kN)t}+\sum_{k=-\infty}^{+\infty}\eta_{i,k}^{-1}e^{-j(\omega_i+kN)t} \\ &= \sum_{k=-\infty}^{+\infty}\sum_{s\in\{-1,1\}}\eta_{i,k}^{s}e^{sj(\omega_i+kN)t}\,, \end{aligned} \tag{12.27}$$

where the $\eta_{i,k}^s$ are complex coefficients which depend on ω_i, N, k, and s. Similarly,

$$\sin_N\omega_i t=\sum_{k=-\infty}^{+\infty}\sum_{s\in\{-1,1\}}\eta_{i,k}^{s}e^{sj(\omega_i+kN)t}\,, \tag{12.28}$$

where the $\eta_{i,k}^s$ are other complex coefficients. In view of (12.27) and (12.28), we can rewrite (12.25) as

$$\int_0^T\alpha_{N,I}^q=\sum_{(k_1,\ldots,k_{|I|})\in\mathbf{Z}^{|I|}}J_{N,I}(k_1,\ldots,k_{|I|})\,,$$

with

$$\begin{aligned} J_{N,I}(k_1,\ldots,k_{|I|}) &\triangleq \int_0^{2\pi}\sum_{s_{|I|}\in\{-1,1\}}\eta_{i_{|I|},k_{|I|}}^{s_{|I|}}e^{s_{|I|}j(\omega_{i_{|I|}}+k_{|I|}N)\tau_{|I|}} \\ &\int_0^{\tau_{|I|}}\cdots\int_0^{\tau_2}\sum_{s_1\in\{-1,1\}}\eta_{i_1,k_1}^{s_1}e^{s_1 j(\omega_{i_1}+k_1N)\tau_1}\,d\tau_1\ldots d\tau_{|I|}\,. \end{aligned} \tag{12.29}$$

The above expression is to be compared with the following one, derived when the sine and cosine functions are not sampled:

$$\begin{aligned}\int_0^T \alpha_I^q = \int_0^{2\pi} \sum_{s_{|I|}\in\{-1,1\}} \eta_{i_{|I|}}^{s_{|I|}} e^{s_{|I|} j\omega_{i_{|I|}}\tau_{|I|}} \int_0^{\tau_{|I|}} \ldots \\ \int_0^{\tau_2} \sum_{s_1\in\{-1,1\}} \eta_{i_1}^{s_1} e^{s_1 j\omega_{i_1}\tau_1}\, d\tau_1 \ldots d\tau_{|I|}\,,\end{aligned} \tag{12.30}$$

with $\eta_i^1 = \eta_i^{-1} = 1/2$ if α_i is a cosine function, and $\eta_i^1 = -\eta_i^{-1} = -i/2$ if α_i is a sine function. We have proved in [11, Lemma 2] that, when the condition of Claim1 is satisfied, the integral (12.30) is zero. We claim that each iterated integral (12.29) is also equal to zero provided that

$$N > \sum_{i=1}^{|I|} |\omega_i|\,. \tag{12.31}$$

This condition is needed in order to ensure the following property:

$$\left.\begin{aligned}\sum_{p=1}^{|I|} \lambda_p(\omega_{i_p} + k_p N) = 0 \\ \lambda_p \in \{-1,0,1\}\end{aligned}\right\} \Longrightarrow \sum_{p=1}^{|I|} \lambda_p \omega_{i_p} = 0\,.$$

We leave to the reader the task of verifying that this property allows a direct transposition of the proof given in [11, Lemma 2] for the integral (12.30). Therefore, both integrals involved in (12.24) are equal to zero when (12.31) holds, and Property c) of Lemma 1 is verified. Note that imposing

$$N > \max_k l(k) \max_{i,k,c} |\omega_i^{k,c}|$$

automatically ensures (12.31) since, from (12.22) and Step 7, $|I| \leq 1/\alpha = \max_k l(k)$. ∎

12.3.2 Control design from a homogeneous approximation

It is often convenient and simpler to work with approximations of control systems. For instance, linear approximations are commonly used for feedback control design when they are controllable (or at least stabilizable). When the linear approximation of the system, evaluated at the equilibrium which feedback control is in charge of stabilizing, is not stabilizable, the extension of the notion of linear approximation yields to homogeneous controllable approximations. Using such an approximation is particularly well adapted to the design of continuous homogeneous feedbacks which render the closed-loop system homogeneous of degree zero. The reason is that asymptotic stabilization of the origin of the homogeneous approximation automatically ensures that the origin of the initial control system is also asymptotically (locally) stabilized by the same feedback control law. It is however important to realize that this property *does not necessarily hold* when using hybrid controllers such as those which we are considering here, and it is not difficult to work out simple examples which illustrate this fact. Nevertheless, it is proved in [11] that a robust controller for the system (S_0) can be derived from the knowledge of a homogeneous approximation of this system, provided that some extra condition is satisfied by the control law. This condition will be stated in a theorem, after recalling a few definitions and properties about homogeneous systems. A complementary proposition will indicate how the control design algorithm previously described can be completed in order to cope with the use of homogeneous approximations.

Given $\lambda > 0$ and a *weight vector* $r = (r_1, \ldots, r_n)$ $(r_i > 0\ \forall i)$, a *dilation* δ^r_λ is a map from $\mathbf{R}^n$ to $\mathbf{R}^n$ defined by

$$\delta^r_\lambda(z_1, \ldots, z_n) = (\lambda^{r_1} z_1, \ldots, \lambda^{r_n} z_n)\,.$$

A function $f \in \mathcal{C}^0(\mathbf{R}^n; \mathbf{R})$ is *homogeneous of degree l with respect to the family of dilations* δ^r_λ $(\lambda > 0)$, or, more concisely, *δ^r-homogeneous of degree l*, if

$$\forall \lambda > 0, \quad f(\delta^r_\lambda(z)) = \lambda^l f(z)\,.$$

A *δ^r-homogeneous norm* can be defined as a positive definite function on $\mathbf{R}^n$, δ^r-homogeneous of degree one. Although this is not a "true" norm when the weight coefficients are not all equal, it still provides a means of "measuring" the size of the state.

A continuous vector field X on $\mathbf{R}^n$ is *δ^r-homogeneous of degree d* if, for all $i = 1, \ldots, n$, the function $z \longmapsto X_i(z)$ is δ^r-homogeneous of degree $r_i + d$. According to these definitions, homogeneity is coordinate dependent, however it is possible to define the above concepts in a coordinate independent framework [5, 13].

Finally, we say that the system

$$\dot{z} = \sum_{i=1}^{m} b_i(z) u_i \tag{12.32}$$

is a *δ^r-homogeneous approximation* of (S_0) if:

1. the change of coordinates $\phi : \ x \longmapsto z$ transforms (S_0) into

$$\dot{z} = \sum_{i=1}^{m} \left(b_i(z) + g_i(z)\right) u_i\,, \tag{12.33}$$

where b_i is δ^r-homogeneous of some degree $d_i < 0$, and g_i denotes higher-order terms, i.e. such that $g_{i,j}$ (the j-th component of g_i) satisfies

$$g_{i,j} = o(\rho^{r_j + d_i}), \quad (j = 1, \dots, n). \tag{12.34}$$

where ρ is a δ^r-homogeneous norm;

2. the system (12.32) is controllable.

Hermes [3] and Stefani [15] have shown that any driftless system (S_0) satisfying the LARC (Lie Algebra Rank Condition) at the origin (12.2) has a homogeneous approximation (which is not unique in general).

Theorem 12.3.2. *Consider a δ^r-homogeneous approximation (12.32) of (S_0), with $d_i \triangleq deg(b_i)$ $(i = 1, \dots, m)$, and a control function*

$$u \in \mathcal{C}^0(U \times [0,T]; \mathbf{R}^m)$$

such that the three assumptions in Theorem 12.2.1 are verified for this approximating system. Assume furthermore that the following assumption, which is a stronger version of the third assumption in Theorem 12.2.1, is also verified for the approximating system:

3-bis. for any multi-index $I = (i_1, \dots, i_{|I|})$ of length $|I| \leq 1/\alpha$,

$$\int_0^T u_I(z) = \sum_{k: r_k \geq \|I\|} a_{I,k} z_k + o(z), \tag{12.35}$$

where $\|I\| \triangleq -\sum_{j=1}^{|I|} d_{i_j}$, and the $a_{I,k}$'s are some scalars.

Then, the three assumptions of Theorem 12.2.1 are verified for the system (12.33).

When applying the algorithm of Section 12.3.1 to the approximation (12.32), the control law u given by (12.20) may not satisfy the extra condition 3-bis of Theorem 12.3.2. However, it is possible to impose extra requirements on the matrix G defined in Step 2 so as to guarantee the satisfaction of this condition. For instance, the following result is proved in [11].

Proposition 3 *Consider a δ^r-homogeneous approximation (12.32) of (S_0), with every control vector field of this system being δ^r-homogeneous of degree -1. Without loss of generality, we assume that the variables z_i are ordered by increasing weight, i.e.*

$$r_1 \leq r_2 \leq \ldots \leq r_n \,,$$

and decompose z as $z = (z^1, \ldots, z^P)$, where each z^p ($1 \leq p \leq P$) is the sub-vector of z whose components have same weight r^p ($r_1 \leq r^p \leq r_n$) with

$$r_1 = r^1 < r^2 < \ldots < r^P = r_n \,.$$

Consider the control design algorithm described in Section 12.3.1 and applied to (12.32). Let $\tilde{b}_j$ ($j \in \{1, \ldots, n\}$) denote the vector fields defined according to Step 1 of the algorithm, and

$$\tilde{B}(z) \overset{\Delta}{=} (\tilde{b}_1(z), \ldots, \tilde{b}_n(z)) \,.$$

Due to the ordering of the variables z_i, the matrix $\tilde{B}(z)$ is block lower triangular, and block diagonal at $z = 0$, i.e.

$$\tilde{B}(0) = \begin{pmatrix} \tilde{B}^{11} & 0 & \cdots & 0 \\ 0 & \tilde{B}^{22} & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \tilde{B}^{PP} \end{pmatrix} .$$

Assume that the control gain matrix G involved in Step 2 of the algorithm is chosen as follows

$$G = \tilde{B}(0)^{-1}(H - I_n)$$

with the matrix H being block upper triangular, i.e.

$$A = \begin{pmatrix} H^{11} & \star & \cdots & \star \\ 0 & H^{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ 0 & \cdots & 0 & H^{PP} \end{pmatrix} ,$$

and discrete-stable ($\Leftrightarrow$ H^{ii} is discrete-stable for $i \in \{1, \ldots, P\}$).
Then, the three assumptions of Theorem 12.2.1 are verified for the system (S_0).

12.3.3 Stabilizers for chained systems

In some cases, it is possible to take advantage of specific structural properties associated with the control system under consideration, in order to derive robust control laws that are simpler than those obtained by application of the general algorithm presented in Section 12.3.1. We illustrate this possibility in the case of the following n-dimensional chained system

$$(S_0) \quad \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = u_1 x_2 \\ \quad \vdots \\ \dot{x}_n = u_1 x_{n-1} \,. \end{cases} \tag{12.36}$$

The next result points out a set of robust exponential stabilizers for this system.

Theorem 12.3.3. *With the control function $u \in \mathcal{C}^0(\mathbf{R}^n \times [0,T]; \mathbf{R}^2)$ defined by*

$$\begin{cases} u_1(x,t) = \dfrac{1}{T}[(g_1 - 1)x_1 + 2\pi\rho_q(x)\sin(\bar{\omega}t)] \\ u_2(x,t) = \dfrac{1}{T}[(g_2 - 1)x_2 \\ \qquad + \sum_{i=3}^{n} 2^{i-2}(i-2)!(g_i - 1)\frac{x_i}{\rho_q^{i-2}(x)}\cos((i-2)\bar{\omega}t)] \,, \end{cases} \tag{12.37}$$

with

$$\begin{aligned} & T = 2\pi/\bar{\omega} \quad (\bar{\omega} \neq 0) \,, \\ & \rho_q(x) = \sum_{j=3}^{n} \alpha_j |x_j|^{\frac{1}{q+j-2}} \,, \qquad (q \geq n-2 \,,\ \alpha_j > 0) \,, \\ & |g_i| < 1 \,, \quad \forall i = 1, \dots, n \,, \end{aligned} \tag{12.38}$$

the three assumptions in Theorem 12.2.1, and the extra assumption in Theorem 12.3.2, are verified for the system (12.36).

Corollary 1 *(of Theorems 12.3.2 and 12.3.3) With the control function (12.37), the three assumptions in Theorem 12.2.1 are verified for any analytic driftless system for which the chained system (12.36) is a δ^r-homogeneous approximation, with $r = (1, q, \dots, q+n-2)$ and $q \geq n-2$.*

12.4 Control laws for a dynamic extension

In mechanics, systems with non-holonomic constraints (wheeled mobile-robots, systems with rolling parts,...) give rise to driftless systems like (S_0). In this case, x represents the configuration vector, and the control,u, is a vector of admissible velocities. In practice, it is however more realistic to consider torque control inputs rather than velocity control inputs. Since torques are homogeneous to accelerations, it is then natural to consider the following system (compare with (S_0))

$$(D_0) : \begin{cases} \dot{x} = \sum_{i=1}^{m} f_i(x) u_i \\ \dot{u} = w \,, \end{cases}$$

where $u = (u_1, \ldots, u_m)$, (x, u) is the state vector, and $w = (w_1, \ldots, w_m)$ is now taken as the control variable. If $\bar{u}$ denotes an exponential robust stabilizer for (S_0) (as derived in the previous section for instance), we would like to deduce an exponential stabilizer w for (D_0), which conserves the robustness properties of $\bar{u}$. More precisely, we look for a feedback $w(y, v, t)$ such that the origin of the controlled system

$$(\bar{D}_\varepsilon) : \begin{cases} \dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\varepsilon, x)) u_i \\ \dot{u} = w(y, v, t) \\ \dot{y} = (\sum_{k \in \mathbf{N}} \delta_{kT})(x - y_{-\alpha}) \\ \dot{v} = (\sum_{k \in \mathbf{N}} \delta_{kT})(u - v_{-\alpha}) \quad 0 < \alpha < T \end{cases}$$

is exponentially stable when $|\varepsilon|$ is small enough. We say that such a controller is an *exponential robust stabilizer* for (D_0). Let us remark that this is a somewhat simplified problem since we do not consider perturbations on the dynamic part. More precisely, having in mind the dynamic equations of mechanical systems, it would be justified to complement the perturbed system (S_ε) with an equation such as

$$\dot{u} = (I_m + g_1(\varepsilon, x, u)) w + g_0(\varepsilon, x, u) \,,$$

with $g_1(0, ., .) = g_0(0., .,) \equiv 0$, and $g_0(., 0, 0) \equiv 0$ (so that $(x, u) = (0, 0)$ remains an equilibrium point). Beside the possibility that there may not exist controllers which ensure robustness with respect to such general perturbations, the analysis appears much more difficulty in this case. For this reason, the present analysis is limited to perturbations on the kinematic part only. Nonetheless, it is not very difficult to show that the control laws proposed below are also robust with respect to less general perturbations (such as these modeled by a function g_1 which depends on ε only).

The following proposition provides exponential robust stabilizers for (D_0).

Proposition 4 *Let $\bar{u} \in \mathcal{C}^0(U \times \mathbf{R}^+; \mathbf{R}^m)$ denote a function Hlder-continuous with respect to x, differentiable and periodic of period T with respect to t. Assume further that $\bar{u}$ is an (hybrid) exponential robust stabilizer for (S_0). Denote α the function*

$$t \longmapsto \alpha(t) = t - \frac{T}{2\pi} \sin \frac{2\pi t}{T}. \tag{12.39}$$

Then,

1. *the function* $\bar{u}_c \in \mathcal{C}^0(U \times \mathbf{R}^+; \mathbf{R}^m)$ *defined by*

$$\bar{u}_c(y,t) = \dot{\alpha}(t)\bar{u}(y,\alpha(t)) \tag{12.40}$$

is also an exponential robust stabilizer for (S_0)*, with the function* $t \longmapsto \bar{u}_c(y(t),t)$ *being continuous along the trajectories of the closed-loop system* $(\bar{S}_0)$*,*

2. *the function* $w \in \mathcal{C}^0(U \times \mathbf{R}^m \times \mathbf{R}^+; \mathbf{R}^m)$ *defined by*

$$w(y,v,t) = \frac{\partial}{\partial t}\bar{u}_c(y,t) - \frac{v}{T} \tag{12.41}$$

is an exponential robust stabilizer for (D_0) .

Proof: First, we show that Property 1 is satisfied. From (12.39), α defines a time-scaling on $\mathbf{R}^+$ which leaves each $t = kT$ invariant (i.e. $\alpha(kT) = kT$ for all $k \in \mathbf{N}$). One readily verifies that this time-scaling maps the solutions of (S_0) controlled by $\bar{u}$ to the solutions of (S_0) controlled by $\bar{u}_c$, i.e.

$$\begin{aligned}\dot{x}(t) = \sum_{i=1}^{m} f_i(x(t))\bar{u}(x_0,t) \Longrightarrow \frac{d}{dt}x(\alpha(t)) &= \sum_{i=1}^{m} f_i(x(t))\dot{\alpha}(t)\bar{u}(x_0,\alpha(t)) \\ &= \sum_{i=1}^{m} f_i(x(t))\bar{u}_c(x_0,\alpha(t))\,.\end{aligned}$$

Since this time-scaling also "preserves" the solutions of the perturbed systems (S_ε), we conclude that $\bar{u}_c$ is a robust exponential stabilizer for (S_0). Finally, $\bar{u}_c$ is continuous along the trajectories of $(\bar{S}_0)$ because $\dot{\alpha}(kT) = 0$ for all k, so that

$$\bar{u}_c(y(kT),kT) = 0 = \lim_{t \to kT} \bar{u}_c(y(t),t)\,.$$

Now we show that Property 2 is verified. We only prove exponential convergence to the origin of the closed-loop systems' solutions. Existence of these solutions and uniform stability of the origin can be proved via a simple adaptation of the proof of [11, Theorem 1], in the case of driftless systems. Let $(x_\varepsilon, u_\varepsilon, y_\varepsilon, v_\varepsilon)(., t_0, x_0, u_0, y_0, v_0)$ denote the solution of the controlled system $(\bar{D}_\varepsilon)$ with initial conditions $(t_0, x_0, u_0, y_0, v_0)$, $t_0 \in [k_0T, (k_0+1)T)$, $k_0 \in \mathbf{N}$. Then, for any $k \in \mathbf{N}$ such that $k_0 < k$, and any $t \in [kT, (k+1)T)$, this solution satisfies

$$\begin{cases} \dot{x} = \sum_{i=1}^{m} (f_i(x) + h_i(\varepsilon,x))u_i(t) \\ \dot{u} = w(x(kT), u(kT), t) \\ \dot{y} = 0\,, \quad y(t) = x(kT) \\ \dot{v} = 0\,, \quad v(t) = u(kT) \end{cases} \tag{12.42}$$

From (12.42) and (12.41),

$$u(t) = u(kT) + \bar{u}_c(x(kT), t) - \bar{u}_c(x(kT), kT) - \frac{u(kT)}{T}(t - kT) . \quad (12.43)$$

Using the fact that $\bar{u}_c(., kT) \equiv 0$ for all k, we deduce that

$$u((k+1)T) = 0 . \quad (12.44)$$

As a consequence, for $t \in [kT, (k+1)T)$ and $k \geq k_0 + 2$, we deduce from (12.43) and (12.44) that

$$u(t) = \bar{u}_c(x(kT), t) . \quad (12.45)$$

Thus, for $t \geq (k_0 + 2)T$, the x component of the solution of (12.42) coincides with the solution of the system $(\bar{S}_\varepsilon)$ controlled by $\bar{u}_c(y, t)$. Since, from Property 1, $\bar{u}_c$ is an exponential stabilizer for $(\bar{S}_0)$, we deduce that $|x(t)|$ converges exponentially to zero. Then, using the fact that $\bar{u}$ (and therefore $\bar{u}_c$) is Hlder-continuous w.r.t. x, we deduce from (12.45) that $|u(t)|$ also converges exponentially to zero. ∎

References

1. M. K. Bennani and P. Rouchon. Robust stabilization of flat and chained systems. In *European Control Conference (ECC)*, pages 2642–2646, 1995.

2. M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *International Journal of Control*, 61:1327–1361, 1995.

3. H. Hermes. Nilpotent and high-order approximations of vector field systems. *SIAM Review*, 33:238–264, 1991.

4. A. Isidori. *Nonlinear Control Systems.* Springer Verlag, third edition, 1995.

5. M. Kawski. Geometric homogeneity and stabilization. In *IFAC Nonlinear Control Systems Design Symp. (NOLCOS)*, pages 164–169, 1995.

6. W. Liu. An approximation algorithm for nonholonomic systems. *SIAM Journal on Control and Optimization*, 35:1328–1365, 1997.

7. D.A. Lizárraga, P. Morin, and C. Samson. Non-robustness of continuous homogeneous stabilizers for affine systems. Technical Report 3508, INRIA, 1998. Available at http://www.inria.fr/RRRT/RR-3508.html.

8. R.T. M'Closkey and R.M. Murray. Exponential stabilization of driftless nonlinear control systems using homogeneous feedback. *IEEE Trans. on Automatic Control*, 42:614–628, 1997.

9. S. Monaco and D. Normand-Cyrot. An introduction to motion planning using multirate digital control. In *IEEE Conf. on Decision and Control (CDC)*, pages 1780–1785, 1991.

10. P. Morin, J.-B. Pomet, and C. Samson. Design of homogeneous time-varying stabilizing control laws for driftless controllable systems via oscillatory approximation of lie brackets in closed-loop. *SIAM Journal on Control and Optimization*. To appear.

11. P. Morin and C. Samson. Exponential stabilization of nonlinear driftless systems with robustness to unmodeled dynamics. Technical Report 3477, INRIA, 1998. To appear in COCV.

12. R.M. Murray and S.S. Sastry. Nonholonomic motion planning: Steering using sinusoids. *IEEE Trans. on Automatic Control*, 38:700–716, 1993.

13. L. Rosier. *Etude de quelques problèmes de stabilisation.* PhD thesis, Ecole Normale de Cachan, 1993.

14. O. J. Sørdalen and O. Egeland. Exponential stabilization of nonholonomic chained systems. *IEEE Trans. on Automatic Control*, 40:35–49, 1995.

15. G. Stefani. Polynomial approximations to control systems and local controllability. In *IEEE Conf. on Decision and Control (CDC)*, pages 33–38, 1985.

16. H. J. Sussmann and W. Liu. Limits of highly oscillatory controls ans approximation of general paths by admissible trajectories. In *IEEE Conf. on Decision and Control (CDC)*, pages 437–442, 1991.

17. H.J. Sussmann. Lie brackets and local controllability: a sufficient condition for scalar-input systems. *SIAM Journal on Control and Optimization*, 21:686–713, 1983.

18. H.J. Sussmann. A general theorem on local controllability. *SIAM Journal on Control and Optimization*, 25:158–194, 1987.