Mathematical modeling of crowds

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Outline of the talk

1. Mathematical models


3. Conclusion and perspectives
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1. Mathematical models


3. Conclusion and perspectives
Mathematical modeling of pedestrian motion: frameworks

Microscopic

- individual agents
- ODEs system
- many parameters
- low and high densities
- comp. cost $\sim$ ped. number.
Mathematical modeling of pedestrian motion: frameworks

Microscopic

- individual agents
- ODEs system
- many parameters
- low and high densities
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Macroscopic

- continuous fluid
- PDEs
- few parameters
- very high densities
- analytical theory
- comp. cost $\sim$ domain size
Macroscopic models

- Pedestrians as "thinking fluid"\(^1\)
- Averaged quantities:
  - \(\rho(t, x)\) pedestrians density
  - \(\vec{v}(t, x)\) mean velocity

\[ \begin{align*}
\partial_t \rho + \text{div}_x (\rho \vec{v}) &= 0 \\
\rho(0, x) &= \rho_0(x)
\end{align*} \]

for \(x \in \Omega \subset \mathbb{R}^2, \ t > 0\)

\(^1\)R.L. Hughes, Transp. Res. B, 2002
Macroscopic models

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### Mass conservation

\[
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \bar{v}) &= 0 \\
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\end{aligned}
\]

for \(x \in \Omega \subseteq \mathbb{R}^2, t > 0\)

### Two classes

- **1st order models**: velocity given by a phenomenological speed-density relation \(\bar{v} = V(\rho)\bar{v}\)
- **2nd order models**: velocity given by a momentum balance equation

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Two classes

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- Density must stay non-negative and bounded: \(0 \leq \rho(t, x) \leq \rho_{\text{max}}\)
- Different from fluid dynamics:
  - preferred direction
  - no conservation of momentum / energy
  - \(n \ll 6 \cdot 10^{23}\)

\(^1\)R.L. Hughes, Transp. Res. B, 2002
Desired direction of motion $\vec{v}^1$

Pedestrians:
- seek the shortest route to destination
- try to avoid high density regions

$\vec{v} = -\frac{\nabla x \phi}{|\nabla x \phi|}$

$\phi: \Omega \to \mathbb{R}$ is given by the Eikonal equation

$|\nabla x \phi| = C(t, x, \rho)$ in $\Omega$

$\phi(t, x) = 0$ for $x \in \Gamma$ outflow

where $C = C(t, x, \rho) \geq 0$ is the running cost

$\Rightarrow$ the solution $\phi(t, x)$ represents the weighted distance of the position $x$ from the target $\Gamma$ outflow

\(^1\)Hughes (macro 1st), Jiang et al. (macro 2nd), Colombo (non-local), Hartmann (micro)…
Desired direction of motion $\vec{v}^1$

Pedestrians:
- seek the shortest route to destination
- try to avoid high density regions

\[ \vec{v} = -\frac{\nabla_x \phi}{|\nabla_x \phi|} \]

The potential $\phi: \Omega \to \mathbb{R}$ is given by the Eikonal equation

\[
\begin{cases}
|\nabla_x \phi| = C(t, x, \rho) & \text{in } \Omega \\
\phi(t, x) = 0 & \text{for } x \in \Gamma_{outflow}
\end{cases}
\]

where $C = C(t, x, \rho) \geq 0$ is the running cost

$\implies$ the solution $\phi(t, x)$ represents the weighted distance of the position $x$ from the target $\Gamma_{outflow}$

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1Hughes (macro 1st), Jiang et al. (macro 2nd), Colombo (non-local), Hartmann (micro)...
Eikonal equation: level set curves for $|\nabla_x \phi| = 1$

In an empty space: potential is proportional to distance to destination
Speed-density relation

Speed function $V(\rho)$:

- decreasing function wrt density
- $V(0) = v_{\text{max}}$ free flow
  
  
  $V(\rho_{\text{max}}) = 0$ congestion

Examples:

![Graphs of speed $V(\rho)$ and flux $\rho V(\rho)$](image)
First order models

- Hughes’ model\(^2\)

\[
V(\rho) = v_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \quad |\nabla_x \phi| = \frac{1}{V(\rho)}
\]

- pedestrians tend to minimize their estimated travel time to the exit
- pedestrians temper their estimated travel time avoiding high densities
- **CRITICISM:** instantaneous global information on entire domain

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\(^3\)Y. Xia, S.C. Wong and C.-W. Shu, Physical Review E, 2009
First order models

- Hughes’ model

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- pedestrians tend to minimize their estimated travel time to the exit
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- CRITICISM: instantaneous global information on entire domain

- Dynamic model with memory effect

\[ V(\rho) = v_{\text{max}} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) \quad \vec{v} = -\frac{\nabla_x (\phi + \omega D)}{|\nabla_x (\phi + \omega D)|} \]

where

\[ |\nabla_x \phi| = \frac{1}{v_{\text{max}}} , \quad D(\rho) = \frac{1}{v(\rho)} + \beta \rho^2 \quad \text{discomfort} \]

- pedestrians seek to minimize their estimated travel time based on their knowledge of the walking domain
- pedestrians temper their behavior locally to avoid high densities

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Other first order models

- Non-local flow\(^4\)

\[
\vec{v} = V(\rho \ast \eta) \vec{\nu}(x) \quad \text{OR} \quad \vec{v} = V(\rho) \left( \vec{\nu}(x) - \varepsilon \frac{\nabla(\rho \ast \eta)}{\sqrt{1 + |\nabla(\rho \ast \eta)|^2}} \right)
\]

\[^4\]R.M. Colombo, Garavello and M. Lécureux-Mercier, M3AS, 2012
\[^5\]E. Cristiani, B. Piccoli and A. Tosin, Multiscale Model. Simul., 2011
Other first order models

- **Non-local flow:**
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  \vec{v} = V(\rho \ast \eta) \vec{v}(x) \quad \text{OR} \quad \vec{v} = V(\rho) \left( \vec{v}(x) - \varepsilon \frac{\nabla(\rho \ast \eta)}{\sqrt{1 + |\nabla(\rho \ast \eta)|^2}} \right)
  \]

- **Multi-scale time-evolving measures:**
  Probability distribution of pedestrians
  \[
  \mu_t = \theta m_t + (1 - \theta) M_t \quad \text{where} \quad \begin{cases} 
  m_t = \sum_{j=1}^{N} \delta_{P_j(t)} \\
  dM_t(x) = \rho(t, x)dx
  \end{cases}
  \]
  microscopic mass
  macroscopic mass

  **Governing equation:** probability conservation deduced from individual-based modeling
  \[
  \partial_t \mu_t + \text{div}_x (\mu_t \vec{v}_t) = 0
  \]
  \[
  \vec{v}_t(x) = v_{\text{max}} \vec{v}(x) + N \int_{B_R(x)} K(x, y)d\mu_t(y)
  \]

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5. E. Cristiani, B. Piccoli and A. Tosin, Multiscale Model. Simul., 2011
Second order model

Momentum balance equation\(^6\)\(^7\)

\[
\partial_t (\rho \vec{v}) + \text{div}_x (\rho \vec{v} \otimes \vec{v}) + \nabla_x P(\rho) = \frac{1}{\tau} \left( \rho V(\rho) \vec{v} - \rho \vec{v} \right)
\]

where

- \( V(\rho) = v_{\text{max}} e^{-\alpha \left( \frac{\rho}{\rho_{\text{max}}} \right)^2} \)
- \( |\nabla_x \phi| = 1/V(\rho) \)
- \( P(\rho) = p_0 \rho^\gamma, \; p_0 > 0, \; \gamma > 1 \) internal pressure
- \( \tau \) response time

\(^6\)Payne-Whitham, 1971
\(^7\)Y.Q. Jiang, P. Zhang, S.C. Wong and R.X. Liu, Physica A, 2010
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Hughes’ VS second order model: obstacles effect

Obstacles:

1. one column $r = 0.3m$
2. three columns $r = 0.2m$
3. two walls $1.2m \times 0.2m$

What is the effect of the obstacles on the outflow?
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Fig. Time evolution of the total mass of pedestrians inside the room.

Hughes’ model

Second order model
Second order model: stop-and-go waves

\[ P(\rho) = 0.005 \rho^2, \quad v_{\text{max}} = 2, \quad \rho_{\text{max}} = 7 \]

**Fig.** Time evolution of density profile at \( x = 64 \) (left exit)
Second order model: dependence on $p_0$

\[ P(\rho) = p_0 \rho^\gamma : \text{total evacuation time optimal for } p_0 \simeq 0.5 \]
Second order model: dependence on $v_{\text{max}}$

Total evacuation time

Social force models\(^8\) show a minimum for $v_{\text{max}} \simeq 1.4 \text{ m/s}$

$\Rightarrow$ faster-is-slower effect\(^9\)

Accounting for inter-pedestrian friction?

\(^8\) D. Helbing, I. Farkas and T. Vicsek, Nature, 2000

Second order model: dependence on \( u_{\text{max}} \)

Total mass evolution

![Graph showing the total mass evolution over time for different values of \( u_{\text{max}} \).]
Evacuation optimization: Braess’ paradox$^{10}$?

Problem: **clogging** at exit

Can obstacles reduce the evacuation time?

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$^{10}$Braess, D. *Über ein Paradoxon aus der Verkehrsplanung*, Unternehmensforschung, 1968
Evacuation optimization: Braess’ paradox?

Time evolution of the total mass of pedestrians inside the room
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Macroscopic models: summary

Strengths:

- lower computational cost for large crowds
- global description of spatio-temporal evolution
- mathematical tools for well-posedness and numerical approximation
- suitable for posing control and optimization problems

Weaknesses:

- only for very high densities
- not all parameters have physical meaning
- able to capture only some features of crowd dynamics

Aspects to be addressed:

- reproduce emerging phenomena observed in real situations
- account for individual choices that may affect the whole system
- validation on empirical data
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Thank you for your attention!