# A DIRICHLET TYPE PROBLEM FOR NONLINEAR DEGENERATE ELLIPTIC EQUATIONS ARISING IN TIME-OPTIMAL STOCHASTIC CONTROL

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**Abstract.** We study general 2nd order fully nonlinear degenerate elliptic equations on an arbitrary closed set with generalized Dirichlet boundary conditions in the viscosity sense. We prove some properties of the maximal subsolution and the minimal supersolution of the Dirichlet type problem. Under a sort of compatibility condition on the boundary data we show that the maximal subsolution is the natural generalized solution of the boundary value problem, even if it is not necessarily continuous, and we give approximation theorems, in particular by penalization. We apply these results to the Hamilton-Jacobi-Bellman-Isaacs equations of time-optimal stochastic control and pursuit-evasion games whose value functions, which are discontinuous in general, are characterized as the unique solution in the previous sense of the appropriate Dirichlet type problem.

**Key words**: viscosity solutions, nonlinear degenerate elliptic equations, Hamilton-Jacobi-Bellman equations, Isaacs equations, Dirichlet problem, time-optimal control, diffusion processes, pursuit-evasion games, penalization.

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# 0 Introduction

In this paper we consider the Dirichlet type boundary value problem

$$(BVP^*) \begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \Omega, \\ u = g \text{ or } F(x,u,Du,D^2u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset I\!\!R^N$ is an open set, $F$ is a fully nonlinear degenerate elliptic operator, increasing in the variable $u$, and $g$ is continuous on $\partial\Omega$. By degenerate elliptic we mean that for any symmetric matrices $X$, $Y$

$$F(x,r,p,X) \le F(x,r,p,Y) \ \text{ whenever } Y - X \le 0,$$

an assumption which is satisfied even by first order operators.

This boundary value problem was introduced by Barles and Perthame [10, 11] and the third author [22] for first order equations, and by the third author and Lions [25] for second order equations. It arises naturally in the vanishing viscosity method as well as in the Dynamic Programming PDE for optimal control problems involving the exit time of the controlled system from $\Omega$ or $\overline{\Omega}$, see the books [8, 4] for 1st order equations and [14, 9] for 2nd order equations, as well as the references therein. In particular it was proved in [25] that there may exist at most one continuous viscosity solution of $(BVP^*)$. However such a solution does not exist in general, and existence results need either some nondegeneracy of $F$ at boundary points, or some coercitivity in the $p$ variables and some compatibility of the boundary data $g$, see [22, 11, 14, 4] and the recent work of Barles and Burdeau [9] on 2nd order quasilinear equations.

On the other hand in the main examples of $(BVP^*)$ arising in stochastic control theory and deterministic differential games, where the operator $F$ is of the form, respectively,

$$F(x,r,p,X) = \sup_{\alpha}[-trace(A^{\alpha}(x)X) + b^{\alpha}(x)\cdot p + c^{\alpha}(x)r - f^{\alpha}(x)], \tag{0.1}$$

and

$$F(x,r,p) = \sup_{\alpha}\inf_{\beta}[b^{\alpha,\beta}(x)\cdot p + c^{\alpha,\beta}(x)r - f^{\alpha,\beta}(x)], \tag{0.2}$$

$(BVP^*)$ is formally solved by the value function, respectively, of a time-optimal control problem for diffusion processes (we describe it later in this introduction), and of a generalized pursuit-evasion game with open target, and these value functions are discontinuous in general, see [36, 32] and [21, 20, 7, 4]. In these applications $g \equiv 0$ and $f^{\alpha}$, $f^{\alpha,\beta} \ge 0$, so the constant 0 is a subsolution of the PDE $F = 0$ in $I\!\!R^N$.

With these motivations we seek a good notion of generalized solution of $(BVP^*)$ which allows it to be discontinuous. Under the assumption that the boundary data $g$ satisfy $F(x,g,Dg,D^2g) \le 0$ in $I\!\!R^N$, we consider the *maximal subsolution* $u$ of $(BVP^*)$ and prove

(i) that $u$ is the infimum of the supersolutions of the PDE $F = 0$ in some open set $\mathcal{O} \supset \overline{\Omega}$, larger than $g$ on $\partial\Omega$;

(ii) the consistency with the notions of viscosity solution of [23] and [25, 14] and of distributional solution in [36, 32];

(iii) a representation formula as the value function of the corresponding control problem when $F$ is of the form (0.1) and (0.2);

(iv) some approximation results, either by smoothing the set $\overline{\Omega}$ or by a penalization method.
We call this generalized viscosity solution *envelope solution*, briefly e-solution, or Perron-Wiener solution, a name proposed in [4, 2, 1] for a similar notion in the case of the standard Dirichlet problem

$$(DP) \begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

see also [37, 3] for first order equations and applications to pursuit-evasion games. We remark that $(BVP^*)$ and $(DP)$ have the same solution under various types of nondegeneracy conditions, but in general their e-solutions are different. In fact they are both obtained by approximating the domain $\Omega$, but this is done from the outside in the case of $(BVP^*)$, and from the inside in the case of $(DP)$. This is the PDE counterpart of the fact that in control problems $(BVP^*)$ and $(DP)$ are naturally associated to cost functionals involving, respectively, the exit time from $\overline{\Omega}$ and the exit time from $\Omega$.

Let us remark that the results (i), (ii) and (iv) work for the larger class of second order Hamilton-Jacobi-Isaacs operators

$$F(x,r,p,X) = \sup_{\alpha} \inf_{\beta} [-trace(A^{\alpha,\beta}(x)X) + b^{\alpha,\beta}(x) \cdot p + c^{\alpha,\beta}(x)r - f^{\alpha,\beta}(x)]$$

arising in stochastic differential games, see [18], and in general for any operator satisfying the Comparison Principle for $(DP)$, see [26, 27, 24, 25, 14, 13] and Section 1. Moreover the property (i) of the maximal subsolution $u$ holds without the assumption that $F(x,g,Dg,D^2g) \leq 0$, and the minimal supersolution $U$ of $(BVP^*)$ satisfies a symmetric property (it is the supremum of subsolutions on some larger set). If $F(x,g,Dg,D^2g) \geq 0$ then $U$, instead of $u$, coincides with a continuous solution of $(BVP^*)$, if this exists; thus in this case $U$ is the correct generalized solution of $(BVP^*)$. This assumption on $g$ is satisfied in control problems with state-space constraints and we believe the value functions of these problems coincide with $U$, but we do not study this issue here. A related result for infinite dimensional deterministic systems was proved by Kocan and Soravia [28].

Our results hold as well for the more general problem

$$(BVP) \begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \overset{\circ}{K}, \\ u = g \text{ or } F(x,u,Du,D^2u) = 0 & \text{on } \partial K, \end{cases}$$

where $K$ is an arbitrary closed set and $\overset{\circ}{K}$ its interior, a generality motivated again by control theory. Even the limit case $\overset{\circ}{K} = \emptyset$ is covered by our analysis, but the trivial solution $u \equiv g$ does not necessarily coincide with the e-solution.

Next we describe the time-optimal stochastic control problem related to $(BVP)$ in the case $F$ is given by (0.1) and $g = 0$. Consider the controlled stochastic differential equation

$$(SDE) \begin{cases} dX_t = \sigma^{\alpha_t}(X_t)dB_t - b^{\alpha_t}(X_t)dt, & t > 0, \\ X_0 = x, \end{cases}$$

where $t \mapsto \alpha_t$ is the control, $\sigma$ is an $N \times M$ matrix such that $\frac{1}{2}\sigma^{\alpha}(\sigma^{\alpha})^T(x) = A^{\alpha}(x)$, $B_t$ is an M-dimensional Brownian motion, and the cost functional

$$J(x,\alpha_.) := E\left(\int_0^{\hat{t}_x(\alpha_.)} f^{\alpha_t}(X_t)e^{-\int_0^t c^{\alpha_s}(X_s)ds}dt\right),$$

where $E$ denotes the expectation and $\hat{t}_x$ is the first exit time of the trajectory of $(SDE)$ from $K$. The value function of this problem is

$$v(x) := \inf_{\alpha} J(x,\alpha).$$

Under standard Lipschitz regularity assumptions on the data and the condition $f^\alpha \geq 0$ for all $\alpha$, we prove that $v$ is the unique e-solution of $(BVP)$.

In the case $K = \overline{\Omega}$ with smooth boundary $\partial\Omega$, the connection between $v$ and the standard Dirichlet problem $(DP)$ was studied in depth by Stroock and Varadhan [36] in the case of linear $F$, when $(SDE)$ is uncontrolled, and by P.L.Lions [32, 33, 34] in the controlled case. Most of Lions' work is devoted to analyzing the case where $v$ is the unique continuous viscosity solution of $(DP)$, or a solution in a stronger sense [33]; in the case that $v$ is not necessarily continuous he proves it is larger than any continuous subsolution in the distributional sense of all the linear operators appearing in the right hand side of (0.1), with a suitable boundary condition.

Let us recall also that in the deterministic case $\sigma \equiv 0$, and for $K = \overline{\Omega}$, $v$ was characterized as the maximal viscosity subsolution of $(BVP^*)$ by various methods; see [11, 8, 7, 35], where different uniqueness theorems are given, and also the connection with the control problem involving the exit time $t_x$ from $\Omega$ is analyzed. However, the results of the present paper seem to be new even in this special case.

We refer to [30, 14, 17, 31] for more informations on nonlinear degenerate elliptic equations, and to [23, 10, 12, 7, 35], the books [8, 4], and the survey [1] for discontinuous viscosity solution.

The paper is organized as follows. Section 1 collects some known definitions and some preliminary results. In Section 2 we prove the property (i) of the maximal subsolution and the symmetric property of the minimal supersolution of $(BVP)$, we study the consistency with earlier definitions of viscosity solution and the approximation by smoothing the domain. Section 3 deals with the applications to deterministic differential games and to stochastic control. Section 4 is about the approximation of $(BVP)$ with a penalized equation in all $\mathbb{R}^N$ which is applied to give an existence theorem in unbounded domains.

Some results of this paper were part of Goatin's thesis [19], in particular Section 4, they were also announced in [5] jointly with the control applications of Section 3.

# 1 Definitions and preliminary results

We consider the partial differential equation

$$F(x, u(x), Du(x), D^2u(x)) = 0 \quad \text{in}\,\Omega, \tag{1.1}$$

where $\Omega$ is an open subset of $\mathbb{R}^N$, the function $u : \Omega \longrightarrow \mathbb{R}$ is the unknown, $Du$ denotes the gradient of $u$ and $D^2u$ denotes the Hessian matrix of the second order derivatives of $u$. The function $F = F(x, r, p, X)$ is a real-valued function on $I\!R^N \times I\!R \times I\!R^N \times S(N)$, where $S(N)$ is the set of symmetric $N \times N$ matrices equipped with its usual order. Following [14] we will assume throughout the paper that $F$ is continuous and *proper*, that is,

$$F(x, r, p, X) \leq F(x, s, p, Y)$$

for all $x \in I\!R^N$, $r, s \in I\!R$, $p \in I\!R^N$, $X, Y \in S(N)$, such that $r \leq s$, $Y \leq X$. In particular this implies that $F$ is *degenerate elliptic* in the sense defined in the introduction. For $S \subseteq I\!R^N$ we will use the following notations

$$\begin{aligned} USC(S) &= \{\text{upper semicontinuous functions}\, u : S \longrightarrow I\!R\}, \\ LSC(S) &= \{\text{lower semicontinuous functions}\, u : S \longrightarrow I\!R\}, \end{aligned}$$

while with $BUSC(S)$, $BLSC(S)$ we will denote the corresponding subsets of bounded functions.

**Definition 1.1** *We will say that $F$ satisfies the* Comparison Principle *in $\Omega$ if for all subsolutions $w \in BUSC(\overline{\Omega})$ and supersolutions $W \in BLSC(\overline{\Omega})$ such that $w \leq W$ on $\partial\Omega$, we have $w \leq W$ in $\overline{\Omega}$.*

We refer to [14] for general structural assumptions on $F$ which imply the Comparison Principle. If such a result holds then the Dirichlet problem $(DP)$ has at most one bounded continuous solution, and any non-continuous (i.e. not necessarily continuous) solution of $(DP)$ in the generalized viscosity sense of [23] is automatically continuous. Let us recall that if $u^*$ and $u_*$ denote, respectively, the smallest u.s.c. function greater than or equal to $u$ and the largest l.s.c. function less than or equal to $u$, then a non-continuous solution of $(DP)$ is a locally bounded function $u$ such that $u^*$ is subsolution and $u_*$ is supersolution of $(DP)$. By the Comparison Principle $u^* \leq u_*$, so $u$ is continuous.

The main example we have in mind is the class of Hamilton-Jacobi-Bellman equations

$$\sup_\alpha \mathcal{L}^\alpha u = 0,$$

where $\alpha$ is a parameter and, for each $\alpha$, $\mathcal{L}^\alpha$ is a linear nondivergence form operator

$$\mathcal{L}^\alpha u(x) = -\sum_{i,j=1}^N a_{ij}^\alpha(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i^\alpha(x) \frac{\partial u}{\partial x_i} + c^\alpha(x) u - f^\alpha(x). \tag{1.2}$$

More generally we are interested in the upper and lower Isaacs equations

$$\sup_\alpha \inf_\beta \mathcal{L}^{\alpha,\beta} u = 0,$$

$$\inf_\beta \sup_\alpha \mathcal{L}^{\alpha,\beta} u = 0,$$

where $\beta$ is a second parameter and $\mathcal{L}^{\alpha,\beta}$ are linear operators of the form (1.2). The corresponding nonlinear operators $F$ have the form

$$F(x, r, p, X) = \sup_\alpha [-trace(A^\alpha(x) X) + b^\alpha(x) \cdot p + c^\alpha(x) r - f^\alpha(x)] \tag{1.3}$$

for the HJB equation, and

$$F(x,r,p,X) = \sup_{\alpha} \inf_{\beta} [-trace(A^{\alpha,\beta}(x)X) + b^{\alpha,\beta}(x)\cdot p + c^{\alpha,\beta}(x)r - f^{\alpha,\beta}(x)] \tag{1.4}$$

for the upper Isaacs equation (respectively, $F(x,r,p,X) = \inf_\beta \sup_\alpha [\ldots]$ for the lower Isaacs equation). If, for all $x \in I\!R^N$, $A^{\alpha,\beta}(x) = \frac{1}{2}\sigma^{\alpha,\beta}(x)(\sigma^{\alpha,\beta})^T(x)$, where $\sigma^{\alpha,\beta}(x)$ is a $N \times M$ matrix, $^T$ denotes the transpose matrix, $\sigma^{\alpha,\beta}$, $b^{\alpha,\beta}$, $c^{\alpha,\beta}$, $f^{\alpha,\beta}$ are bounded and uniformly continuous, uniformly with respect to $\alpha, \beta$, then $F$ is continuous. Moreover $F$ is degenerate elliptic if and only if $A^{\alpha,\beta}(x) \geq 0$ for all $\alpha, \beta$, and it is proper if also $c^{\alpha,\beta}(x) \geq 0$.

HJB and Isaacs equations satisfy the Comparison Principle if, for instance, there exist $C > 0$ and $c_0 > 0$ such that

$$||\sigma^{\alpha,\beta}(x) - \sigma^{\alpha,\beta}(y)|| \leq C|x-y|, \text{ for all } x, y \in \overline{\Omega} \text{ and all } \alpha, \beta \tag{1.5}$$

$$|b^{\alpha,\beta}(x) - b^{\alpha,\beta}(y)| \leq C|x-y|, \text{ for all } x, y \in \overline{\Omega} \text{ and all } \alpha, \beta, \tag{1.6}$$

$$c^{\alpha,\beta}(x) \geq c_0, \text{ for all } x \in \overline{\Omega}, \text{ and all } \alpha, \beta. \tag{1.7}$$

This result can be found in [24] in the case of unbounded domains, for bounded domains see also [25, 14]. Condition (1.7) can be weakened at points where $F$ is uniformly elliptic: see [27].

The assumptions (1.5)-(1.7) imply that the functions $F$ defined by (1.3), (1.4) satisfy the following conditions: for each $R > 0$ there exists $\gamma_R > 0$ such that for $R \geq r \geq s \geq -R$, $(x,p,X) \in \overline{\Omega} \times I\!R^N \times S(N)$

$$\gamma_R(r-s) \leq F(x,r,p,X) - F(x,s,p,X); \tag{1.8}$$

and

$$F(y,r,\alpha(x-y),Y) - F(x,r,\alpha(x-y),X) \leq \omega_R\Big(\alpha|x-y|^2 + \frac{1}{\alpha}\Big) \tag{1.9}$$

whenever $x, y \in \overline{\Omega}$, $\alpha > 1$, $|r| \leq R$, $X, Y \in S(N)$ satisfy

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{1.10}$$

Moreover, $F$ is continuous on $\overline{\Omega} \times I\!R \times I\!R^N \times S(N)$.

Indeed, if a function $F$ is continuous on $\overline{\Omega} \times I\!R \times I\!R^N \times S(N)$ and satisfies (1.8) and (1.9) and $\Omega$ is bounded, then it satisfies the Comparison Principle in $\Omega$.

Next we give the definition of viscosity solution of the problem

$$(BVP) \begin{cases} F(x,u,Du,D^2u) = 0 & \text{in } \Omega := \overset{\circ}{K}, \\ u = g \text{ or } F(x,u,Du,D^2u) = 0 & \text{on } \partial K, \end{cases}$$

where $F$ is continuous and proper, $K \subseteq I\!R^N$ is a closed set, $g \in C(\partial K)$. This is well known in the special case $K = \overline{\Omega}$, see [10, 22, 25, 14].

**Definition 1.2** *A function* $u \in USC(K)$ *(respectively* $LSC(K)$*) is a* subsolution *(respectively, a* supersolution*) of (BVP) if it is a viscosity subsolution (respectively, supersolution) of* (1.1) *and, for any* $\phi \in C^2(I\!R^N)$ *and* $x \in \partial K$ *such that* $u - \phi|_K$ *has a local maximum (respectively minimum) at* $x$,

$$(u-g)(x) \wedge F(x, u(x), D\phi(x), D^2\phi(x)) \le 0$$

$$(\textit{respectively}\ \ (u-g)(x) \vee F(x, u(x), D\phi(x), D^2\phi(x)) \ge 0);$$

$u \in C(K)$ *is a (viscosity)* solution *of* $(BVP)$ *if it is a sub- and a supersolution.*

It is well known that in general a subsolution of $(BVP)$ is not necessarily smaller than any supersolution of $(BVP)$. The standard comparison theorem for $(BVP)$ (see Weak Comparison Principle Theorem 1.7) requires the continuity of at least one semisolution and some regularity of $\Omega$. Now we present three comparison results that need less assumptions and are useful in Section 2.

**Theorem 1.3** *Let* $K$ *be a compact set and* $\mathcal{O} \supset K$ *a bounded open set. Suppose that* $F$ *satisfies* (1.8) *and* (1.9)*, with* $\Omega$ *replaced by* $\mathcal{O}$*. If* $w \in BUSC(K)$ *is a subsolution (respectively* $W \in BLSC(K)$ *is a supersolution) of* $(BVP)$ *and* $\tilde{W} \in BLSC(\overline{\mathcal{O}})$ *is a supersolution (respectively* $\tilde{w} \in BUSC(\overline{\mathcal{O}})$ *is a subsolution) of* $F = 0$ *in* $\mathcal{O}$*, and* $\tilde{W} \ge g$ *(respectively* $\tilde{w} \le g$*) on* $\partial K$*, then* $w \le \tilde{W}$ *(respectively* $\tilde{w} \le W$*) in* $K$*.*

**Proof.** The proof is an adaptation of standard arguments of usual comparison results that can be found for example in [14]. We sketch here the main idea. We only prove that $w \le \tilde{W}$ in $K$. We assume that

$$\max_K (w - \tilde{W}) > 0,$$

which is a typical assumption leading to a contradiction in the standard proof of comparison. Let $\alpha > 0$ and let $(x_\alpha, y_\alpha) \in K \times \overline{\mathcal{O}}$ be a maximum point of the function

$$w(x) - \tilde{W}(y) - \alpha|x-y|^2$$

on $K \times \overline{\mathcal{O}}$. It is clear that as $\alpha \to \infty$, $|x_\alpha - y_\alpha| \to 0$. By compactness, we may assume that for some sequence $\alpha_j \to \infty$ and point $z \in K$, $x_\alpha \to z$ as $\alpha = \alpha_j$ and $j \to \infty$. Arguing as usual we see that $y_\alpha \to z$, $w(x_\alpha) \to w(z)$, and $\tilde{W}(y_\alpha) \to \tilde{W}(z)$ as $\alpha = \alpha_j$ and $j \to \infty$ and that $w(z) - \tilde{W}(z) = \max_K(w - \tilde{W})$.

Note that $y_\alpha \notin \partial\mathcal{O}$ for $\alpha = \alpha_j$ with $j$ large enough. Note also that if $z \in \partial K$, then $w(z) > \tilde{W}(z) \ge g(z)$ and hence $w(x_{\alpha_j}) > g(x_{\alpha_j})$ for $j$ large enough. From now on we proceed as usual. □

**Theorem 1.4** *Let* $K$ *be a compact set and* $\mathcal{O} \supset K$ *a bounded open set. Suppose that for any continuous extension of* $g$ *to* $\overline{\mathcal{O}}$*, the function*

$$G(x, r, p, X) := \min\{r - g(x),\ F(x, r, p, X)\}$$

*satisfies the Comparison Principle in* $\mathcal{O}$*. Then the same conclusion holds as that of Theorem 1.3.*

**Proof.** We only prove that $w \le \tilde{W}$ in $K$.

Since $\tilde{W} \in BLSC(\overline{\Omega})$ and $\tilde{W} \ge g$ on $\partial K$, we can choose a continuous extension of $g$ to $\overline{\mathcal{O}}$, denoted again by $g$, such that $\tilde{W} \ge g$ in $\overline{\mathcal{O}}$.

Define $G(x, r, p, X)$ as above with the current choice of $g$. Observe that $\tilde{W}$ is a supersolution of $G = 0$ in $\mathcal{O}$ and that if we define the function $v$ on $\overline{\mathcal{O}}$ by

$$v(x) = \begin{cases} \max\{w(x), g(x)\}, & x \in K, \\ g(x), & x \notin K, \end{cases}$$

then $v \in BUSC(\overline{\mathcal{O}})$ and is a subsolution of $G = 0$ in $\mathcal{O}$. Since $v(x) = g(x) \le \tilde{W}(x)$ on $\partial\mathcal{O}$, we conclude by the Comparison Principle that $v \le \tilde{W}$ in $\mathcal{O}$. This shows that $w \le \tilde{W}$ in $K$.
□

**Theorem 1.5** *Let $K$ be a compact set and $\mathcal{O} \supset K$ a bounded open set. Assume that $g$ is defined and continuous on $\overline{\mathcal{O}}$ and is a subsolution of $F = 0$ in $\mathcal{O}$ and that $F$ satisfies the Comparison Principle in every open subset of $\mathcal{O}$ containing $K$. If $w \in BUSC(K)$ is a subsolution of $(BVP)$ and $\tilde{W} \in BLSC(\overline{\mathcal{O}})$ is a supersolution of $F = 0$ in $\mathcal{O}$, and $\tilde{W} \ge g$ on $\partial K$, then we have $w \le \tilde{W}$ in $K$.*

*If we assume instead that $g$ is a supersolution of $F = 0$ in $\mathcal{O}$ and if $W \in BLSC(K)$ is a supersolution of $(BVP)$ and $\tilde{w} \in BUSC(\overline{\mathcal{O}})$ is a subsolution of $F = 0$ in $\mathcal{O}$, and $\tilde{w} \le g$ on $\partial K$, then we have $\tilde{w} \le W$ in $K$.*

**Proof.** Fix $\varepsilon > 0$ and set $V(x) = \tilde{W}(x) + \varepsilon$ for $x \in \overline{\mathcal{O}}$.

Since $V > g$ on $\partial K$ and $V \in LSC(\overline{\mathcal{O}})$, we may assume by replacing $\mathcal{O}$ by a smaller set that $V > g$ on $\partial\mathcal{O}$.

Define $v \in BUSC(\overline{\mathcal{O}})$ by

$$v(x) = \begin{cases} \max\{w(x), g(x)\}, & x \in K, \\ g(x), & x \notin K. \end{cases}$$

It is easy to see that the function $g$ on $K$ is a subsolution of (BVP). Hence, the function $\max\{w(x), g(x)\}$ on $K$ is a subsolution of (BVP). Moreover, we see that $v$ is a subsolution of $F = 0$ in $\mathcal{O}$.

Now, we apply the Comparison Principle in $\mathcal{O}$ to $v$ and $V$, to conclude that $v \le V$ in $\mathcal{O}$, which implies that $w(x) \le \tilde{W}(x) + \varepsilon$. The proof is complete. □

Next we report the standard comparison result for $(BVP)$ that allows to compare semisolutions of $(BVP)$ with semisolutions of $(DP)$ that are continuous at the boundary, in the case $K = \overline{\Omega}$.

**Definition 1.6** *We will say that $F$ satisfies the* Weak Comparison Principle *in $K = \overline{\Omega}$ if for all subsolutions $w \in BUSC(\overline{\Omega})$ (respectively supersolutions $W \in BLSC(\overline{\Omega})$) of (BVP) and all supersolutions $W \in BLSC(\overline{\Omega})$ (respectively subsolutions $w \in BUSC(\overline{\Omega})$) of (DP) and continuous at the boundary, we have $w \le W$ in $\overline{\Omega}$.*

Here we give a list of conditions under which the Weak Comparison Principle holds.

**Theorem 1.7** *Let $\Omega$ be a bounded open set, $\partial\Omega$ a Lipschitz surface, or, more precisely,*

*there is $c > 0$ and $\eta : \overline{\Omega} \longrightarrow I\!R^N$ continuous such that $B(x + t\eta(x), ct) \subseteq \Omega$ for all $x \in \overline{\Omega}$, $0 < t \leq c$.*

*Suppose that $F$ satisfies (1.8) and the following conditions: for each $R > 0$ there exists a modulus of continuity $\omega_R$ and a neighborhood $V$ of $\partial\Omega$ relative to $\overline{\Omega}$ such that*

$$|F(x, r, p, X) - F(x, r, q, Y)| \leq \omega_R(|p - q| + \|X - Y\|) \tag{1.11}$$

*for $x \in V$, $p, q \in I\!R^N$, $X, Y \in S(N)$;*

$$F(y, r, p, Y) - F(x, r, p, X) \leq \omega_R(\alpha|x - y|^2 + |x - y|(|p| + 1)) \tag{1.12}$$

*whenever $x, y \in \overline{\Omega}$, $\alpha \geq 0$, $|r| \leq R$, $X, Y \in S(N)$ such that (1.10) holds.*

*Then $F$ satisfies the Weak Comparison Principle in $\overline{\Omega}$.*

The proof is obtained by combining standard arguments from [13, 14] and [22, 25], see [19] for the details. A Weak Comparison Principle for 1st order equations in unbounded sets can be found in [6].

We end this section with a stability property that will be useful in the sequel.

**Proposition 1.8** *Let $\{w_n\} \subset USC(\overline{\Omega})$ be a sequence of subsolutions (respectively $\{W_n\} \subset LSC(\overline{\Omega})$ a sequence of supersolutions) of (1.1), such that $w_n(x) \searrow u(x)$ for all $x \in \Omega$ (respectively $W_n(x) \nearrow u(x)$) and $u$ is a locally bounded function. Then $u$ is a subsolution (respectively supersolution) of (1.1).*

For the proof see for instance [4, Chapter V, Proposition 2.16].

# 2 Maximal subsolution and minimal supersolution

Let $K$ and $\mathcal{O}$ denote respectively a closed and an open subset of $I\!R^N$. In the following we will assume $g$ to be a continuous function on $\partial K$. We denote by $\mathcal{S}_v^-$, $\mathcal{S}_v^+$ respectively the sets of all subsolutions and supersolutions of $(BVP)$, that are

$$\begin{aligned} \mathcal{S}_v^- &= \{w \in BUSC(K) : w \text{ is subsolution of } (BVP)\}, \\ \mathcal{S}_v^+ &= \{W \in BLSC(K) : W \text{ is supersolution of } (BVP)\}, \end{aligned}$$

and by $\mathcal{S}_e^-$, $\mathcal{S}_e^+$ the sets of all subsolutions, respectively supersolutions of the PDE on some open set $\supset K$ and smaller, respectively larger, than or equal to $g$ on the boundary of $K$, i.e.

$$\begin{aligned} \mathcal{S}_e^- = \; & \{\tilde{w} \in BLSC(\overline{\mathcal{O}}) : \mathcal{O} \text{ is open, } K \subset \mathcal{O}, \\ & \tilde{w} \text{ is subsolution of } F = 0 \text{ in } \mathcal{O},\ \tilde{w} \leq g \text{ on } \partial K\}. \end{aligned}$$

$$\begin{aligned} \mathcal{S}_e^+ = \; & \{\tilde{W} \in BLSC(\overline{\mathcal{O}}) : \mathcal{O} \text{ is open, } K \subset \mathcal{O}, \\ & \tilde{W} \text{ is supersolution of } F = 0 \text{ in } \mathcal{O},\ \tilde{W} \geq g \text{ on } \partial K\}. \end{aligned}$$

Here the subscript $v$ stands for "viscosity" and $e$ for "extended".

Next, our main assumptions are

$(CP^-)$ if $w \in \mathcal{S}_v^-$ and $\tilde{W} \in \mathcal{S}_e^+$ then $w \leq \tilde{W}$ in $K$;

$(CP^+)$ if $\tilde{w} \in \mathcal{S}_e^-$ and $W \in \mathcal{S}_v^+$ then $\tilde{w} \leq W$ in $K$.

See Theorems 1.3-1.5 for some cases where $(CP^-)$ and $(CP^+)$ hold.

Now we can give the main result of this section.

**Theorem 2.1** *(i) Assume that $\mathcal{S}_v^- \neq \emptyset$ and $\mathcal{S}_e^+ \neq \emptyset$. If $(CP^-)$ holds, then*

$$u(x) := \max_{w \in \mathcal{S}_v^-} w(x) = \inf_{\tilde{W} \in \mathcal{S}_e^+} \tilde{W}(x). \tag{2.1}$$

*(ii) Assume that $\mathcal{S}_e^- \neq \emptyset$ and $\mathcal{S}_v^+ \neq \emptyset$. If $(CP^+)$ holds, then*

$$U(x) := \min_{W \in \mathcal{S}_v^+} W(x) = \sup_{\tilde{w} \in \mathcal{S}_e^-} \tilde{w}(x). \tag{2.2}$$

**Proof.** We will give only the proof of $(i)$, the proof of the symmetric case $(ii)$ being completely similar.

By assumption $(CP^-)$, we have

$$\sup_{w \in \mathcal{S}_v^-} w(x) \leq \inf_{\tilde{W} \in \mathcal{S}_e^+} \tilde{W}(x) \quad \text{in } K.$$

To show the reverse inequality, define $V : K \to I\!\!R$ by

$$V(x) = \inf_{\tilde{W} \in \mathcal{S}_e^+} \tilde{W}^*(x).$$

Since $\tilde{W} \leq \tilde{W}^*$, we have

$$\inf_{\tilde{W} \in \mathcal{S}_e^+} \tilde{W}(x) \leq V(x) \quad \text{in } K.$$

So it is enough to prove that $V(x) \leq \sup_{w \in \mathcal{S}_v^-} w(x)$ in $K$. It is clear that $V \in BUSC(K)$. It suffices to show that $V$ is a subsolution of $(BVP)$, which implies that $V \in \mathcal{S}_v^-$ and then $V = u$. We argue by contradiction and hence suppose that $V$ is not a subsolution of $(BVP)$. Then we find a function $\phi \in C^2(K)$ and a point $z \in K$ such that

1. $V - \phi$ has a strict maximum at $z$,
2. $V(z) = \phi(z)$ (and hence, $V(x) < \phi(x)$ for all $x \in K$, $x \neq z$),
3. $F(z, V(z), D\phi(z), D^2\phi(z)) > 0$ if $z \in \overset{\circ}{K}$, and $F(z, V(z), D\phi(z), D^2\phi(z)) > 0$ and $V(z) > g(z)$ if $z \in \partial K$.

We choose $\varepsilon > 0$ and $\delta > 0$ so that

$$F(x, \phi(x) - \varepsilon, D\phi(x), D^2\phi(x)) \geq 0 \quad \text{in } B(z, 2\delta) \cap K,$$

$$V(x) < \phi(x) - \varepsilon \quad \text{in } (B(z, 2\delta) \cap K) \setminus \overset{\circ}{B}(z, \delta).$$

We now treat the case where $z \in \overset{\circ}{K}$ and the case where $z \in \partial K$ separately. Consider first the case where $z \in \overset{\circ}{K}$. We may assume that $B(z, 2\delta) \subset \overset{\circ}{K}$. For each $x \in \partial B(z, \delta)$ we select $\tilde{W}_x \in \mathcal{S}_e^+$ so that

$$\tilde{W}_x^*(x) < \phi(x) - \varepsilon.$$

By semicontinuity, this inequality holds in a neighborhood of $x$. Thus, by the standard compactness argument, we find a finite sequence of points $x_i$, $i = 1, ..., N$, such that

$$\min_{1 \leq i \leq N} \tilde{W}_{x_i}^*(x) < \phi(x) - \varepsilon$$

holds for all $x$ in a neighborhood of $\partial B(z, \delta)$. Let $\mathcal{O}_i$ be the domain of definition of $\tilde{W}_{x_i}$ and set $\mathcal{O} = \cap_{1 \leq i \leq N} \mathcal{O}_i$. Define $U_0 : \mathcal{O} \to I\!R$ by

$$U_0(x) = \begin{cases} \min_{1 \leq i \leq N} \tilde{W}_{x_i}(x), & x \in \mathcal{O} \setminus B(z, \delta) \\ \min\{\phi(x) - \varepsilon, \ \min_{1 \leq i \leq N} \tilde{W}_{x_i}(x)\}, & x \in B(z, \delta) \end{cases}$$

This function $U_0$ is a supersolution of $F = 0$ in $\mathcal{O}$ and, moreover, satisfies $U_0 \geq g$ on $\partial K$. Hence, $U_0 \in \mathcal{S}_e^+$. However, we have

$$V(z) = \phi(z) > \phi(z) - \varepsilon \geq U_0^*(z) \geq V(z).$$

This is a contradiction.

Next consider the case where $z \in \partial K$. For each $x \in K \cap \partial B(z, \delta)$ we select $\tilde{W}_x \in \mathcal{S}_e^+$ so that

$$\tilde{W}_x^*(x) < \phi(x) - \varepsilon.$$

As above, we can select a finite sequence $x_i, ..., x_N$ of points of $K \cap \partial B(z, \delta)$ so that

$$\min_{1 \leq i \leq N} \tilde{W}_{x_i}^*(x) < \phi(x) - \varepsilon$$

in an open neighborhood $\mathcal{N}$ of $K \cap \partial B(z, \delta)$. Choose an open neighborhood $\mathcal{M}$ of $\partial B(z, \delta) \setminus \mathcal{N}$ so that $\overline{\mathcal{M}} \cap K = \emptyset$. Define $\mathcal{O}$ as before and set $\mathcal{O}_0 = \mathcal{O} \setminus \overline{\mathcal{M}}$ and define $U_0 : \mathcal{O}_0 \to I\!R$ by

$$U_0(x) = \begin{cases} \min_{1 \leq i \leq N} \tilde{W}_{x_i}(x), & x \in \mathcal{O}_0 \setminus B(z, \delta), \\ \min\{\phi(x) - \varepsilon, \ \min_{1 \leq i \leq N} \tilde{W}_{x_i}(x)\}, & x \in B(z, \delta). \end{cases}$$

This function $U_0$ is a supersolution of $F = 0$ in $\mathcal{O}_0$ and, moreover, we may assume that $U_0 \geq g$ on $\partial K$. Hence, $U_0 \in \mathcal{S}_e^+$. However, we have

$$V(z) = \phi(z) > \phi(z) - \varepsilon \geq U_0^*(z) \geq V(z).$$

This is a contradiction. The proof is now complete. □

Theorem 2.1 provides two candidates as generalized viscosity solutions of $(BVP)$. Next we study their consistency properties. We consider first the notion of non-continuous viscosity solution (see [23]), namely, $u$ is a solution of $(BVP)$ if $u^*$ is subsolution and $u_*$ is supersolution.

**Corollary 2.2** *Under the hypotheses of Theorem 2.1, the maximal subsolution $u$ and the minimal supersolution $U$ are non-continuous solutions of $(BVP)$.*

**Proof.** We observe that $u = u^*$ is a subsolution of $(BVP)$ by Theorem 2.1. On the other side, since $u = \inf_{\tilde{W} \in \mathcal{S}_e^+} \tilde{W}$, $u_*$ is a supersolution of (1.1) in $\overset{\circ}{K}$, and $\tilde{W} \geq g$ on $\partial K$ implies $u_* \geq g_* = g$.

The proof for $U$ is the same. □

Now we consider the case that $(BVP)$ has a continuous solution $v$, and we wonder if either $u$ or $U$ coincides with $v$. We can answer this question in the following two cases

**(H0)** $g \in C(\mathcal{O}')$ is a bounded subsolution of $F = 0$ in $\mathcal{O}'$, for some open set $\mathcal{O}' \supset K$.

**(H1)** $g \in C(\mathcal{O}')$ is a bounded supersolution of $F = 0$ in $\mathcal{O}'$, for some open set $\mathcal{O}' \supset K$.

**Theorem 2.3** *(**Consistency**) Let $K = \overline{\Omega}$, $\Omega$ open. Assume there exists $v \in C(\overline{\Omega})$ solution of $(BVP)$ and $F$ satisfies the Weak Comparison Principle in $\overline{\Omega}$.*

*(i) If $(CP^-)$ and $(H0)$ hold, then $u = v$.*

*(ii) If $(CP^+)$ and $(H1)$ hold, then $U = v$.*

**Proof.** Let us consider case $(i)$, case $(ii)$ being again similar. Observe that $v \geq g$ in $\overline{\Omega}$ by standard comparison between continuous solutions of $(BVP)$. Therefore $v \geq w$ for all $w \in \mathcal{S}_v^-$ by the Weak Comparison Principle, so $v$ is the maximal subsolution of $(BVP)$. □

From now on we study the maximal subsolution $u$ under the assumptions $(CP^-)$ and $(H0)$. In view of the previous results we consider $u$ as the correct generalized solution of $(BVP)$ in this case, and we call it the *envelope solution*, briefly *e-solution*, or the Perron-Wiener solution of $(BVP)$, by analogy with a similar generalized solution of the Dirichlet problem studied in [4, 2, 1].

We observe here that the e-solution of $(BVP)$ depends only on the restriction of the operator $F$ to $K \times \mathbb{R} \times \mathbb{R}^N \times S(N)$ in the following sense: if two different operators coincide on this set and both satisfy the assumptions of Theorem 2.1, then the e-solution of $(BVP)$ is the same, because it is characterized as the maximal element of $\mathcal{S}_v^-$.

**Example 2.4** The Isaacs equations $\sup_\alpha \inf_\beta \mathcal{L}^{\alpha,\beta} u = 0$ and $\inf_\beta \sup_\alpha \mathcal{L}^{\alpha,\beta} u = 0$, where $\mathcal{L}^{\alpha,\beta}$ are linear operators of the form (1.2), fit into the assumptions of Theorem 2.1 if their coefficients satisfy (1.5)-(1.7). If $f^{\alpha,\beta} \geq 0$ for all $\alpha, \beta$, then $g \equiv 0$ is a subsolution in $\mathbb{R}^N$ and Theorem 2.3 (i) applies for any compact set $K$ with Lipschitz boundary. These conditions of nonnegative running cost and null terminal cost are satisfied in time-optimal control and differential games. A result for $(BVP)$ associated to the Isaacs equation and unbounded $K$ is given in Section 4.

A similar characterization of the e-solution can be given considering the set

$$\begin{aligned} \mathcal{Z}_e &= \{W \in BLSC(\overline{\mathcal{O}}) : \mathcal{O} \text{ is open, } K \subset \mathcal{O} \subseteq \mathcal{O}', \\ &\quad W \text{ is supersolution of } F = 0 \text{ in } \mathcal{O}, \, W \geq g \text{ on } \partial\mathcal{O}\}, \end{aligned}$$

instead of $\mathcal{S}_e^+$. Then we have the following

**Theorem 2.5** *Assume (H0), $\mathcal{Z}_e \neq \emptyset$ and that $F$ satisfies the Comparison Principle on every open set $\mathcal{O}$, $K \subset \mathcal{O} \subseteq \mathcal{O}'$. Then*

$$u(x) := \max_{w \in \mathcal{S}_v^-} w(x) = \inf_{W \in \mathcal{Z}_e} W(x).$$

The proof of this theorem parallels that of Theorem 2.1, (i) and we leave it to the reader to check the details. Note that under the hypotheses (H0) and the Comparison Principle we have $W \geq g$ in $\mathcal{O}$ for $W \in \mathcal{Z}_e$, i.e., $\mathcal{S}_e^+ \supseteq \mathcal{Z}_e$.

The e-solution can also be characterized as the infimum of supersolutions of $(BVP)$, instead of $(DP)$, in larger sets. We consider

$$\begin{aligned} \tilde{\mathcal{Z}}_e &= \{W \in BLSC(\overline{\mathcal{O}}) : \; \mathcal{O} \text{ is open, } K \subset \mathcal{O} \subseteq \mathcal{O}', \\ &\quad W \text{ is supersolution of } F = 0 \text{ in } \mathcal{O} \text{ and of } W = g \text{ or } F = 0 \text{ on } \partial\mathcal{O}\}. \end{aligned}$$

We see that $\mathcal{S}_e^+ \supseteq \tilde{\mathcal{Z}}_e \supseteq \mathcal{Z}_e$ under the hypotheses $(H0)$ and $(CP^+)$ with $K$ and $\mathcal{O}$ replaced by $\mathcal{O}$ and $\mathcal{O}'$ respectively. Therefore, under the assumptions of Theorem 2.5 the e-solution of $(BVP)$ satisfies also

$$u(x) = \inf_{W \in \tilde{\mathcal{Z}}_e} W(x) \text{ for all } \; x \in K.$$

We end this section with two results on the approximation of the e-solution.

**Theorem 2.6** *Assume (H0) and that $F$ satisfies the Comparison Principle in every open $\mathcal{O} \subset \mathcal{O}'$. Suppose there exist a nonincreasing sequence of open sets $\mathcal{O}_n \subseteq \mathcal{O}'$ and a sequence of functions $v_n \in C(\overline{\mathcal{O}})$ such that $\bigcap_n \mathcal{O}_n = K$ and $v_n$ is a solution of $F = 0$ in $\mathcal{O}_n$ and $v_n = g$ on $\partial\mathcal{O}_n$. Then $v_n(x) \searrow u(x)$ for all $x \in K$, where $u$ is the e-solution of $(BVP)$.*

**Proof.** By Comparison Principle between a subsolution $g$ and a supersolution $v_n$ which satisfy $g \leq v_n$ on $\partial\mathcal{O}_n$,we have $v_n \geq g$ on $\overline{\mathcal{O}}_n$, in particular $v_n \geq g$ on $\partial K$, so $v_n \in \mathcal{S}_e^+$. Then by Theorem 1.5 we have $u(x) \leq \inf_n v_n(x)$.

Next we prove that $\{v_n\}$ is a nonincreasing sequence, and that it converges to a subsolution of $(BVP)$, so that the theorem is proved. To this end, we extend $v_{n+1}$ equal to $g$ in $\mathcal{O}_n \setminus \mathcal{O}_{n+1}$, call it $\overline{v}_{n+1}$. We claim that $\overline{v}_{n+1}$ is a subsolution of $(DP)$ in $\mathcal{O}_n$ which takes up continuously the boundary data $g$. Then $v_n \geq \overline{v}_{n+1}$ by comparison, so in particular $v_n \geq v_{n+1}$ in $\mathcal{O}_{n+1}$. To prove the claim we note that the PDE is satisfied in $\mathcal{O}_{n+1}$ and in $\mathcal{O}_n \setminus \overline{\mathcal{O}}_{n+1}$. We have only to check on $\partial\mathcal{O}_{n+1}$, so let $x \in \partial\mathcal{O}_{n+1}$, $\phi \in C^2(I\!R^N)$ such that $x$ is a maximum point of $\overline{v}_{n+1} - \phi$ in $B(x, r)$ for some $r > 0$. Note that $v_{n+1}(x) = g(x)$, then $x$ is a maximum point of $g - \phi$ in $B(x, r)$ and we know that $g$ is subsolution of $F = 0$, so the proof of the claim is complete.

Next we extend $v_n$ equal to $g$ in $\mathcal{O}_1 \setminus \mathcal{O}_n$, and we define

$$V(x) = \inf_n v_n(x). \tag{2.3}$$

Note that, by Proposition 1.8, $V$ is subsolution of $F = 0$ in $\overset{\circ}{K}$. To check the boundary condition, let $x \in \partial K$, $\phi \in C^2(I\!R^N)$ such that $x$ is a maximum point of $V - \phi$ in $B(x,r) \cap K$, $r > 0$, and $V(x) > g(x)$. Then $x$ is a maximum point of $V - \phi$ in $B(x,R) \subseteq \mathcal{O}_1$ for some $R > 0$, because $V = g$ in $I\!R^N \setminus K$ and $g$ is continuous, so

$$F(x, V(x), D\phi(x), D^2\phi(x)) \leq 0.$$

□

**Theorem 2.7** *Under the assumptions of Theorem 2.6, let $\mathcal{O}_n$ be any nonincreasing sequence of open sets with $C^2$ boundary such that $\mathcal{O}_n \subseteq \mathcal{O}'$, $\bigcap_n \mathcal{O}_n = K$. Then for all $n$ there exists a unique $v_n \in C(\overline{\mathcal{O}}_n)$ which solves*

$$\begin{cases} -dist(x,K)^2 \Delta v + F(x, v, Dv, D^2 v) = 0 & \text{in } \mathcal{O}_n, \\ v = g & \text{on } \partial \mathcal{O}_n, \end{cases} \tag{2.4}$$

*and $v_n(x) \searrow u(x)$ for all $x \in K$ as $n \to +\infty$, where $u$ is the e-solution of $(BVP)$.*

**Proof.** The existence of a nonincreasing sequence of smooth open sets whose intersections is a given closed set is well-known. The PDE appearing in (2.4) satisfies the Comparison Principle, so the Dirichlet problem (2.4) has at most one continuous solution. The existence of such a solution follows from standard methods in viscosity theory [13, 14]: since the PDE is nondegenerate near $\partial \mathcal{O}_n$ and $\mathcal{O}_n$ satisfies an exterior sphere condition one can build a supersolution of (2.4) attaining the boundary data, then $v_n$ is found by Perron's method. By Theorem 2.6, $v_n$ converge to the e-solution $\tilde{u}$ of $(BVP)$ for the operator $\tilde{F} := -dist^2(\cdot, K)\Delta +$ $F$. Since $\tilde{F} = F$ on $K \times I\!R \times I\!R^N \times S(N)$, $\tilde{u} = u$ as we remarked earlier. □

# 3 Applications to differential games and stochastic control

## 3.1 A pursuit-evasion game

In this section we consider a two-players zero-sum deterministic differential game, see [21, 20, 15, 16, 37]. We are given a controlled dynamical system

$$\begin{cases} y' = f(y(t), a(t), b(t)), & t > 0 \\ y(0) = x, \end{cases} \tag{3.1}$$

where $f : I\!R^N \times A \times B \to I\!R^N$ is continuous, $A$, $B$ are compact metric spaces and $a = a(\cdot) \in \mathcal{A} := \{\text{measurable functions } [0, +\infty) \to A\}$ is the control function of the first player. For the second player we will use relaxed controls $b = b(\cdot) \in \mathcal{B}^r := \{\text{measurable functions } [0, +\infty) \to B^r\}$ where $B^r$ is the set of Radon probability measures on $B$. For the definitions of relaxed trajectories of (3.1) we refer for instance to [38],[4]. Throughout this section we will always assume that the system satisfies, for some constant $L$,

$$(f(x,a,b) - f(y,a,b)) \cdot (x-y) \le L|x-y|^2 \tag{3.2}$$

for all $x, y \in I\!R^N$, $a \in A$, $b \in B$.

The cost functional, which the first player wants to minimize and the second player wants to maximize, is

$$J(x,a,b) = \int_0^{\hat{t}_x(a,b)} e^{-t} dt = 1 - e^{-\hat{t}_x(a,b)},$$

where $\hat{t}_x(a,b)$ is the first exit time from a given closed set $K \subseteq I\!R^N$, i.e. $\hat{t}_x(a,b) := \inf\{t \in [0,+\infty) : y_x(t,a,b) \notin K\}$, $y_x(t,a,b)$ being the solution of (3.1) corresponding to $a \in \mathcal{A}$, $b \in \mathcal{B}^r$. Note that $\hat{t}_x$ is the time taken by the system to reach the *open target* $I\!R^N \setminus K$, and that $J$ is a bounded and increasing rescaling of $\hat{t}_x$.

A relaxed strategy for the second player is a map $\beta : \mathcal{A} \to \mathcal{B}^r$; it is nonanticipating if, for any $t > 0$ and $a, \tilde{a} \in \mathcal{A}$, $a(s) = \tilde{a}(s)$ for all $s \le t$ implies $\beta[a](s) = \beta[\tilde{a}](s)$ for all $s \le t$, see [15, 16]. We will denote with $\Delta^r$ the set of nonanticipating relaxed strategies for the second player.

Now we can define the upper value of this differential game, which is

$$\hat{u}(x) = \sup_{\beta \in \Delta^r} \inf_{a \in \mathcal{A}} J(x,a,\beta[a]).$$

It is well known by the Dynamic Programming Principle that the upper value is a viscosity solution of the upper Hamilton-Jacobi-Isaacs (briefly HJI) equation

$$u(x) + \tilde{H}(x, Du(x)) := u(x) + \max_{a \in A} \min_{b \in B^r} \{-f(x,a,b) \cdot Du(x) - 1\} = 0 \text{ in } \overset{\circ}{K}. \tag{3.3}$$

We are going to prove, by means of the results of the previous section, that the upper value $\hat{u}(x)$ is the e-solution of

$$(BV\tilde{P_0}) \begin{cases} u + \tilde{H}(x, Du) = 0 & \text{in } \overset{\circ}{K}, \\ u = 0 \text{ or } u + \tilde{H}(x, Du) = 0 & \text{on } \partial K. \end{cases} \tag{3.4}$$

The idea is to approximate from above $\hat{u}(x)$ with the upper values on larger sets. Note that $g \equiv 0$ is a subsolution of the HJI equation in $I\!R^N$.

**Theorem 3.1** *Assume* (3.2). *Then the upper value of the minimum time problem*

$$\hat{u}(x) = \sup_{\beta \in \Delta^r} \inf_{a \in \mathcal{A}} \int_0^{\hat{t}_x(a,\beta[a])} e^{-t} dt$$

*is the maximal subsolution of* (3.4) *and therefore the e-solution of* $(BV\tilde{P_0})$.

**Proof.** Since $\hat{u}(x) = 1 - e^{-\hat{T}(x)}$, where $\hat{T}(x) = \sup_{\beta \in \Delta^r} \inf_{a \in \mathcal{A}} \hat{t}_x(a, \beta[a])$, in the proof we will refer to the upper minimum time function $\hat{T}(\cdot)$, and then apply the transformation. We may assume without loss of generality $\hat{T}(x) < +\infty$. It is well known that $\hat{T}^* \in \mathcal{S}_v^-$. We want to build a sequence of supersolutions on larger sets converging to $\hat{T}$ from above. We consider the sequence $\Omega_h = \{x \in I\!R^N : dist(x, K) < h\}$ of open sets converging to $K$ from outside, and let $T_h$ be the corresponding minimum time function, i.e.

$$T_h(x) = \sup_{\beta \in \Delta^r} \inf_{a \in \mathcal{A}} t_x^h(a, \beta[a]),$$

where $t_x^h(a, \beta[a]) := \inf\{t \in [0, +\infty) : y_x(t, a, \beta[a]) \notin \Omega_h\}$. Then $T_h(x) \geq \hat{T}(x)$ and $(T_h)_*$ is supersolution of (3.3) in $\Omega_h$ and $(T_h)_* \geq 0$ on $\partial K$. For each $h$ there exists $\beta_h \in \Delta^r$ such that $T_h(x) \leq \inf_{a \in \mathcal{A}} t_x^h(a, \beta_h[a]) + h$. By compactness of the set of relaxed strategies (see [15, 20, 38]), there exists $\overline{\beta} \in \Delta^r$ such that for each $a \in \mathcal{A}$ there exists $h_n = h_{n(a)} \searrow 0$ such that

$$\beta_{h_n}[a] \rightharpoonup \overline{\beta}[a] \text{ in the weak star topology of } B^r,$$

$$y_x(\cdot; a, \beta_{h_n}[a]) \to y_x(\cdot; a, \overline{\beta}[a]) \text{ uniformly in } [0, \tilde{T}], \text{ for any } \tilde{T}.$$

We now keep $a$ fixed and call $\beta_n = \beta_{h_n}$, $\tau_n(a) = t_x^{h_n}(a, \beta_{h_n}[a])$, so that $y_x(\tau_n(a); a, \beta_n[a]) \in \partial\Omega_{h_n}$ and

$$\tau_n(a) \geq T_{h_n}(x) - h_n. \tag{3.5}$$

If the sequence $\{\tau_n\}$ is unbounded, so that $\tau_n \to +\infty$ up to a subsequence, then $\hat{t}_x(a, \overline{\beta}[a]) = +\infty$. In fact, if we assume by contradiction $\hat{t}_x(a, \overline{\beta}[a]) = \theta < +\infty$, $y_x(\overline{\theta}; a, \overline{\beta}[a]) \notin \Omega_{\overline{h}}$ for some $\overline{\theta} > \theta$ and some $\overline{h}$ small enough. Then, by uniform convergence of the trajectories, $\tau_n < \overline{\theta}$ for $n$ large enough. Now we suppose $\{\tau_n\}$ bounded, $\tau_n(a) \to \overline{\tau}(a)$ up to a subsequence. By uniform convergence $y_x(\overline{\tau}(a); a, \overline{\beta}[a]) \in \partial K$. We claim that $\overline{\tau}(a) \leq \hat{t}_x(a, \overline{\beta}[a])$. Assume by contradiction $\overline{\tau}(a) > \hat{t}_x(a, \overline{\beta}[a])$, so that there exists $\theta < \overline{\tau}$ such that $y_x(\theta; a, \overline{\beta}[a]) \notin K$. If $\overline{h} = dist(y_x(\theta), K)$, then by uniform convergence of the trajectories $y_x(\theta; a, \beta_{h_n}[a]) \notin \Omega_{\overline{h}/2}$ for $n$ large enough, a contradiction to the fact that $\tau_n > \theta$ for $n$ large. To conclude, if $\hat{t}_x(a, \overline{\beta}[a]) = +\infty$ for every $a \in \mathcal{A}$, then $\hat{T}(x) = +\infty$, otherwise call $\mathcal{A}' := \{a \in \mathcal{A} : \hat{t}_x(a, \overline{\beta}[a]) < +\infty\}$. Then

$$\hat{T}(x) \geq \inf\{\hat{t}_x(a, \overline{\beta}[a]) : a \in \mathcal{A}'\}$$

and we have just proved that $\overline{\tau}(a) \leq \hat{t}_x(a, \overline{\beta}[a])$. So, by inequality (3.5), we have

$$\hat{T}(x) \geq \inf_{a \in \mathcal{A}'} \hat{t}_x(a, \overline{\beta}[a]) \geq \inf_{a \in \mathcal{A}'} \overline{\tau}(a) \geq T_{h_n}(x) - h_n,$$

and, passing to the limit as $n \to +\infty$, we obtain

$$\hat{T}(x) = \inf_n T_{h_n}(x).$$

Now the statement is an easy consequence of Theorem 2.1. □

## 3.2 Time-optimal control of diffusion processes

In this section we study a stochastic optimal control problem following [32, 34], see also [29, 17]. We consider a probability space $(\Omega', \mathcal{F}, P)$ with a right-continuous increasing filtration of complete sub-$\sigma$ fields $\{\mathcal{F}_t\}$, a Brownian motion $B_t$ in $I\!R^M$ $\mathcal{F}_t$-adapted, a compact set $A$, and call $\mathcal{A}$ the set of progressively measurable processes $\alpha_t$ taking values in $A$. We are given bounded maps $\sigma$ from $I\!R^N \times A$ into the set of $N \times M$ matrices and $b : I\!R^N \times A \to I\!R^N$ satisfying (1.5),(1.6) and consider the controlled stochastic differential equation

$$(SDE) \begin{cases} dX_t = \sigma^{\alpha_t}(X_t)dB_t - b^{\alpha_t}(X_t)dt, & t > 0, \\ X_0 = x. \end{cases}$$

This has a pathwise unique solution $X_t$ which is $\mathcal{F}_t$-progressively measurable and has continuous sample paths. We are given also two bounded and uniformly continuous maps $f, c : I\!R^N \times A \to I\!R$, $c^a(x) \geq c_0 > 0$ for all $x, a$, and consider the cost functional

$$J(x, \alpha_.) := E\left(\int_0^{\hat{t}_x(\alpha_.)} f^{\alpha_t}(X_t)e^{-\int_0^t c^{\alpha_s}(X_s)ds}dt\right),$$

where $E$ denotes the expectation and

$$\hat{t}_x(\alpha_.) := \inf\{t \geq 0 : X_t \notin K\}$$

for a given compact set $K$ (of course $\hat{t}_x(\alpha_.) = +\infty$ if $X_t \in K$ for all $t \geq 0$). We define the value function

$$v(x) := \inf_{\alpha_. \in \mathcal{A}} J(x, \alpha_.),$$

and consider the Bellman equation

$$F(x, u, Du, D^2u) := \max_{\alpha \in A}\{-a^{\alpha}_{ij}(x)u_{x_ix_j} + b^{\alpha}(x) \cdot Du + c^{\alpha}(x)u - f^{\alpha}(x)\} = 0$$

where the matrix $(a_{ij})$ is $\frac{1}{2}\sigma\sigma^T$. We consider the boundary value problem

$$(BVP_0) \begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \overset{\circ}{K}, \\ u = 0 \text{ or } F(x, u, Du, D^2u) = 0 & \text{on } \partial K, \end{cases}$$

under the additional assumption

$$f^{\alpha}(x) \geq 0 \;\text{ for all } x \in I\!R^N,\ \alpha \in A. \tag{3.6}$$

Note that in this case $g \equiv 0$ is a subsolution of $F = 0$ in $I\!R^N$.

**Theorem 3.2** *Under the previous assumptions the value function $v$ is the unique e-solution of $(BVP_0)$.*

**Proof.** We consider a Brownian motion $\tilde{B}_t$ in $I\!R^N$ $\mathcal{F}_t$-adapted and the stochastic differential equation

$$(SDE') \begin{cases} dX_t = \sigma^{\alpha_t}(X_t)dB_t - b^{\alpha_t}(X_t)dt + \sqrt{2}dist(x, K)d\tilde{B}_t, & t > 0, \\ X_0 = x. \end{cases}$$

Note that the solutions of $(SDE')$ and $(SDE)$ are the same as long as they remain in $K$, and it is easy to see that the exit time from $K$, $\hat{t}_x$, is also the same for $(SDE)$ and $(SDE')$. We take a decreasing sequence $\{\mathcal{O}_n\}$ of bounded open sets with $C^2$ boundary such that $\bigcap_n \mathcal{O}_n = K$ and define the cost functionals

$$J_n(x, \alpha_.) := E\left(\int_0^{t_x^{(n)}} f^{\alpha_t}(X_t) e^{-\int_0^t c^{\alpha_s}(X_s) ds} dt\right),$$

where $t_x^{(n)}(\alpha_.) := \inf\{t \geq 0 : X_t \notin \overline{\mathcal{O}}_n\}$ and $X_t$ is the solution of $(SDE')$. By the results of P.L. Lions [32] the value function

$$v_n(x) := \inf_{\alpha_.} J_n(x, \alpha_.)$$

is continuous in $\overline{\mathcal{O}}_n$, $v_n = 0$ on $\partial\mathcal{O}_n$, and it is the unique solution of

$$-dist^2(x, K)\Delta w + F(x, w, Dw, D^2 w) = 0 \text{ in } \mathcal{O}_n$$

taking up the null boundary data continuously. Therefore, by Theorem 2.7, $v_n(x) \searrow u(x)$ for all $x \in K$ as $n \to \infty$, where $u$ is the e-solution of $(BVP_0)$.

It remains to prove that $u = v$ in $K$. We fix $\alpha_.$ such that $\hat{t}_x(\alpha_.) < +\infty$, and observe that $t_x^{(n)} \searrow \bar{t}$ implies $X_{t_x^{(n)}} \to X_{\bar{t}} \in \partial K$, thus $t_x^{(n)} \searrow \hat{t}_x$ as $n \to \infty$. Therefore, by (3.6),

$$J_n(x, \alpha_.) \searrow J(x, \alpha_.) \text{ as } n \to \infty.$$

Then, for all $x \in K$,

$$u(x) = \inf_n \inf_{\alpha_.} J_n(x, \alpha_.) = \inf_{\alpha_.} \inf_n J_n(x, \alpha_.) = v(x),$$

and the proof is complete. □

**Remark 3.3** In the case $K = \overline{\Omega}$ with smooth boundary the function $v$ was studied in detail by Stroock and Varadhan [36] in the uncontrolled case and by P.L. Lions [32] in general. They characterized it as the solution, or maximal subsolution, of $(DP)$ in a suitable weak sense based on the theory of distributions. As a consequence of Theorem 3.2, the e-solution solves $(DP)$ in that sense as well. In the case $v$ is continuous it is also characterized in [32, 34] as the unique solution of the Bellman equation which is null on the "usable part" of the boundary.

# 4 Approximation by penalization

In this section we characterize the e-solution of $(BVP)$ as the limit of a sequence of solutions of equations in all $\mathbb{R}^N$ involving a term which penalizes the distance between the unknown $u$ and the boundary data $g$ of $(BVP)$. This was originally motivated by a similar result in [36] for linear equations. Here we apply the penalization theorem to prove an existence result for $(BVP)$ in unbounded sets. At the end of the section we give a control theoretic interpretation of the penalization procedure.

Throughout this section we assume that

$$g \text{ is a bounded and continuous subsolution of } F = 0 \text{ in } I\!\!R^N$$

and $\zeta \in C(I\!\!R^N)$ is a function such that

$$\begin{cases} \zeta(x) = 0 & \text{in } K, \\ 0 < \zeta(x) \leq 1 & \text{in } I\!\!R^N \setminus K. \end{cases}$$

We can take for instance

$$\zeta(x) = \begin{cases} 0 & \text{if } x \in K \\ cd(x) & \text{if } d(x) \leq 1/c,\ c > 0 \\ 1 & \text{if } d(x) > 1/c \end{cases}$$

where $d(x) = dist(x, K)$. For each $n \in I\!\!N$, we consider the equation

$$n\zeta(x)(u - g) + F(x, u, Du, D^2u) = 0 \text{ in } I\!\!R^N \tag{4.1}$$

and we assume that there exists a continuous solution $v_n \geq g$ such that

$$\sup_{\mathcal{K}} |v_n(x)| \leq C_{\mathcal{K}} \text{ for each compactum } \mathcal{K} \subseteq I\!\!R^N, \text{ for all } n \tag{4.2}$$

(we refer to [24, 13, 14] for results ensuring the existence of such $v_n$, see also the examples at the end of this section).

Now we give the main result of this section which asserts that the sequence $\{v_n\}$ converges to the e-solution of $(BVP)$.

**Theorem 4.1** *Assume the equation* (4.1) *satisfies the Comparison Principle and has a continuous solution $v_n$ for all $n$ verifying* (4.2). *Moreover suppose $F$ satisfies the Comparison Principle for every open set $\mathcal{O} \supseteq K$. Then*

$$v_n(x) \searrow u(x) := \max_{w \in \mathcal{S}_v^-} w(x) = \inf_{\tilde{W} \in \mathcal{S}_e^+} \tilde{W}(x), \quad x \in K. \tag{4.3}$$

It is not hard to show that $v_n \to g$ uniformly on compacta of $K^c$ under the assumptions of the preceding theorem.

**Proof.** We first claim that

$$v_n(x) \geq \sup_{w \in \mathcal{S}_v^-} w(x) \text{ for all } x \in K.$$

We can assume without loss of generality $w \geq g$ and we can extend $w$ equal to $g$ in $K^c$, obtaining a subsolution of (4.1). In fact this is obvious in $\overset{\circ}{K}$ and $K^c$. To check it on $\partial K$, take $\phi \in C^2(I\!\!R^N)$ and $x \in \partial K$ local maximum point of $w - \phi$. If $w(x) > g(x)$ we can conclude since $w \in \mathcal{S}_v^-$. If $w(x) = g(x)$, then $x$ is a maximum point of $g - \phi$ as well, and we obtain the claim because $g$ is a subsolution of $F = 0$. Since $v_n$ is in particular supersolution of (4.1), the Comparison Principle yields $v_n \geq w$ in $K$.

Next we show that

$$\overline{v} := \limsup_n {}^* v_n \in \mathcal{S}_v^- .$$

By a well-known stability property of viscosity solutions [10, 14], $\overline{v}$ is subsolution of $F = 0$ in $\overset{\circ}{K}$. We need only to check the boundary condition. Let $x \in \partial K$ and $\phi \in C^2(I\!\!R^N)$ such that $\overline{v} - \phi$ attains a strict maximum point in $B(x, r) \cap K$ at $x$ for some $r > 0$, and $\overline{v}(x) > g(x)$. We may assume that $\overline{v}(x) = \phi(x)$. Moreover, we may assume by adding a smooth nonnegative function vanishing near the point $x$ to $\phi$ that $\max_{\partial B(x,r)}(\overline{v} - \phi) < (\overline{v} - \phi)(x)$ and that $\phi > g$ on $B(x, r)$. The first of these assumptions guarantees that there is a point $y \in \overset{\circ}{B}(x, r)$ where $(\overline{v} - \phi)|_{B(x,r)}$ attains its maximu at $y$. Note that $\overline{v}(y) \geq \phi(y) > g(y)$. Again, by Lemma 6.1 in [14], there is a sequence of points $x_k$ such that $(v_{n_k} - \phi)|_{B(x,r)}$ attains a local maximum at $x_k$ and $x_k \to y$, $v_{n_k}(x_k) \to \overline{v}(y)$. We claim that $y \in K$. Indeed, we have

$$n_k \zeta(x_k)(v_{n_k} - g)(x_k) + F(x_k, v_{n_k}(x_k), D\phi(x_k), D^2\phi(x_k)) \leq 0 \quad \text{for all } k. \tag{4.4}$$

Letting $k \to \infty$, we see that $\zeta(y)(\overline{v} - g)(y) \leq 0$. Therefore, since $\overline{v}(y) > g(y)$, we see that $y \in K$. It is now clear that $y = x$. Since $v_{n_k}(x_k) > g(x_k)$ for $k$ large enough, we conclude by letting $k \to \infty$ in (4.4) that

$$F(x, \overline{v}(x), D\phi(x), D^2\phi(x)) \leq 0. \tag{4.5}$$

This shows that $\overline{v} \in \mathcal{S}_v^-$. Therefore

$$\overline{v}(x) = \max_{w \in \mathcal{S}_v^-} w(x).$$

Moreover, $v_n$ is supersolution of (4.1) with $n$ replaced by $n + 1$, so by the Comparison Principle $v_n \geq v_{n+1}$. Then it is not hard to check that

$$\overline{v}(x) = \inf_n v_n(x) = \lim_n v_n(x),$$

see, e.g., [4, Chapter V, Lemma 2.18]. □

Next we apply the penalization theorem to prove the existence and uniqueness of the e-solution of $(BVP)$ in any unbounded $K$ for the Isaacs operator (1.4).

**Corollary 4.2** *Assume $A^{\alpha,\beta} = \frac{1}{2}\sigma^{\alpha,\beta}(\sigma^{\alpha,\beta})^T$ for some $N \times M$ matrix valued functions $\sigma^{\alpha,\beta}$, and $\sigma^{\alpha,\beta}, b^{\alpha,\beta}, c^{\alpha,\beta}, f^{\alpha,\beta}$ be uniformly bounded with respect to $\alpha, \beta$ and satisfy* (1.5), (1.6),(1.7) *with $c_0 > 0$. Suppose also $f^{\alpha,\beta}$ are uniformly continuous, uniformly in $\alpha, \beta$, and $g$ is a bounded and uniformly continuous subsolution of $F = 0$ in $I\!\!R^N$. Then for any closed set $K$ there is a unique e-solution $u$ of $(BVP)$. Moreover for all $n$ there is a continuous solution $v_n$ of the penalized equation* (4.1), *$|v_n| \leq C$ for all $n$, and $v_n$ converge to $u$, i.e.* (4.3) *holds.*

**Proof.** Under the current assumptions the operator $F$ defined by (1.4) satisfies the Comparison Principle in every open set $\mathcal{O}$ by Theorem 7.3 in [24], and the same is true for the operator in (4.1). Define $M = \sup_{x,\alpha,\beta} |f^{\alpha,\beta}(x)|$ and $W(x) \equiv \max\{\sup |g|, M/c_0\}$. Then $W$ is a supersolution of (4.1) and we can apply Theorem 7.5 in [24] to produce a solution $v_n \in C(I\!\!R^N)$ of (4.1) such that $g \leq v_n \leq W$. Therefore the sequence $\{v_n\}$ satisfies the uniform bound (4.2) and we can apply Theorem 4.1 to get all the remaining conclusions. □

We end this section with some comments on the penalization procedure in connection with Bellman-Isaacs equations where the operator $F$ is given by (1.3) or (1.4). As we saw in Section 3 on two examples, the boundary value problem $(BVP)$ for these equations is solved by the value function of an optimal control problem or a differential game whose cost functional is

$$J(x, \alpha_., \beta_.) := E\left(\int_0^{\hat{t}_x} f^{\alpha_t,\beta_t}(X_t) e^{-\int_0^t c^{\alpha_s,\beta_s}(X_s)ds} dt + g(X_{\hat{t}_x})\right).$$

Here $f^{\alpha,\beta}$ is the running cost, $c^{\alpha,\beta}$ the interest rate, the system $X_.$ is stopped at time $\hat{t}_x$ when it exits $K$ and the terminal cost $g$ is paid at the exit point. The penalized equation (4.1) is itself the Bellman-Isaacs equation of a control problem or a game with the same controlled system, but with the infinite horizon cost functional

$$J_n(x, \alpha_., \beta_.) := E\left(\int_0^{\infty} (f^{\alpha_t,\beta_t} + n\zeta g)(X_t) e^{-\int_0^t (c^{\alpha_s,\beta_s} + n\zeta)(X_s)ds} dt\right),$$

where the running cost is $f^{\alpha,\beta} + n\zeta g$ and the interest rate is $c^{\alpha,\beta} + n\zeta$. The reader can check that the $\int_0^\infty$ appearing in $J_n$ converges, as $n \to \infty$, to the cost $\int_0^{\hat{t}_x} + g(X_{\hat{t}_x})$ which appears in $J$, and this gives a control theoretic interpretation of the penalization Theorem 4.1.

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