CONFERENCE OF THE EUROPEAN GDR CONTROL OF PDES

Locally constrained conservation laws in traffic management

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Outline of the talk

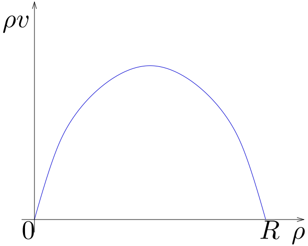
LWR traffic flow model

Lighthill-Whitham '55 and Richards '56:

$$\partial_t \rho + \partial_x (\rho v) = 0$$

$t \in [0, +\infty[$	time	$\rho = \rho(t,x)$	car density
$x \in \mathbb{R}$	space	$v = v(t,x)$	velocity

the number of cars is conserved and $v = v(\rho) = \left(1 - \frac{\rho}{R}\right) V$



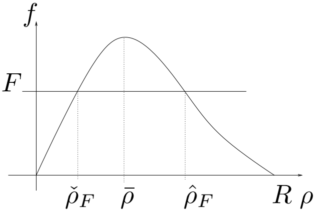
V maximal speed
R maximal density
ρv traffic flow

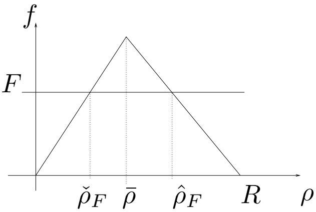
Conservation laws with unilateral constraints

Consider

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 & x \in \mathbb{R},\ t > 0, \\ \rho(0,x) = \rho_0(x) & x \in \mathbb{R}, \\ f(\rho(t,0)) \leq F(t) & t > 0, \end{cases}$$

with $f : [0,R] \to \mathbb{R}^+$ Lip., $f(0) = f(R) = 0$, $\exists\ \bar\rho$ s.t. $f'(\rho)(\bar\rho - \rho) > 0$,
$\rho_0 \in \mathbf{L}^\infty(\mathbb{R}; [0,R])$,
$F \in \mathbf{L}^\infty(\mathbb{R}^+; [0, f(\bar\rho)])$.





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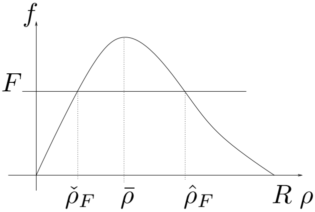
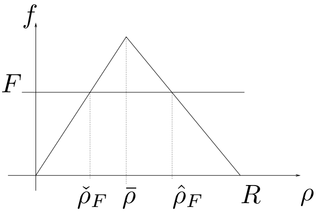
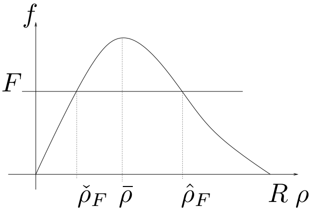


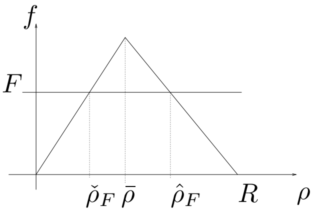


Applications: toll gate, construction site, traffic light...

The Riemann Solver $\mathcal{R}^F$

$$(\mathrm{CRP}) \quad \begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0,x) = \rho_0(x) \\ f(\rho(t,0)) \leq F \end{cases} \qquad \rho_0(x) = \begin{cases} \rho^l & \text{if } x < 0 \\ \rho^r & \text{if } x > 0 \end{cases}$$





Definition (Colombo-Goatin '07)

If $f\big(\mathcal{R}(\rho^l,\rho^r))(0)\big) \leq F$, then $\mathcal{R}^F(\rho^l,\rho^r) = \mathcal{R}(\rho^l,\rho^r)$.

Otherwise, $\mathcal{R}^F(\rho^l,\rho^r)(x) = \begin{cases} \mathcal{R}(\rho^l,\hat{\rho}_F)(x) & \text{if } x < 0\,, \\ \mathcal{R}(\check{\rho}_F,\rho^r)(x) & \text{if } x > 0\,. \end{cases}$

$\Longrightarrow$ non-classical shock at $x = 0$

Entropy conditions

Definition 1 (Colombo-Goatin '07)

$\rho \in \mathbf{L}^\infty$ is **weak entropy solution** if

- $\forall \phi \in \mathcal{C}_c^1$, $\phi \geq 0$, and $\forall k \in [0, R]$

$$\int_0^{+\infty}\int_{\mathbb{R}} \left(|\rho - \kappa|\partial_t + \Phi(\rho,\kappa)\partial_x\right) \phi \, dx \, dt + \int_{\mathbb{R}} |\rho_0 - \kappa| \, \phi \, dx + 2\int_0^{+\infty} \left(1 - \frac{F(t)}{f(\bar\rho)}\right) f(\kappa) \, \phi(t,0) \, dt \geq 0$$

- $f(\rho(t,0-)) = f(\rho(t,0+)) \leq F(t)$ a.e. $t > 0$

where $\Phi(a,b) = \mathrm{sgn}(a-b)(f(a) - f(b))$

(Cfr. conservation laws with discontinuous flux function: Karlsen-Risebro-Towers '03, Karlsen-Towers '04, Coclite-Risebro '05...)

Entropy conditions

Definition 2 (Andreianov-Goatin-Seguin '10)

$\rho \in \mathbf{L}^\infty$ is **weak entropy solution** if $\exists\ M > 0$ s.t. $\forall \phi \in \mathcal{C}^1_c$, $\phi \geq 0$, and $\forall (c_l, c_r) \in [0, R]^2$

$$\int_0^{+\infty} \int_{\mathbb{R}} \left(|\rho - c|\partial_t + \Phi(\rho, c)\partial_x\right) \phi \; dx \; dt + \int_{\mathbb{R}} |\rho_0 - c| \; \phi \; dx$$
$$+ M \int_0^{+\infty} \mathrm{dist}((c_l, c_r), \mathcal{G}(F(t)))\phi(t, 0) \; dt \geq 0$$

where $c = c(x) = \begin{cases} c_l & \text{if } x < 0 \\ c_r & \text{if } x > 0 \end{cases}$

(cfr. *adapted entropies* of Baiti-Jenssen '97, Audusse-Perthame '05...)

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$$\int_0^{+\infty} \int_{\mathbb{R}} \left(|\rho - c| \partial_t + \Phi(\rho, c) \partial_x \right) \phi \; dx \; dt + \int_{\mathbb{R}} |\rho_0 - c| \; \phi \; dx + M \int_0^{+\infty} \operatorname{dist}((c_l, c_r), \mathcal{G}(F(t))) \phi(t, 0) \; dt \geq 0$$

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(cfr. *adapted entropies* of Baiti-Jenssen '97, Audusse-Perthame '05...)

$\longrightarrow$ **no explicit conditions on traces!**

Admissibility germ

$\mathcal{G}(F) = \mathcal{G}_1(F) \cup \mathcal{G}_2(F) \cup \mathcal{G}_3(F)$ where

- $\mathcal{G}_1(F) = \{(c_l, c_r) \in [0,R]^2 \colon\ c_l > c_r, f(c_l) = f(c_r) = F\}$
- $\mathcal{G}_2(F) = \{(c, c) \in [0,R]^2 \colon\ f(c) \leq F\}$
- $\mathcal{G}_3(F) = \{(c_l, c_r) \in [0,R]^2 \colon\ c_l < c_r, f(c_l) = f(c_r) \leq F\}$

(cfr. (A, B)-*connection*:
Adimurthi-Mishra-Veerappa Gowda '05, Bürgers-Karlsen-Towers '09,
Andreianov-Karlsen-Risebro '09...)

Equivalence

Theorem (Andreianov-Goatin-Seguin '10)

$\rho \in \mathbf{L}^\infty$ is **weak entropy solution** iff

- $\forall \phi \in \mathcal{C}_c^1(\mathbb{R}^+ \times \mathbb{R} \setminus \{0\})$, $\phi \geq 0$, and $k \in [0, R]$

$$\int_0^{+\infty} \int_{\mathbb{R}} (|\rho - \kappa| \partial_t + \Phi(\rho, \kappa) \partial_x) \ \phi \ dx \ dt + \int_{\mathbb{R}} |\rho_0 - \kappa| \ \phi \ dx \geq 0$$

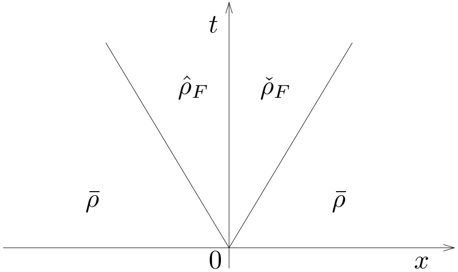
- $(\rho(t, 0-), \rho(t, 0+)) \in \mathcal{G}(F(t))$ a.e. $t > 0$

Dissipation condition:

$$\forall \ (c_l, c_r) \in \mathcal{G}(F), \quad \Phi(\rho(t, 0-), c_l) \geq \Phi(\rho(t, 0+), c_r)$$

Well-posedness in BV

constraint $\longrightarrow$ TV(ρ) explosion



We consider the function

$$\Psi(\rho) = \mathrm{sgn}(\rho - \bar{\rho})(f(\bar{\rho}) - f(\rho))$$

(cfr. Temple '82, Coclite-Risebro '05 ...)

Well-posedness in BV

Theorem (Colombo-Goatin '07)

$F \in \mathrm{BV}$. There exists a semigroup $S^F : \mathbb{R}^+ \times \mathcal{D} \mapsto \mathcal{D}$ s.t.

- $\mathcal{D} \supseteq \{\rho \in \mathbf{L^1} : \Psi(\rho) \in \mathrm{BV}\}$;
- $\left\| S_t^F \rho_1 - S_t^F \rho_2 \right\|_{\mathbf{L^1}} \leq \|\rho_1 - \rho_2\|_{\mathbf{L^1}} \quad \forall \rho_1, \rho_2 \in \mathcal{D}$.

Well-posedness in BV

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Proof

1. Wave-front tracking.
2. Glimm functional *ad hoc*

$$\Upsilon(\rho^n, F^n) = \sum_{\alpha} \left| \Psi(\rho^n_{\alpha+1}) - \Psi(\rho^n_\alpha) \right| + 5 \sum_{t_\beta \geq 0} \left| F^n_{\beta+1} - F^n_\beta \right| + \gamma$$

3. Doubling of variables method with constraint.

Well-posedness in $\mathbf{L}^\infty$

If $F^1, F^2 \in \mathbf{L}^\infty$, $\rho_1, \rho_2 \in \mathbf{L}^\infty$ and $\rho_1 - \rho_2 \in \mathbf{L}^1$:

$$\int_{\mathbb{R}} |\rho^1 - \rho^2|(T,x)\, dx \;\leq\; 2\int_0^T |F^1 - F^2|(t)\, dt \,+\, \int_{\mathbb{R}} |\rho_0^1 - \rho_0^2|(x)\, dx$$

Theorem (Andreianov-Goatin-Seguin '10)

$\forall \rho_0 \in \mathbf{L}^\infty$ and $\forall F \in \mathbf{L}^\infty$ $\exists!$ weak entropy solution.

Well-posedness in $\mathbf{L}^\infty$

If $F^1, F^2 \in \mathbf{L}^\infty$, $\rho_1, \rho_2 \in \mathbf{L}^\infty$ and $\rho_1 - \rho_2 \in \mathbf{L}^\mathbf{1}$:

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Theorem (Andreianov-Goatin-Seguin '10)

$\forall \rho_0 \in \mathbf{L}^\infty$ and $\forall F \in \mathbf{L}^\infty$ $\exists!$ weak entropy solution.

Proof

Truncation + regularization + finite propagation speed.

The Initial-Boundary Value Problem

Previous results can be generalized to

$$\text{(CIBVP)}\begin{cases} \partial_t\rho + \partial_x f(\rho) = 0 & x \in \mathbb{R}^+,\ t>0, \\ \rho(0,x) = \rho_0(x) & x\in\mathbb{R}^+, \\ f(\rho(t,0)) = q(t) & t>0 \\ f(\rho(t,\bar{x})) \leq F(t) & \bar{x}>0,\ t>0. \end{cases}$$

(CIBVP) can be used as basic brick to describe:

- road merging;
- sequence of traffic lights;
- work sites;

and optimization of related cost functionals.

Well-posedness for the IBVP

Definition (Colombo-Goatin-Rosini '11)

$\rho \in \mathbf{L}^\infty$ is **weak entropy solution** to (CIBVP) if

- $\forall \phi \in \mathcal{C}_c^1$, $\phi \geq 0$, and $\forall k \in [0, R]$

$$\int_0^{+\infty} \int_{\mathbb{R}} (|\rho - \kappa| \partial_t + \Phi(\rho, \kappa) \partial_x) \; \phi \; dx \; dt + \int_{\mathbb{R}} |\rho_0 - \kappa| \; \phi \; dx$$
$$+ \int_0^{+\infty} \operatorname{sgn}(f_*^{-1}(q(t)) - k)(f(\rho(t, 0+)) - f(k)) \; \phi(t, 0) \; dt$$
$$+ 2 \int_0^{+\infty} \left(1 - \frac{F(t)}{f(\bar{\rho})}\right) f(\kappa) \; \phi(t, \bar{x}) \; dt \geq 0$$

- $f(\rho(t, \bar{x}-)) = f(\rho(t, \bar{x}+)) \leq F(t)$ a.e. $t > 0$

Well-posedness for the IBPV

Definition (Colombo-Goatin-Rosini '11)

$\rho \in \mathbf{L}^\infty$ is **weak entropy solution** to (CIBVP) if

- $\forall \phi \in \mathcal{C}_c^1$, $\phi \geq 0$, and $\forall k \in [0, R]$

$$\int_0^{+\infty} \int_{\mathbb{R}} (|\rho - \kappa| \partial_t + \Phi(\rho, \kappa) \partial_x) \, \phi \, dx \, dt + \int_{\mathbb{R}} |\rho_0 - \kappa| \, \phi \, dx$$
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$$+ 2 \int_0^{+\infty} \left(1 - \frac{F(t)}{f(\bar{\rho})}\right) f(\kappa) \, \phi(t, \bar{x}) \, dt \geq 0$$

- $f(\rho(t, \bar{x}-)) = f(\rho(t, \bar{x}+)) \leq F(t)$ a.e. $t > 0$

Theorem (Colombo-Goatin-Rosini '11)

$F,\ q \in \mathrm{BV}$, $\rho_0 \in \mathcal{D}$. There exists a unique weak entropy solution to IBVP. Moreover,

$$\|\rho(t) - \rho'(t)\|_{\mathbf{L}^1} \leq \|\rho_0 - \rho_0'\|_{\mathbf{L}^1} + \|q - q'\|_{\mathbf{L}^1} + 2\|F - F'\|_{\mathbf{L}^1} .$$

Cost functional: Queue length

Queue length for BV data and $F(t) \equiv F = const$:

$$A_c(\rho(t)) = \{x \in [0, \bar{x}] \colon \rho(t,\xi) = \hat{\rho}_F \text{ a.e. } \xi \in [x, \bar{x}]\}$$

and

$$\mathcal{L}(\rho(t)) = \begin{cases} \bar{x} - \inf A_c(\rho(t)) & \text{if } A_c(\rho(t)) \neq \emptyset \\ 0 & \text{if } A_c(\rho(t)) = \emptyset \end{cases}$$

Upper semicontinuity (Colombo-Goatin-Rosini '11)

The map $\mathcal{L}$ is upper semicontinuous with respect to the $\mathbf{L^1}$–norm.

$\longrightarrow$ **no existence of minimizers for queue length!**

Cost functional: Stop & Go waves

Minimize the total variation of traffic speed (weighted by $p(x) \in [0,1]$)

$$J(\rho) = \int_0^T \int_{\mathbb{R}^+} p(x)\, d|\partial_x v(\rho)|\, dt$$

Lower semicontinuity (Colombo-Groli '04)

The map $\mathcal{J}$ is lower semicontinuous with respect to the $\mathbf{L^1}$–norm.

Cost functional: Travel times

If $\rho_0 = 0$ and $\mathrm{supp}(q) \subseteq [0, \tau_0]$, then $Q_{\text{in}} = \int_0^{\tau_0} q(t)\, dt$ and

mean **arrival** time $$T_a(x) = \frac{1}{Q_{\text{in}}} \int_0^{+\infty} t\, f(\rho(t,x))\, dt$$

mean **travel** time $$T_t(x) = \frac{1}{Q_{\text{in}}} \int_0^{+\infty} t\, \left(f(\rho(t,x)) - f(\rho(t,0))\right) dt$$

Lipschitz continuity (Colombo-Goatin-Rosini '11)

The maps $T_a(x)$ and $T_t(x)$ are Lipschitz continuous with respect to the $\mathbf{L^1}$–norm.

Cost functional: Density dependent functionals

Fix $T > 0$ and $b > a > 0$:

$$\mathcal{F}(\rho) = \int_0^T \int_a^b \varphi\left(\rho(t,x)\right) w(t,x) dx\ dt$$

where φ can be chosen

- $\varphi(\rho) = \left(v(\rho) - \bar{v}\right)^2$, to have vehicles travelling at a speed as near as possible to a desired optimal speed $\bar{v}$ along a given road segment $[a, b]$
- $\varphi(\rho) = f(\rho)$, to maximize the traffic flow along $[a, b]$

Finite volume schemes

Constraint at $i = 0$:

$$u_i^{n+1} = u_i^n - \frac{k}{h_i}\left(g(u_i^n, u_{i+1}^n, F_{i+1/2}^n) - g(u_{i-1}^n, u_i^n, F_{i-1/2}^n)\right)$$

with numerical flux

$$g(u,v,F) = \begin{cases} \min(h(u,v), F) & \text{if interface } i = 0 \\ h(u,v) & \text{otherwise} \end{cases}$$

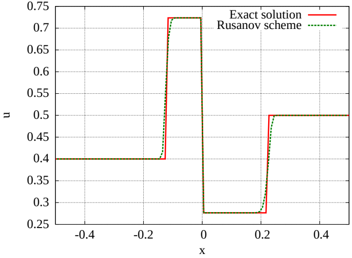
h classical numerical flux:

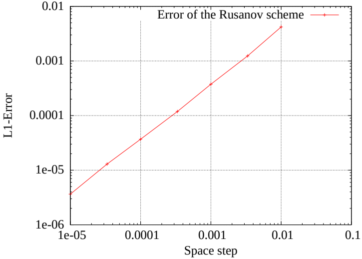
- **regular:** Lipschitz L;
- **consistent:** $h(s,s) = f(s)$;
- **monotone:** $u \nearrow$, $v \searrow$.

Numerical test

$$f(u) = u(1-u), \quad F = 0.2, \quad u_0(x) = \begin{cases} 0.4 & \text{if } x < 0 \\ 0.5 & \text{if } x > 0 \end{cases}$$

$$\text{Rusanov flux: } h(u,v) = \frac{f(u)+f(v)}{2} - \frac{\max(|f'(u)|,|f'(v)|)}{2}(v-u)$$

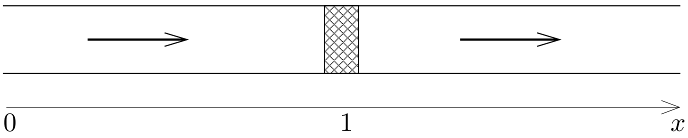




(Andreianov-Goatin-Seguin ’10)

Example: toll gate

Colombo-Goatin-Rosini '09:

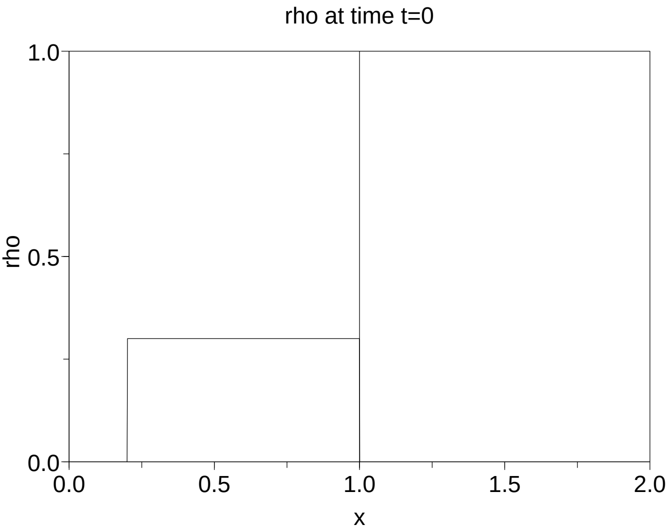


$$\partial_t \rho + \partial_x \left(\rho(1-\rho)\right) = 0 \qquad \text{(LWR)}$$

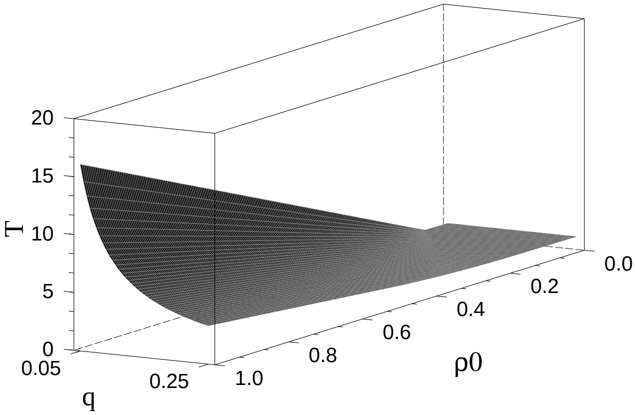
$$\rho(0,x) = 0.3\chi_{[0.2,1]}(x)$$

$$f\left(\rho(t,1)\right) \le 0.1$$

Example: toll gate



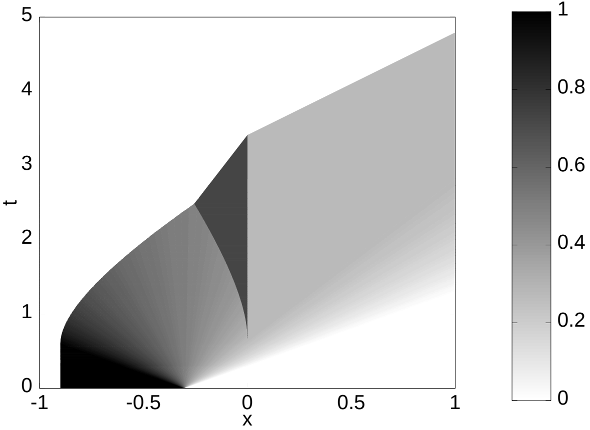
Example: toll gate



Exit times for $\rho_0 \in [0.1, 1]$ and $F \in [0.05, 0.25]$

Wave-front tracking scheme

For simple initial data is a good alternative to precisely compute shock positions and exit times:



WFT solution with $\bar{x} = 0$, $u_0 = \chi_{[-0.9,-0.3]}$, $F = 0.2$.

Wave-front tracking VS Lax-Friedrichs

Wave Front Tracking

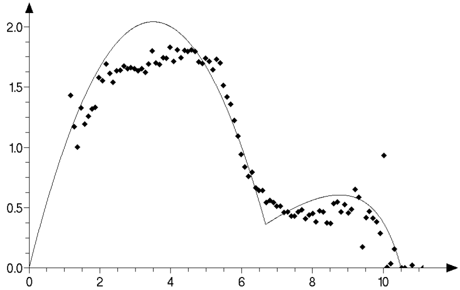
$\Delta\rho$	Exit Time	CPU Time (s)	Relative Error
4.00e-03	4.79564272	0.32	-1.90e-02 %
2.00e-03	4.79615273	0.59	-8.40e-03 %
1.00e-03	4.79640870	1.18	-3.07e-03 %
5.00e-04	4.79653693	2.36	-3.94e-04 %
2.50e-04	4.79660132	4.95	9.49e-04 %
1.25e-04	4.79656903	10.60	2.76e-04 %
6.25e-05	4.79655291	24.48	-6.06e-05 %

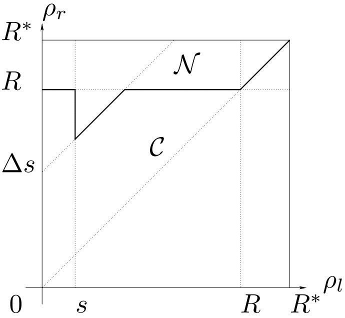
Lax–Friedrichs

Δx	"Exit Time"	CPU Time (s)	Relative Error
4.00e-03	4.94600000	1.69	3.12e-00 %
2.00e-03	4.87000000	5.18	1.53e-00 %
1.00e-03	4.83300000	18.90	7.60e-01 %
5.00e-04	4.81475000	73.40	3.79e-01 %
2.50e-04	4.80562500	295.99	1.89e-01 %
1.25e-04	4.80100000	1213.41	9.27e-02 %
6.25e-05	4.79878125	5264.29	4.64e-02 %

(Colombo-Goatin-Rosini '11)

Pedestrian flow modeling





Panic $\Leftrightarrow$ non-classical shock

(Colombo-Rosini '05)

Example: Braess paradox

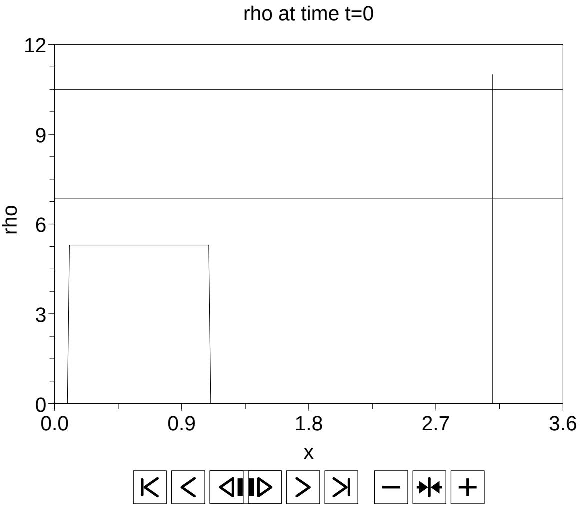
Colombo-Goatin-Rosini '09:

$$\begin{array}{l} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(0,x) = \rho_0(x) \end{array} \qquad \begin{array}{rcl} f\left(\rho(t,d)\right) & \leq & q\left(\rho(t,d)\right) \\ f\left(\rho(t,D)\right) & \leq & Q\left(\rho(t,D)\right) \end{array}$$

$$q(\rho) = \left\{ \begin{array}{llll} \hat{q} & \text{if} & \rho & \in [0,R] \\ \check{q} & \text{if} & \rho & \in \,]R,R^*] \end{array} \right. \qquad Q(\rho) = \left\{ \begin{array}{llll} \hat{Q} & \text{if} & \rho & \in [0,R] \\ \check{Q} & \text{if} & \rho & \in \,]R,R^*] \end{array} \right.$$

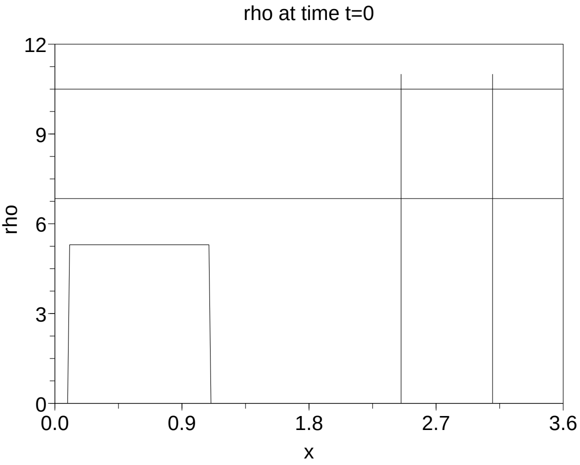
$$\hat{q} > \check{q} \qquad\qquad \hat{Q} > \check{Q}$$

Application:pedestrians



$$\text{Exit time} = 30.807771$$

Application:pedestrians



$\text{Exit time} = 28.206514$

Perspectives

- Rigorous study of general fluxes and non-classical problems.
- Improve numerical techniques for non-classical problems.
- Control problems.

Thanks for your attention!