A PDE-ODE model for a junction with ramp buffer

Maria Laura Delle Monache

Email: maria-laura.delle_monache@inria.fr.

Workshop Traffic Modeling and Management: trends and perspectives

March 21, 2013

Outline

Outline

"ORESTE" Associated team

This work is the result of the collaboration between Inria and UC Berkeley:

- Alexandre Bayen
- Walid Krichene
- Jack Reilly
- Samitha Samaranayake

- Paola Goatin
- Maria Laura Delle Monache

Why a macroscopic ramp model?

- Develop a general optimization framework for many highway problems: partial rerouting, variable speed limit and ramp metering
- Extend to the continuous setting problems addressed in the engineering community
- Address specific shortcomings for control needs

Why a macroscopic ramp model?

- Develop a general optimization framework for many highway problems: partial rerouting, variable speed limit and ramp metering
- Extend to the continuous setting problems addressed in the engineering community
- Address specific shortcomings for control needs

Why a macroscopic ramp model?

- Develop a general optimization framework for many highway problems: partial rerouting, variable speed limit and ramp metering
- Extend to the continuous setting problems addressed in the engineering community
- Address specific shortcomings for control needs

Outline

Junction

- Two incoming links:
  - Upstream mainline $I_1 =]-\infty, 0[$
  - Onramp R_1
- Two outgoing links:
  - Downstream mainline $I_2 =]0, +\infty[$
  - Offramp R_2

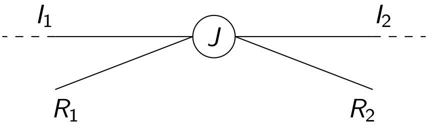


Figure : Junction modeled

Junction

- Two incoming links:
  - Upstream mainline $I_1 =]-\infty, 0[$
  - Onramp R_1
- Two outgoing links:
  - Downstream mainline $I_2 =]0, +\infty[$
  - Offramp R_2

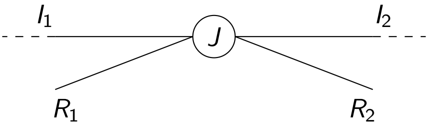


Figure : Junction modeled

Model

- Classical LWR on each mainline I_1, I_2

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad (t, x) \in \mathbb{R}^+ \times I_i,$$

  - $\rho = \rho(t,x) \in [0, \rho_{\max}]$ mean traffic density
  - $\rho_{\max}$ maximal density allowed on the road
  - $f : [0, \rho_{\max}] \to \mathbb{R}^+$ given by $f(\rho) = \rho v(\rho)$, flux function
  - $v(\rho)$ mean traffic speed

- Dynamics of the onramp described by a buffer

$$\frac{dl(t)}{dt} = F_{\mathrm{in}}(t) - \gamma_{\mathrm{r}1}(t), \quad t \in \mathbb{R}^+,$$

  - $l(t) \in [0, +\infty[$ length of the queue
  - $F_{\mathrm{in}}(t)$ flux that enters the onramp
  - $\gamma_{\mathrm{r}1}(t)$ flux that exits the onramp

Model

- Classical LWR on each mainline I_1, I_2

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad (t,x) \in \mathbb{R}^+ \times I_i,$$

  - $\rho = \rho(t,x) \in [0, \rho_{\max}]$ mean traffic density
  - $\rho_{\max}$ maximal density allowed on the road
  - $f : [0, \rho_{\max}] \to \mathbb{R}^+$ given by $f(\rho) = \rho v(\rho)$, flux function
  - $v(\rho)$ mean traffic speed

- Dynamics of the onramp described by a buffer

$$\frac{dl(t)}{dt} = F_{\mathrm{in}}(t) - \gamma_{\mathrm{r1}}(t), \quad t \in \mathbb{R}^+,$$

  - $l(t) \in [0, +\infty[$ length of the queue
  - $F_{\mathrm{in}}(t)$ flux that enters the onramp
  - $\gamma_{\mathrm{r1}}(t)$ flux that exits the onramp

Dynamics of the onramp described by a buffer

$$\frac{dl(t)}{dt} = F_{\mathrm{in}}(t) - \gamma_{\mathrm{r1}}(t), \quad t \in \mathbb{R}^{+},$$

Why this choice?

- Boundary conditions usually apply weakly and backward moving shock waves can happen at the boundary
- Lost information on the flux that actually enters the onramp, i.e. **demand not always satisfied for control schemes**
- **The buffer accounts for all the flow that enters the onramp**

Dynamics of the onramp described by a buffer

$$\frac{dl(t)}{dt} = F_{\mathrm{in}}(t) - \gamma_{\mathrm{r1}}(t), \quad t \in \mathbb{R}^+,$$

Why this choice?

- Boundary conditions usually apply weakly and backward moving shock waves can happen at the boundary
- Lost information on the flux that actually enters the onramp, i.e. **demand not always satisfied for control schemes**
- **The buffer accounts for all the flow that enters the onramp**

Dynamics of the onramp described by a buffer

$$\frac{dl(t)}{dt} = F_{\mathrm{in}}(t) - \gamma_{\mathrm{r1}}(t), \quad t \in \mathbb{R}^+,$$

Why this choice?

- Boundary conditions usually apply weakly and backward moving shock waves can happen at the boundary
- Lost information on the flux that actually enters the onramp, i.e. **demand not always satisfied for control schemes**
- **The buffer accounts for all the flow that enters the onramp**

Dynamics of the onramp described by a buffer

$$\frac{dl(t)}{dt} = F_{\mathrm{in}}(t) - \gamma_{\mathrm{r1}}(t), \quad t \in \mathbb{R}^+,$$

Why this choice?

- Boundary conditions usually apply weakly and backward moving shock waves can happen at the boundary
- Lost information on the flux that actually enters the onramp, i.e. **demand not always satisfied for control schemes**
- **The buffer accounts for all the flow that enters the onramp**

Cauchy Problem

$$
\begin{cases}
\partial_t \rho_i + \partial_x f(\rho_i) = 0, & (t,x) \in \mathbb{R}^+ \times I_i, \; i = 1,2 \\
\dfrac{dl(t)}{dt} = F_{\mathrm{in}}(t) - \gamma_{\mathrm{r1}}(t), & t \in \mathbb{R}^+, \\
\rho_i(0,x) = \rho_{i,0}(x), & \text{on } I_i \; i = 1,2 \\
l(0) = l_0,
\end{cases}
$$

Coupled with the following **Junction Problem**

$$
\begin{aligned}
d(F_{\mathrm{in}}, l) &= \begin{cases} \gamma_{\mathrm{r1}}^{\max} & \text{if } l(t) > 0, \\ \min\left(F_{\mathrm{in}}(t), \gamma_{\mathrm{r1}}^{\max}\right) & \text{if } l(t) = 0, \end{cases} \\
\delta(\rho_1) &= \begin{cases} f(\rho_1) & \text{if } 0 \le \rho_1 < \rho^{\mathrm{cr}}, \\ f^{\max} & \text{if } \rho^{\mathrm{cr}} \le \rho_1 \le 1, \end{cases} \\
\sigma(\rho_2) &= \begin{cases} f^{\max} & \text{if } 0 \le \rho_2 \le \rho^{\mathrm{cr}}, \\ f(\rho_2) & \text{if } \rho^{\mathrm{cr}} < \rho_2 \le 1, \end{cases} \\
\gamma_{\mathrm{r2}}(t) &= \beta f(\rho_1),
\end{aligned}
$$

Junction Assumptions

1. $f(\rho_1(t,0-)) + \gamma_{\mathrm{r1}}(t) = f(\rho_2(t,0+)) + \gamma_{\mathrm{r2}}(t)$
2. $f(\rho_2(t,0+))$ is maximum subject to 1 and

$$f(\rho_2(t,0+)) = \min\Big((1-\beta)\delta(\rho_1(t,0-)) + d(F_{\mathrm{in}}(t), l(t)), \sigma(\rho_2(t,0+))\Big)$$

3. To ensure uniqueness of the solution a **right of way parameter** $P \in]0,1[$ is introduced such that

$$f_1(\rho(t,0-)) = \frac{P}{1-P}\gamma_{r1}$$

4. No flux from the onramp is allowed on the offramp

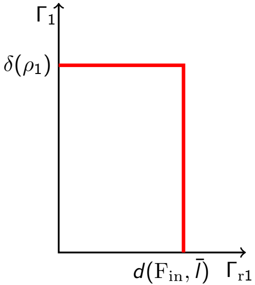
Outline

Solution of the Riemann problem: steps

To find a solution of the problem we will take the following steps.

1. Define $\Gamma_1 = f(\rho_1(t, 0-))$, $\Gamma_2 = f(\rho_2(t, 0+))$, $\Gamma_{\mathrm{r1}} = \gamma_{\mathrm{r1}}(t)$
2. Consider the space $(\Gamma_1, \Gamma_{\mathrm{r1}})$ and the sets $\mathcal{O}_1 = [0, \delta(\rho_1)]$, $\mathcal{O}_{\mathrm{r1}} = [0, d(F_{\mathrm{in}}, \bar{l})]$
3. Trace the line $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r1}} = \Gamma_2$
4. Consider the region

$$\Omega = \left\{(\Gamma_1, \Gamma_{\mathrm{r1}}) \in \mathcal{O}_1 \times \mathcal{O}_{\mathrm{r1}} : (1-\beta)\Gamma_1 + \Gamma_{\mathrm{r1}} \in [0, \Gamma_2]\right\}.$$

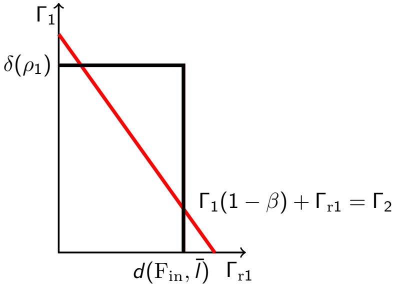


Solution of the Riemann problem: steps

To find a solution of the problem we will take the following steps.

1. Define $\Gamma_1 = f(\rho_1(t, 0-))$, $\Gamma_2 = f(\rho_2(t, 0+))$, $\Gamma_{r1} = \gamma_{r1}(t)$
2. Consider the space (Γ_1, Γ_{r1}) and the sets $\mathcal{O}_1 = [0, \delta(\rho_1)]$, $\mathcal{O}_{r1} = [0, d(F_{in}, \bar{l})]$
3. Trace the line $(1-\beta)\Gamma_1 + \Gamma_{r1} = \Gamma_2$
4. Consider the region

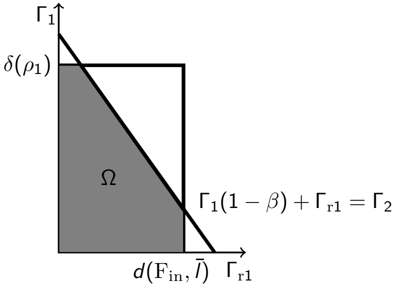
$$\Omega = \left\{ (\Gamma_1, \Gamma_{r1}) \in \mathcal{O}_1 \times \mathcal{O}_{r1} : (1-\beta)\Gamma_1 + \Gamma_{r1} \in [0, \Gamma_2] \right\}.$$



To find a solution of the problem we will take the following steps.

1. Define $\Gamma_1 = f(\rho_1(t, 0-))$, $\Gamma_2 = f(\rho_2(t, 0+))$, $\Gamma_{\mathrm{r1}} = \gamma_{\mathrm{r1}}(t)$
2. Consider the space $(\Gamma_1, \Gamma_{\mathrm{r1}})$ and the sets $\mathcal{O}_1 = [0, \delta(\rho_1)]$, $\mathcal{O}_{\mathrm{r1}} = [0, d(F_{\mathrm{in}}, \bar{l})]$
3. Trace the line $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r1}} = \Gamma_2$
4. Consider the region

$$\Omega = \Big\{(\Gamma_1, \Gamma_{\mathrm{r1}}) \in \mathcal{O}_1 \times \mathcal{O}_{\mathrm{r1}} : (1-\beta)\Gamma_1 + \Gamma_{\mathrm{r1}} \in [0, \Gamma_2]\Big\}.$$



Solution of the Riemann problem: steps

To find a solution of the problem we will take the following steps.

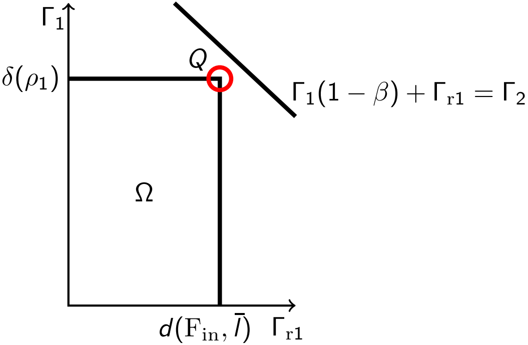
1. Define $\Gamma_1 = f(\rho_1(t, 0-))$, $\Gamma_2 = f(\rho_2(t, 0+))$, $\Gamma_{\mathrm{r1}} = \gamma_{\mathrm{r1}}(t)$
2. Consider the space $(\Gamma_1, \Gamma_{\mathrm{r1}})$ and the sets $\mathcal{O}_1 = [0, \delta(\rho_1)]$, $\mathcal{O}_{\mathrm{r1}} = [0, d(F_{\mathrm{in}}, \bar{l})]$
3. Trace the line $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r1}} = \Gamma_2$
4. Consider the region
$$\Omega = \Big\{(\Gamma_1, \Gamma_{\mathrm{r1}}) \in \mathcal{O}_1 \times \mathcal{O}_{\mathrm{r1}} : (1-\beta)\Gamma_1 + \Gamma_{\mathrm{r1}} \in [0, \Gamma_2]\Big\}.$$
5. Different situations can occur depending on the value of Γ_2

- **Demand limited case**: $\Gamma_2 = (1-\beta)\delta(\rho_1(t, 0-)) + d(F_{\mathrm{in}}, \bar{l})$
- **Supply limited case**: $\Gamma_2 = \sigma(\rho_2(t, 0+))$

Solution of the Riemann problem: Demand limited case

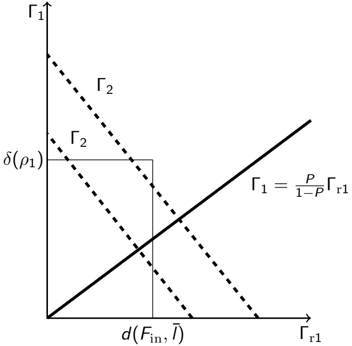
We set the **optimal point** Q to be the point $(\hat{\Gamma}_1, \hat{\Gamma}_{r1})$ such that

$\hat{\Gamma}_1 = \delta(\rho_1(t, 0-))$, $\hat{\Gamma}_{r1} = d(F_{in}, \bar{l})$ and $\hat{\Gamma}_2 = (1-\beta)\delta(\rho_1(t, 0-)) + d(F_{in}, \bar{l})$



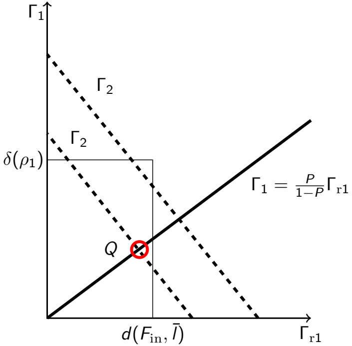
Solution of the Riemann problem: Supply limited case

- We introduce the **right of way parameter** , i.e., we trace the line $\Gamma_1 = \frac{P}{1-P}\Gamma_{\mathrm{r}1}$
- We set optimal point Q to be the point of intersection of $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r}1} = \Gamma_2$ and $\Gamma_1 = \frac{P}{1-P}\Gamma_{\mathrm{r}1}$.
- Different situations can occur depending on the value of the intersection point
    - $Q \in \Omega$
    - $Q \notin \Omega$



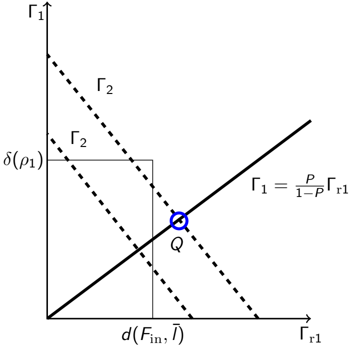
Solution of the Riemann problem: Supply limited case

- We introduce the **right of way parameter** , i.e., we trace the line $\Gamma_1 = \frac{P}{1-P}\Gamma_{\mathrm{r1}}$
- We set optimal point Q to be the point of intersection of $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r1}} = \Gamma_2$ and $\Gamma_1 = \frac{P}{1-P}\Gamma_{\mathrm{r1}}$.
- Different situations can occur depending on the value of the intersection point
  - $Q \in \Omega$
  - $Q \notin \Omega$



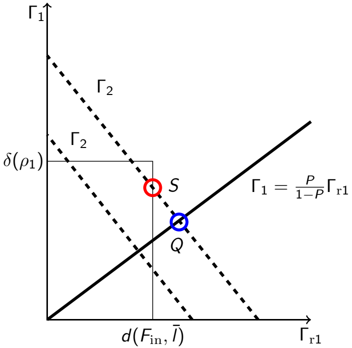
Solution of the Riemann problem: Supply limited case

- We introduce the **right of way parameter** , i.e., we trace the line $\Gamma_1 = \frac{P}{1-P}\Gamma_{\mathrm{r}1}$
- We set optimal point Q to be the point of intersection of $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r}1} = \Gamma_2$ and $\Gamma_1 = \frac{P}{1-P}\Gamma_{\mathrm{r}1}$.
- Different situations can occur depending on the value of the intersection point
  - $Q \in \Omega$
  - $Q \notin \Omega$



Solution of the Riemann problem: Supply limited case

- We introduce the **right of way parameter** , i.e., we trace the line $\Gamma_1 = \frac{P}{1-P}\Gamma_{\mathrm{r}1}$
- We set optimal point Q to be the point of intersection of $(1-\beta)\Gamma_1 + \Gamma_{\mathrm{r}1} = \Gamma_2$ and $\Gamma_1 = \frac{P}{1-P}\Gamma_{\mathrm{r}1}$.
- Different situations can occur depending on the value of the intersection point
  - $Q \in \Omega$
  - $Q \notin \Omega \implies$ **Optimal point**: S

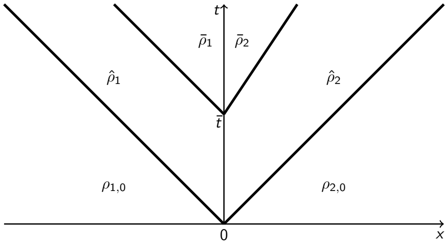


Theorem of consistency

Theorem

Consider a junction J and fix a priority parameter $P \in]0,1[$. For every $\rho_{1,0}$, $\rho_{2,0} \in [0,1]$ and $l_0 \in [0,+\infty[$, there exists a unique admissible solution $(\rho_1(t,x), \rho_2(t,x), l(t))$ satisfying the priority (possibly in an approximate way). Moreover, for a.e. $t > 0$, it holds

$$(\rho_1(t,0-), \rho_2(t,0+)) = \mathcal{R}_{l(t)}(\rho_1(t,0-), \rho_2(t,0+)).$$



Theorem of consistency

Theorem

Consider a junction J and fix a priority parameter $P \in]0,1[$. For every $\rho_{1,0}$, $\rho_{2,0} \in [0,1]$ and $l_0 \in [0,+\infty[$, there exists a unique admissible solution $(\rho_1(t,x), \rho_2(t,x), l(t))$ satisfying the priority (possibly in an approximate way). Moreover, for a.e. $t > 0$, it holds

$$(\rho_1(t,0-), \rho_2(t,0+)) = \mathcal{R}_{l(t)}(\rho_1(t,0-), \rho_2(t,0+)).$$

Sketch of the proof: using the following lemma

Lemma

If $(\hat{\rho}_1, \hat{\rho}_2)$ is a solution of the Riemann problem with initial data $(\rho_{1,0}, \rho_{2,0})$, then the following holds:

$$\begin{aligned} \delta(\rho_{1,0}) &\leq \delta(\hat{\rho}_1), \\ \sigma(\rho_{2,0}) &\leq \sigma(\hat{\rho}_2), \\ d(F_{\mathrm{in}}, l_0) &\leq d(F_{\mathrm{in}}, l). \end{aligned}$$

Outline

Modified Godunov Scheme

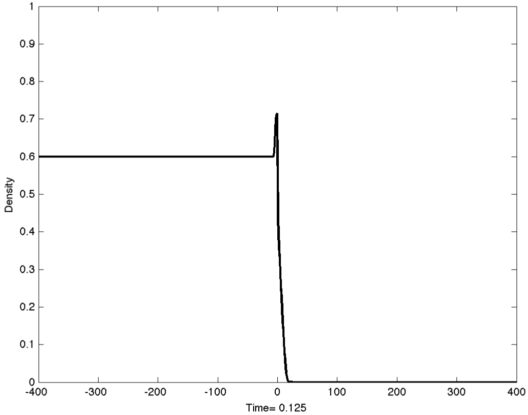
- At some time step Δt^n, we might have multiple shocks exiting the junction
- We divide the time step $\Delta t^n = (t^n, t^{n+1})$ into two sub-intervals $\Delta t_a = (t^n, \bar{t})$ and $\Delta t_b = (\bar{t}, t^{n+1})$
- We solve in one time step two different Riemann Problems at the junction
  - For Δt_a: **Classical Godunov flux update**

$$\begin{aligned} v_J^{n+1} &= v_J^n - \frac{\Delta t_a}{\Delta x}(\hat{\Gamma}_1 - g(v_{J-1}^n, v_J^n)) \\ v_0^{n+1} &= v_0^n - \frac{\Delta t_a}{\Delta x}(g(v_0^n, v_1^n) - \hat{\Gamma}_2) \end{aligned}$$

  - For Δt_b: **Modified flux update**

$$\begin{aligned} v_J^{n+1} &= v_J^{\bar{t}} - \frac{\Delta t_b}{\Delta x}\left(\hat{\Gamma}_1^{\bar{t}} - g(v_{J-1}^n, v_J^{\bar{t}})\right) \\ v_0^{n+1} &= v_0^{\bar{t}} - \frac{\Delta t_b}{\Delta x}\left(g(v_0^{\bar{t}}, v_1^n) - \hat{\Gamma}_2^{\bar{t}}\right) \end{aligned}$$

Numerical Simulations



Current work & Perspectives

- Corresponding discrete optimization problem solved using the adjoint method
- Production-scale implementation in the framework of the Berkeley Connected-Corridors traffic system
- Extension to optimal rerouting with multi-commodity flow and partial control
- Extension to traffic flow modeling on roundabouts

Thank you for your attention

Maria Laura Delle Monache
PhD student in "Traffic flow by conservation laws"
Inria Sophia Antipolis-Méditerranée
Email: maria-laura.delle_monache@inria.fr