Optimization based control of networks of discretized PDEs: application to traffic engineering

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Workshop BIS 2014

Paris, June 17, 2014
1. Associated Team ORESTE
2. A macroscopic model for ramp metering
3. Discretized system
4. Quick review of the adjoint method
5. Numerical results
6. Application to MPC-based ramp metering control
7. Application to optimal re-routing
8. Perspectives
Outline

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ORESTE (Optimal RErouting Strategies for Traffic mangEment) is an associated team between Inria project-team OPALE and the Connected Corridors project at UC Berkeley.

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- Maria Laura Delle Monache
- Alexandre Bayen (PI)
- Jack Reilly
- Samitha Samaranayake
- Walid Krichene
- Nikolaos Bekiaris-Liberis

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- ERC Starting Grant TRAM3
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Workshop TRAM2

Traffic Modeling and Management: Trends and Perspectives
Inria Sophia Antipolis, March 20-22, 2013

- 14 invited speakers
- 10 contributed talks
- ~40 participants
Road traffic congestion

  - $121 billions in wasted time and fuel (1% of GDP)
  - 5.5 billions hours of delay
  - 56 billions pounds of additional CO2 emissions

- Federal Highway Administration forecast

  ![Map of the US in 2002](image1.png) ![Map of the US in 2035](image2.png)

  - 37% of total delay occurs outside the peak hours
New opportunities

- Spread of smart phones and GPS navigation devices
  - US smartphone penetration is at 65%

- Real-time information
  - accurate live information (<5 minute delay)
  - high granularity (>1 minute update frequency)

- Adaptive control
  - stochastic routing
  - dynamic re-routing
ORESTE project goals

- Optimize traffic flow in corridors
  - ramp metering
  - re-routing strategies

- Modeling approach:
  - macroscopic traffic flow models
  - discrete adjoint method for gradient computation
Other ramp-metering models

- **ALINEA**  
  [Papageorgiou - Hadj-Salem - Blosseville, TRB, 1991]  
  - Closed-loop  
  - Only uses state estimation (no forward-sim model)  
  - Local policy (single ramp), extensions (HERO) for multi-ramp using heuristics

- **Continuous adjoint approach**  
  [Jacquet - Canudas de Wit - Koenig, Proc IFAC, 2005]  
  - Derive and discretize  
  - Onramps modeled as source terms  
  - Control the number of car entering the highway

- **Link-node CTM with ramp metering problem as LP**  
  [Muralidharana - Horowitz, ACC, 2012]  
  - For $n \times m$ junctions  
  - Priorities are proportional to demand (junction solver not "self-similar")  
  - Relaxation of the junction model
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Traffic flow models

Three possible approaches

- Microscopic Models
  - ODE system: each single car is modeled
  - Large number of parameters
  - Numerical simulations

- Kinetic Models
  - Distribution functions of the macroscopic quantities
  - Boltzmann-like equations

- Macroscopic Models
  - Traffic flow modeled as a fluid, i.e., PDEs from fluid dynamics
  - Few parameters
  - Analytical solutions
  - Suitable for optimization and control problems
LWR model

[Lighthill-Whitham ’55, Richards ’56]

Non-linear transport equation: PDE for mass conservation

\[ \partial_t \rho + \partial_x f(\rho) = 0 \quad x \in \mathbb{R}, \ t > 0 \]

- \( \rho \in [0, \rho_{\text{max}}] \) mean traffic density
- \( f(\rho) = \rho v(\rho) \) flux function

Empirical flux-density relation: fundamental diagram
Extension to networks

- LWR on arcs
- Maximization of flux through the junction subject to:
  - Mass conservation
  - Distribution/priority parameters

A model for ramp metering

- Two incoming roads:
  - Upstream mainline $I_1 = ]-\infty, 0[$
  - Onramp $R_1$

- Two outgoing roads:
  - Downstream mainline $I_2 = ]0, +\infty[$
  - Offramp $R_2$

Figure: Junction modeled
Coupled PDE-ODE model:

- Classical LWR on each mainline $l_1, l_2$

$$\partial_t \rho + \partial_x f(\rho) = 0, \quad (t, x) \in \mathbb{R}^+ \times l_i,$$

- Dynamics of the onramp described by an ODE (buffer)

$$\frac{dl(t)}{dt} = F_{\text{in}}(t) - \gamma_{r1}(t), \quad t \in \mathbb{R}^+,$$

- $l(t) \in [0, +\infty[$ length of the onramp queue
- $F_{\text{in}}(t)$ flux entering the onramp
- $\gamma_{r1}(t)$ flux leaving the onramp (through the junction)

- Coupled at junction
A model for ramp metering

Coupled PDE-ODE model:

- Classical LWR on each mainline $l_1, l_2$

\[
\partial_t \rho + \partial_x f(\rho) = 0, \quad (t, x) \in \mathbb{R}^+ \times l_i,
\]

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\frac{dl(t)}{dt} = F_{in}(t) - \gamma_{r1}(t), \quad t \in \mathbb{R}^+,
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Coupled at junction
A model for ramp metering

Coupled PDE-ODE model:

- Classical LWR on each mainline $I_1, I_2$

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- $F_{\text{in}}(t)$ flux entering the onramp
- $\gamma_{r1}(t)$ flux leaving the onramp (through the junction)

- Coupled at junction
\[ \delta(\rho_1) = \begin{cases} f(\rho_1) & \text{if } 0 \leq \rho_1 < \rho_{cr}, \\ f_{\text{max}} & \text{if } \rho_{cr} \leq \rho_1 \leq 1, \end{cases} \]

\[ d(F_{\text{in}}, l) = \begin{cases} \gamma_{r1}^{\max} & \text{if } l(t) > 0, \\ \min(F_{\text{in}}(t), \gamma_{r1}^{\max}) & \text{if } l(t) = 0, \end{cases} \]

\[ \sigma(\rho_2) = \begin{cases} f_{\text{max}} & \text{if } 0 \leq \rho_2 \leq \rho_{cr}, \\ f(\rho_2) & \text{if } \rho_{cr} < \rho_2 \leq 1, \end{cases} \]
Junction solver: flux maximization

1. Mass conservation: 
   \[ f(\rho_1(t, 0^-)) + \gamma_{r1}(t) = f(\rho_2(t, 0^+)) + \gamma_{r2}(t) \]

2. \( f(\rho_2(t, 0^+)) \) maximum subject to 1 and
   \[ f(\rho_2(t, 0^+)) = \min \left( (1 - \beta)\delta(\rho_1(t, 0^-)) + d(F_{in}(t), l(t)), \sigma(\rho_2(t, 0^+)) \right) \]

3. **Right of way parameter** \( P \in ]0, 1[ \) to ensure uniqueness:
   \[ f(\rho_2(t, 0^+)) = P f_1(\rho(t, 0^-)) + (1 - P) \gamma_{r1} \]

4. Offramp treated as a sink (infinite capacity)

5. No flux from the onramp to the offramp is allowed
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Ramp metering discretized system $H(\vec{\rho}, \vec{u}) = 0$

Dynamics and junctions solutions based on the model described earlier

- Piecewise affine system
- Control parameter $u_i$ is a constraint on the ramp inflow $r_i(k)$
Lower triangular forward system $H(\vec{\rho}, \vec{u}) = 0$:

- $h_{i,k} = \rho_i(k) - \rho_i(k - 1) - \{\text{flux update equations for cell } i \text{ and time step } k\}$
- $H$ system of concatenated $h_{i,k}$
- $H$ lower triangular
- Very efficient to solve $H^T x = b$
Ramp metering discretized system $H(\rho, \bar{u}) = 0$

Figure: Dependency diagram of the variables in the system.
Lower triangular forward system $H(\vec{ρ}, \vec{u}) = 0$

Figure: $\frac{∂H}{∂ρ}$ matrix
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Optimization of a PDE-constrained system

Optimization problem

\[
\begin{align*}
\text{minimize}_{\vec{u} \in U} & \quad J(\vec{\rho}, \vec{u}) \\
\text{subject to} & \quad H(\vec{\rho}, \vec{u}) = 0
\end{align*}
\]

- \( \vec{\rho} \in \mathcal{X} \subseteq \mathbb{R}^{N_T} \): state variables
- \( \vec{u} \in \mathcal{U} \subseteq \mathbb{R}^{M_T} \): control variables

Want to do gradient descent: How to compute the gradient \( \frac{\partial J}{\partial \vec{\rho}} \nabla \vec{u} \vec{\rho} + \frac{\partial J}{\partial \vec{u}} \)?

On trajectories, \( H(\vec{\rho}, \vec{u}) = 0 \) constant, thus \( \frac{\partial H}{\partial \vec{\rho}} \nabla \vec{u} \vec{\rho} + \frac{\partial H}{\partial \vec{u}} = 0 \)

Adjoint system:

\[
\frac{\partial H}{\partial \vec{\rho}}^T \lambda = -\frac{\partial J}{\partial \vec{u}}^T
\]

Then

\[
\frac{\partial J}{\partial \vec{\rho}} \nabla \vec{u} \vec{\rho} = -\lambda^T \frac{\partial H}{\partial \vec{\rho}} \nabla \vec{u} \vec{\rho} = \lambda^T \frac{\partial H}{\partial \vec{u}}
\]
The structure of the forward system influences the efficiency of the adjoint system solving.

Solving for $\lambda$

\[
\frac{\partial H^T}{\partial \rho} \lambda = -\frac{\partial J^T}{\partial \rho}
\]

Since $\frac{\partial H}{\partial \rho}$ is lower triangular, $\frac{\partial H^T}{\partial \rho}$ is an upper triangular matrix.

$\implies$ The adjoint system can be solved efficiently using backward substitution.

Complexity reduction from $O((NT)^2 MT)$ to $O((NT)^2 + (NT)(MT))$

Sparsity $\implies O(NT + MT)$
Exploiting system structure

The structure of the forward system influences the efficiency of adjoint system solving.

Solving for $\lambda$

$$\frac{\partial H^T}{\partial \rho} \lambda = -\frac{\partial J^T}{\partial \rho}$$

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Complexity reduction from $O((NT)^2MT)$ to $O((NT)^2 + (NT)(MT))$

Sparsity $\implies O(NT + MT)$
Algorithm 1 Gradient descent loop

Pick initial control \( \vec{u}_{\text{init}} \in U_{\text{ad}} \)

while not converged do

\[ \vec{\rho} = \text{forwardSim}(\vec{u}, IC, BC) \] solve for state trajectory (forward system)

\[ \lambda = \text{adjointSln}(\vec{\rho}, \vec{u}) \] solve for adjoint parameters (adjoint system)

\[ \nabla_{\vec{u}} J = \lambda^T \frac{\partial H}{\partial \vec{u}} + \frac{\partial J}{\partial \vec{u}} \] compute the gradient (search direction)

\[ \vec{u} \leftarrow \vec{u} - \alpha \nabla_{\vec{u}} J, \; \alpha \in (0, 1) \text{ s.t. } \vec{u} \in U_{\text{ad}} \] update the control \( \vec{u} \) (update step)

end while
Remarks on discrete adjoint

Strengths:
- Requires only one solution of adjoint system, independent of $\dim(\vec{u})$ (unlike finite differences or sensitivity analysis).
- Extends existing simulation code
- General technique, can be applied in very different settings

Weaknesses:
- Optimization of the discretized problem, rather than approximation of the optimal solution of the continuous problem
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Numerical Results: case study

Figure: I15 South, San Diego: 31 km

\[ N = 125 \text{ links} \]
\[ M = 9 \text{ onramps} \]
\[ T = 1800 \text{ time-steps} \]
\[ \Delta t = 4 \text{ seconds (120 minutes time interval)} \]
Figure: Density and queue lengths without control
Figure: Density and queue difference with control
Model Predictive Control

Performance under noisy input data: MPC loop
- initial conditions at time $t$ and boundary fluxes on $T_h$ (noisy inputs)
- optimal control policy on $T_h$
- forward simulation on $T_u \leq T_h$ using optimal controls and exact initial and boundary data
- $t \rightarrow t + T_u$

Comparison with ALINEA (local feedback control without boundary conditions)

Figure: Congestion reduction and noise robustness
Running time

Convergence

Simulation time
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Predictive/Coordinated Ramp Metering in Connected Corridors

UC Berkeley - PATH
Connected Corridors Framework

Estimation
- Real time Occupancy

Prediction
- Boundary Flows

Calibration
- Macroscopic Network

Model Predictive Control

Decision Support
- Dynamic Reroute
- Ramp Metering
- Queue-limited Metering
Simulation Framework

Microscopic Simulator

Ramp Meter 
Rates

Loop Detector 
Data

Vehicle Occupancies

Actuate Metering

CC Input

CC Output

Model Predictive Control

Estimation

Real time Occupancy

Prediction

Boundary Flows

Calibration

Macroscopic Network

Ramp Metering Engine
Model Predictive Control: Ramp Metering

5 AM
5:30 AM
6 AM

5 AM Estimated Occupancies
5 AM Forecasted Boundary Flows
5 AM Generated Metering Policies

5:15 AM Estimated Occupancies
5:15 AM Forecasted Boundary Flows
5:15 AM Generated Metering Policies

5:30 AM Estimated Occupancies
5:30 AM Forecasted Boundary Flows
5:30 AM Generated Metering Policies

5:45 AM Estimated Occupancies
5:45 AM Forecasted Boundary Flows
5:45 AM Generated Metering Policies

Actuated Metering Policy
I15 South Simulation

Contour Summaries

Density (veh / km)

Speed (km / hr)

Forward In Time

Downstream

MPC Ramp Metering

Carroll Canyon

No Control

No Control
I15 South: Total Travel Time- Mainline
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Multi-commodity flow:

\[ \rho_i(k) = \sum_{c \in C} \rho_{i,c}(k) \]

accounting for compliant \( c_c \in C \) and non-compliant \( c_n \) users

Goal:

Control compliant users to optimize traffic flow
Application to optimal re-routing

Example [Ziliaskopoulos, 2001]:

- **Normal conditions**: link capacity between cell 3 and cell 4 = 6
  Optimal total travel time: 190.9875

- **Incident between cell 3 and cell 4**: capacity goes to 0
  Optimal total travel time: 229.1973 (otherwise 254.758)
Application to optimal re-routing

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With partial control:

Total travel time reduction as a function of the percentage of vehicles that are rerouted: almost optimal allocation can be achieved by controlling \(~60\%\) of the demand.
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California Connected Corridors project

- Ramp Metering
- Smartphone enabled reroute
- Parking
- CMS
- Express Lanes
- BART
- Local Arterial Traffic Signals
Testbed for deployment in California: SCOPE

Phase 1 Area of Interest

[Map of the area with major roads and intersections marked.]
CPS operational decision support system: SENSING

Elements

1 - ARM
- Coordinated Ramp Meters
- Maximum Queue Detectors
- Adaptive Ramp Metering
- Incident Management

2 - ATM
- Lane Use Signals
- Variable Advisory Speed Signs
- Flush Plans
- Trailblazer Signs

3 - Transit
- Changeable Message Signs
- Traveler Information

4 - Arterial
- Transit Signal Priority
- Expanded Signal Coordination
- CCTV Cameras
- Local Street Improvements
- TransLiterate Information

System Integration

The 80 Integrated Corridor Mobility Project
CPS operational decision support system: CONTROL
Models developed as part of the project feed "estimation" and "prediction"

Accurate modeling is required to perform efficient traffic management

Perspectives for 2014 - 2015:
- New member on secondment at Inria Sophia
- 2015: IPAM workshop at UCLA, co-organized by Berkeley and Inria Sophia as part of "New Directions in Mathematical Approaches for Traffic Flow Management" trimester

~10 joint publications (journals, conferences)

Thank you for your attention!
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