Macroscopic traffic flow models on networks - III

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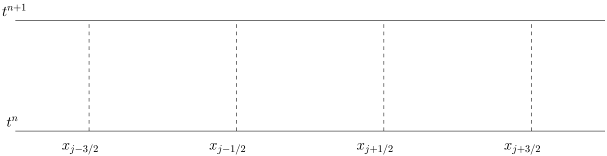
Outline of the talk

1. Numerical schemes
2. Examples of particular applications

Finite volume schemes

Denote Δx and Δt the space and time steps, $x_{j+1/2} = j\Delta x$ for $j \in \mathbb{Z}$ the mesh interfaces and $t^n = n\Delta t$ for $n \in \mathbb{N}$ the intermediate times

$$+ \text{ stability condition (CFL): } \frac{\Delta t}{\Delta x} \max_{\mathbf{u},i}\{|\lambda_i(\mathbf{u})|\} < \frac{1}{2}$$

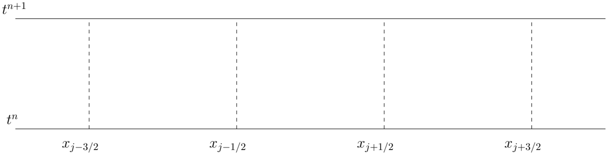


$$U_j^n \simeq \mathbf{u}(t^n, x) \quad \forall x \in C_j = [x_{j-1/2}, x_{j+1/2}[$$

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1/2}^n - F_{j-1/2}^n\right)$$

Conservative methods

Applying the conservation equation in the cell $[t^n, t^{n+1}] \times [x_{j-1/2,j+1/2}]$ we get



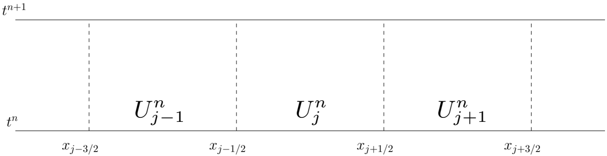
$$
\begin{aligned}
\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}(t^{n+1}, x)dx \;=\; & \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}(t^n, x)dx \\
& + \int_{t^n}^{t^{n+1}} f(\mathbf{u}(t, x_{j-1/2}))dt - \int_{t^n}^{t^{n+1}} f(\mathbf{u}(t, x_{j+1/2}))dt
\end{aligned}
$$

set

$$
U_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}(t^n, x)dx \qquad F_{j+1/2}^n = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(\mathbf{u}(t, x_{j+1/2}))dt
$$

Conservative methods

F numerical flux function s.t.



$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F(U_j^n, U_{j+1}^n) - F(U_{j-1}^n, U_j^n)\right)$$

- **Consistency**: $F(\mathbf{u}, \mathbf{u}) = f(\mathbf{u})$
- **Lipschitz continuity**: $|F(\mathbf{v}, \mathbf{w}) - f(\mathbf{u})| \leq K \max\{|\mathbf{v} - \mathbf{u}|, |\mathbf{w} - \mathbf{u}|\}$

Conservative methods

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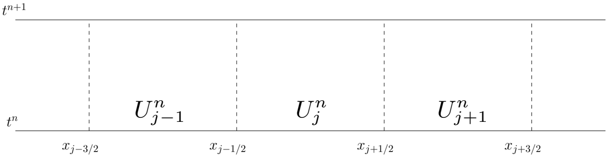

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Theorem (Lax-Wendroff, 1960)

Let $U_\Delta(t, x)$ be a numerical approximation computed with a consistent and conservative method. If $U_\Delta(t, x) \to \mathbf{u}$ as $\Delta x \to 0$ (in $\mathbf{L}^1$), then $\mathbf{u}$ is a weak solution of

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$$

Conservative methods: counterexample

Consider the Burgers' equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) = 0$$

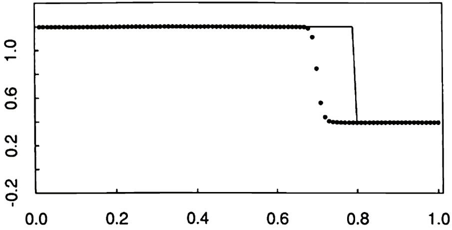
For smooth solutions, this is equivalent to

$$\partial_t u + u \partial_x u = 0$$

If $u \geq 0$, a natural discretization is

$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x} U_j^n \left(U_j^n - U_{j-1}^n \right)$$

but



Discrete entropy condition

If the scheme satisfies

$$E(U_j^{n+1}) \leq E(U_j^n) - \frac{\Delta t}{\Delta x}\left(\Psi(U_j^n, U_{j+1}^n) - \Psi(U_{j-1}^n, U_j^n)\right)$$

for some Ψ s.t. $\Psi(\mathbf{u}, \mathbf{u}) = Q(\mathbf{u})$

then the limit $\mathbf{u}$ is an entropic solution of $\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0$

$$\partial_t E(\mathbf{u}) + \partial_x Q(\mathbf{u}) \leq 0$$

Numerical fluxes

- **Lax-Friedrichs:**

$$F(U_j^n, U_{j+1}^n) = \frac{1}{2}\left(f(U_j^n) + f(U_{j+1}^n)\right) + \frac{\Delta t}{2\Delta x}\left(U_j^n - U_{j+1}^n\right)$$

or

$$F(U_j^n, U_{j+1}^n) = \frac{1}{2}\left(f(U_j^n) + f(U_{j+1}^n)\right) + \alpha\left(U_j^n - U_{j+1}^n\right)$$

- **Flux-splitting (upwind):** $f = f^- + f^+$ s.t. $Df^- \leq 0$, $Df^+ \geq 0$

$$F(U_j^n, U_{j+1}^n) = f^+(U_j^n) + f^-(U_{j+1}^n)$$

- **Godunov:**

$$F(U_j^n, U_{j+1}^n) = f(\mathcal{RS}(U_j^n, U_{j+1}^n)(0))$$

Godunov for scalar case

Compact formula

$$F(U_j^n, U_{j+1}^n) = \begin{cases} \min\limits_{U_j^n \le u \le U_{j+1}^n} f(u) & \text{if } U_j^n < U_{j+1}^n \\ \max\limits_{U_{j+1}^n \le u \le U_j^n} f(u) & \text{if } U_j^n > U_{j+1}^n \end{cases}$$

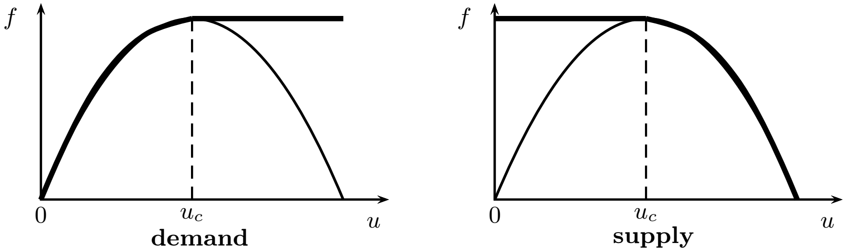
Godunov for scalar case

For concave flux functions f we can define

$$F^n_{j+1/2} = F(u^n_j, u^n_{j+1}) = \min\left\{D(u^n_j), S(u^n_{j+1})\right\}$$

where

- $D(u) = \begin{cases} f(u) & \text{if } u < u_c \\ f(u_c) = f_{\max} & \text{if } u \geq u_c \end{cases}$ is the **demand function**
- $S(u) = \begin{cases} f(u_c) = f_{\max} & \text{if } u < u_c \\ f(u) & \text{if } u \geq u_c \end{cases}$ is the **supply function**



(cfr. Cell Transmission Model, Godunov scheme)

Scalar case: convergence

Scheme properties: $\partial_1 F(u,v) \geq 0$, $\partial_2 F(u,v) \leq 0$

1. **Monotone:** $U_j^{n-1} \leq V_j^{n-1} \; \forall j \quad \Longrightarrow \quad U_j^n \leq V_j^n \; \forall j$
2. **$\mathbf{L^1}$-contracting:** $\sum_j \left|U_j^n - V_j^n\right| \leq \sum_j \left|U_j^{n-1} - V_j^{n-1}\right|$
3. **TVD:** $\sum_j \left|U_j^n - U_{j-1}^n\right| \leq \sum_j \left|U_j^{n-1} - U_{j-1}^{n-1}\right|$
4. **Monotonicity preserving:** $U_j^{n-1} \leq U_{j-1}^{n-1} \; \forall j \quad \Longrightarrow \quad U_j^n \leq U_{j-1}^n \; \forall j$

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Theorem (Harten-Hyman-Lax, CPAM 1976)

A monotone method is at most first order accurate.

Outline of the talk

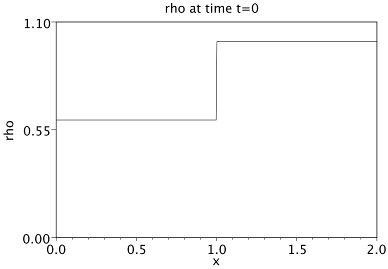
Example: red light

May be written as a Riemann problem:

$$\partial_t \rho + \partial_x[\rho(1-\rho)] = 0$$

$$\rho_0(x) = \begin{cases} \bar{\rho} & \text{si } x < 0 \\ 1 & \text{si } x > 0 \end{cases} \qquad 0 < \bar{\rho} < 1$$

$\bar{\rho} < 1 \Rightarrow$ shock of speed $\lambda = 1 - \bar{\rho} - 1 < 0$

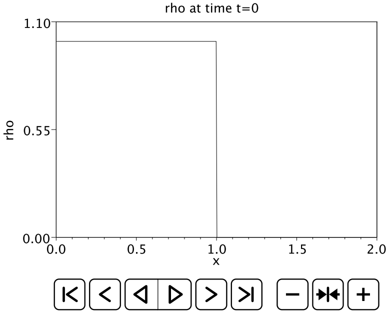


Example: green light

May be written as a Riemann problem:

$$\partial_t \rho + \partial_x[\rho(1-\rho)] = 0$$
$$\rho_0(x) = \begin{cases} 1 & \text{si } x < 0 \\ 0 & \text{si } x > 0 \end{cases} \qquad 0 < \bar{\rho} < 1$$

$1 > 0 \Rightarrow$ rarefaction with profile $\rho(t,x) = \frac{1}{2} - \frac{x}{2t}, \quad -t \leq x \leq t$



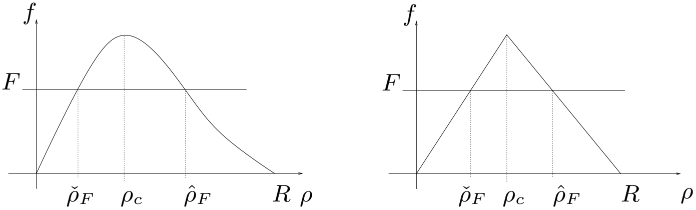
Example: toll gate

May be written as a conservation law with unilateral constraint on flux:

$$\partial_t \rho + \partial_x f(\rho) = 0 \qquad x \in \mathbb{R},\ t > 0$$

$$\rho(0,x) = \rho_0(x) \qquad x \in \mathbb{R}$$

$$f(\rho(t,0)) \leq F(t) \qquad t > 0$$



Example: toll gate

Constraint at $i = 0$:

$$u_i^{n+1} = u_i^n - \frac{k}{h_i}\left(g(u_i^n, u_{i+1}^n, F_{i+1/2}^n) - g(u_{i-1}^n, u_i^n, F_{i-1/2}^n)\right)$$

with numerical flux

$$g(u, v, F) = \begin{cases} \min(h(u,v), F) & \text{if interface } i = 0 \\ h(u,v) & \text{otherwise} \end{cases}$$

h classical numerical flux:

- **regular:** Lipschitz L;
- **consistent:** $h(s, s) = f(s)$;
- **monotone:** $u \nearrow$, $v \searrow$.

(Andreianov-Goatin-Seguin, 2010)

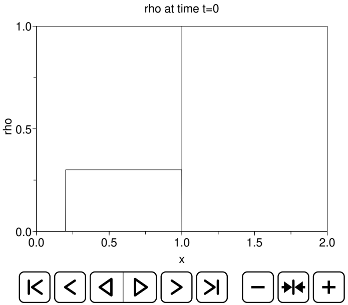
Example: toll gate

We consider

$$\partial_t \rho + \partial_x \left(\rho(1-\rho)\right) = 0$$

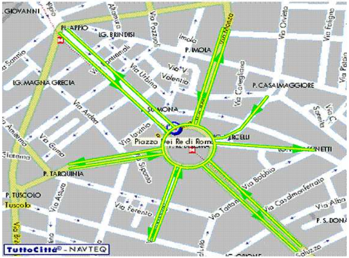
$$\rho(0,x) = 0.3\chi_{[0.2,1]}(x)$$

$$f(\rho(t,1)) \leq 0.1$$



Example: road network

A roundabout in Rome:



(Bretti-Natalini-Piccoli '06)

or a whole *town*, a conglomeration (see `http://traffic.berkeley.edu/`) ...

References

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