# On Pure and (approximate) Strong Equilibria of Facility Location Games

Thomas Dueholm Hansen ⋆ and Orestis A. Telelis ⋆

Department of Computer Science, University of Aarhus, Denmark
{tdh,telelis}@cs.au.dk

**Abstract.** We study social cost losses in Facility Location games, where $n$ selfish agents install facilities over a network and connect to them, so as to forward their local demand (expressed by a non-negative weight per agent). Agents using the same facility share fairly its installation cost, but every agent pays individually a (weighted) connection cost to the chosen location. We study the Price of Stability (PoS) of *pure* Nash equilibria and the Price of Anarchy of *strong* equilibria (SPoA), that generalize pure equilibria by being resilient to coalitional deviations. For unweighted agents on metric networks we prove upper and lower bounds on $PoS$, while an $O(\ln n)$ upper bound implied by previous work is tight for non-metric networks. We also prove a constant upper bound for the SPoA of metric networks when strong equilibria exist. For the weighted game on general networks we prove existence of $e$-approximate ($e = 2.718\ldots$) strong equilibria and an upper bound of $O(\ln W)$ on SPoA ($W$ is the sum of agents' weights), which becomes tight $\Theta(\ln n)$ for unweighted agents.

## 1 Introduction

We study *Facility Location* games played by $n$ selfish agents residing on the nodes of a network. Eash agent chooses strategically a certain network location to connect and forward its local demand to (expressed by a non-negative weight $w_i$ for agent $i$), so as to minimize its individual facility installation and (weighted) connection costs to the chosen location. We use Shapley (fair) cost-sharing [1] for facility installation costs; agents connecting to the same location $v$ share facility installation cost at $v$, so that each pays an amount proportional to the fraction of total demand that it forwards to $v$. This game models *Content Distribution Network* creation, and distributed *selfish caching* [2]. We study the social cost (sum of individual agents' costs) of *stable* network infrastructures, represented by *pure* Nash equilibria and *strong* equilibria of the game. Strong equilibria - introduced by Aumann in [3] - extend pure equilibria by being resilient to *pure* coalitional deviations: no subset of agents can deviate so that all of its members are better off. We prove bounds on the *Price of Stability* (PoS) of pure equilibria, i.e. the cost of the cheapest equilibrium relative to the socially optimum cost [1], and on the *Price of Anarchy* of strong equilibria (SPoA), the cost of the most expensive strong equilibrium relative to the socially optimum cost [4].

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Anshelevich *et al.* [1] first studied the Price of Stability for network design games with fair cost sharing. In these games $n$ agents wish to connect node pairs in a network, by sharing fairly installation costs of links and paying individually link delays. The authors showed that for unweighted agents these games are *potential* games (see [5]), hence they have pure equilibria. They proved logarthmic (in $n$) upper bounds on the $PoS$. Research thereafter was focused on games *without non-shareable delays*. For weighted agents it was shown in [6] that pure equilibria do not always exist. The authors studied approximate equilibria. Albers [7] recently considered *strong* equilibria: though they do not always exist, she showed that $O(\ln W)$-approximate equilibria do exist ($W$ is the sum of the agents' weights). She proved polylogarithmic upper and lower bounds on the PoS and the SPoA in the weighted and the unweighted case. However, strong equilibria in the context of (single-sink) unweighted network design games with fair cost sharing were first studied in [8]. The authors gave topological characterizations for the existence of strong equilibria and proved that $SPoA = \Theta(\log n)$. The Facility Location game is a special case of the model studied in [1], that is interesting on its own right: it finds numerous applications and exhibits intriguing characteristics. It embodies non-shareable delays explicitly and in a sense specializes single-sink network design considered in [6, 8]: augment the network with a node $t$ and set links $(v, t)$ to have fairly shareable cost equal to the facility opening cost at $v$. The original network links have a delay cost only. Then every agent needs to choose at most two edges from the node it resides on, to $t$.

**Results** For unweighted agents on metric networks we prove constant upper and lower bounds on the PoS, by analyzing the social cost increase caused by an iterative best response procedure. Strong equilibria do not always exist, but their SPoA is constant upper-bounded when they do. For weighted agents on general networks we prove that $\alpha$-approximate strong equilibria exist for $\alpha \geq e = 2.718\ldots$ (no subset deviation causes factor $\alpha$ improvement to all of its members), and that their SPoA is at most $\alpha(1+\ln W)$. This becomes $\Theta(\ln n)$ for unweighted agents on general networks. See [9] for additional results, omitted proofs and technical details. Refer to [2, 10, 11] for related work on facility location game models. [2] is similar to ours, but does not incorporate fair cost-sharing of facility costs. [11] specializes the model of [1], but does not incorporate delays.

**Definitions** The network will be a complete graph $G(V, E)$, having each edge $(u, v)$ associated to a non-negative cost $d(u, v)$. We consider a set $A$ of $n$ agents; agent $i$ resides on $u_i \in V$ and has a non-negative demand weight $w_i$. The strategy space of agent $i$ is $V$: $i$ chooses a location $v \in V$ to receive service from. Opening a facility at node $v \in V$ costs $\beta_v$. Denote a strategy profile (configuration) by $s = (s_1, \ldots, s_n)$, $s_i \in V$. We define $W_s(v) = \sum_{i:s_i=v} w_i$. The cost $c_i(s)$ experienced by agent $i$ in $s$ is $c_i(s) = w_i\left(d(u_i, s_i) + \frac{\beta_{s_i}}{W_s(s_i)}\right)$. Agent $i$ pays a fraction $\frac{w_i}{W_s(s_i)}$ of the facility installation cost at $s_i$. We denote facility locations specified in $s$ by $F_s \subseteq V$. The *social cost* $c(s)$ is:

$$c(s) = \sum_i c_i(s) = \sum_i w_i d(u_i, s_i) + \sum_i \frac{w_i \beta_{s_i}}{W_s(s_i)} = \sum_i w_i d(u_i, s_i) + \sum_{v \in F_s} \beta_v$$

We use $W(I)$ for the sum of weights of agents in the set $I$. $c_I(s)$ is the social cost of agents in $I$, and $c_v(s)$ is the social cost of agents connected to $v \in V$ under $s$.

The *unweighted* Facility Location game is a potential game [5] specializing the network design games of [1], and therefore has pure Nash equilibria reachable by iterative best response performed by the players. For unweighted agents the PoA of pure equilibria can be $n$, while the PoS upper bound of $H(n)$ ($n$-th harmonic number) from [1] is tight for non-metric networks [9].

**Definition 1.** *For $\alpha \geq 1$, a strategy profile $s$ is an $\alpha$-approximate strong equilibrium if no subset of agents can perform a pure deviation, and each of its members be better off by a factor more than $\alpha$. If $\alpha = 1$, $s$ is a strong equilibrium [3, 8].*

## 2 Unweighted Agents on Metric Networks

We analyze evolution of an equilibrium through iterative best response executed by the agents, when the initial configuration is the social optimum. The following lemma charges any specific agent $i$ a bounded amount of social cost increase during the algorithm's execution.

**Lemma 1.** *Let $A_{s^*}(v)$ be the subset of agents that are connected to $v$ in $s^*$. For any $i \in A_{s^*}(v)$ that deviates from $v$ during iterative best response let $A^i_{s^*}(v) \subseteq A_{s^*}(v)$ be the subset of agents that have not yet deviated from $v$ exactly before the first deviation of $i$. Then we can charge $i$ with a total increase contribution to the social cost at most $\beta_v/|A^i_{s^*}(v)|$, throughout the algorithm's execution.*

*Proof.* For simplicity let $|A^i_{s^*}(v)| = k_i(v)$. Clearly $i \in A^i_{s^*}(v)$. Let us analyze contribution of $i$ to social cost increase during its first deviation. By deviating $i$ reduces its individual cost from $c_i(v) = x_i(v) + \frac{\beta_v}{k_i(v)}$ to $c_i(v') = x_i(v') + \frac{\beta_{v'}}{\lambda_i(v')}$ by joining another facility node $v'$. $x_i(v)$ and $x_i(v')$ is the connection cost payed by $i$ before and after its first deviation. $\lambda_i(v')$ is the number of agents sharing facility cost at $v'$, including $i$. Since $c_i(v') < c_i(v)$, we get $x_i(v') - x_i(v) \leq \frac{\beta_v}{k_i(v)} - \frac{\beta_{v'}}{\lambda_i(v')}$. Let $\Delta sc_i(v)$ be the social cost difference caused by $i$. There are four cases:

1. $k_i(v) > 1$, $\lambda_i(v') > 1$: Then $\Delta sc_i(v) = x_i(v') - x_i(v) \leq \frac{\beta_v}{k_i(v)} - \frac{\beta_{v'}}{\lambda_i(v')}$.
2. $k_i(v) = 1$, $\lambda_i(v') > 1$: Then $\Delta sc_i(v) = -\beta_v + x_i(v') - x_i(v) \leq -\frac{\beta_{v'}}{\lambda_i(v')}$.
3. $k_i(v) > 1$, $\lambda_i(v') = 1$: Then $\Delta sc_i(v) = \beta_{v'} + x_i(v') - x_i(v) \leq \frac{\beta_v}{k_i(v)}$.
4. $k_i(v) = 1$, $\lambda_i(v') = 1$: Then $\Delta sc_i(v) = \beta_{v'} - \beta_v + x_i(v') - x_i(v) \leq 0$.

Clearly the above hold in general for any agent deviating from any node $v$ to any node $v'$. Now we implement a charging procedure along with iterative best response. Give all agents an initial label $l(i) = i$, before executing iterative best response. The current label $l(i)$ of $i$ will denote the agent to which an increase caused by $i$ is charged. Initialize $\Delta sc_{l(i)} = 0$. For every facility node $v \in F_{s^*}$ and every $i \in A_{s^*}(v)$ initialize $\lambda_{l(i)} = \lambda_i$ to a distinct value from $\{1, 2, \ldots, |A_{s^*}(v)|\}$. Charging is implemented by relabeling deviating agents in the following manner.

For an agent $i$ that deviates from node $v$ to node $v'$ set $k_{l(i)}(v)$ to the number of agents connected to $v$ exactly before deviation of $i$. If $k_{l(i)}(v) = \lambda_{l(i)}(v)$ no relabeling is needed. Otherwise there must be some $j \neq i$ connected to $v$ such that $\lambda_{l(j)}(v) = k_{l(i)}(v)$. In this case exchange labels of $i$ and $j$. Subsequently add the increase caused by deviation of $i$ to $\Delta sc_{l(i)}(v)$. Finally, set $\lambda_{l(i)}(v')$ equal to the number of agents connected to $v'$ right after $i$ has joined $v'$.

By the previous definitions it follows that if $k_{l(i)}(v) \neq \lambda_{l(i)}(v)$, then it is always $k_{l(i)}(v) > \lambda_{l(i)}(v)$, i.e. $i$ has joined $v$ before some agent $j$ with $\lambda_{l(j)}(v) = k_{l(i)}(v)$, but leaves $v$ before $i$ leaves. By exchanging labels of $i$,$j$ we add the increase caused by $i$ to the agent that previously labeled $j$. Possible increases in 1.,2.,3.,4., imply that any agent is charged by the end of iterative best response at most $\frac{\beta_v}{|A^i_{s^*}(v)|}$ for some $i$. Initializing $\lambda_i(v) = k_i(v)$ in $s^*$ charges exactly $i$. $\square$

**Note:** In the following we assume an order of agents, so that agents of the same facility in the initial configuration best-respond consecutively.

**Theorem 1.** *The Price of Stability for the unweighted metric Facility Location game is upper bounded by a constant, strictly less than* 2.36*.*

*Proof.* Let $\Delta sc_i$ denote the increase contributed by agent $i$ to the social optimum $c(s^*)$, during iterative best response initialized at $s^*$. Assume an order of agents, such that agents $i \in A_{s^*}(v)$ "best-respond" consecutively, for each $v \in F_{s^*}$. Define $c_v(s^*) = \beta_v + \sum_{i:s^*_i=v} d(u_i, v)$. Then $c(s^*) = \sum_{v \in F_{s^*}} c_v(s^*)$. We will upper bound the $PoS$ by $\max_{v \in F_{s^*}} \frac{c_v(s^*) + \sum_{i:s^*_i=v} \Delta sc_i}{c_v(s^*)}$. We focus on the *first* deviation of $i \in A_{s^*}(v)$, for any facility $v \in F_{s^*}$. Let $v'$ be the node that $i$ deviates to, and $\delta x^*_i = d(u_i, v') - d(u_i, v)$. We also use $x^*_i = d(u_i, v)$ for convenience. Let $\lambda_i$ be the number of agents serviced at $v'$ right after deviation of $i$. The new cost of $i$ right after its first deviation is: $d(u_i, v) + \delta x^*_i + \frac{\beta_{v'}}{\lambda_i}$. For a second agent $j \in A_{s^*}(v)$ deviating from $v$ to some node $v''$ after $i$, we have:

$$d(u_j, v) + \delta x^*_j + \frac{\beta_{v''}}{\lambda_j} \leq d(u_j, v') + \frac{\beta_{v'}}{\lambda_i} \tag{1}$$

Substitute $d(u_j, v')$ in (1) by triangle inequality: $d(u_j, v') \leq d(u_j, v) + d(u_i, v) + d(u_i, v')$. Also, by lemma 1 $\delta x^*_i + \frac{\beta_{v'}}{\lambda_i} \leq \frac{\beta_v}{k^*_i}$, where $k^*_i = |A^i_{s^*}(v)|$ ($A^i_{s^*}(v)$ is defined as in lemma 1). Thus:

$$d(u_i, v) \geq \frac{1}{2}\Big(\delta x^*_j - \delta x^*_i + \frac{\beta_{v''}}{\lambda_j} - \frac{\beta_{v'}}{\lambda_i}\Big) \geq \frac{1}{2}\Big(\delta x^*_j + \frac{\beta_{v''}}{\lambda_j} - \frac{\beta_v}{k^*_i}\Big) \tag{2}$$

The latter has to hold for every pair of distinct agents $i, j \in A_{s^*}(v)$, hence:

$$d(u_i, v) \geq \max\Big\{0, \frac{1}{2}\Big(\max_{j:s^*_j=v}\Big(\delta x^*_j + \frac{\beta_{v''}}{\lambda_j}\Big) - \frac{\beta_v}{k^*_i}\Big)\Big\} \tag{3}$$

We use (3) for the connection cost of agents in $A_{s^*}(v)$ under $s^*$, and consider two complementary cases: either some agents never deviate from $v$, or all of them do.

Let $n_{s^*}(v) = |A_{s^*}(v)|$. We only analyze the first case here (the second is similar - see [9]). If $r$ agents never deviate from $v$, then trivially in (3) we set $v'' = v$, $\delta x_j^* = 0$, and $\lambda_j = r$, whereas $k_i^* \geq r+1$. The cost $c_v(s^*)$ is:

$$c_v(s^*) \geq \beta_v + \frac{\beta_v}{2} \sum_{k=r+1}^{n_{s^*}(v)} \Big(\frac{1}{r} - \frac{1}{k}\Big) = \beta_v + \frac{\beta_v}{2} \Big(\frac{n_{s^*}(v) - r}{r} - H(n_{s^*}(v)) + H(r)\Big) \quad (4)$$

By lemma 1 it is $\sum_{i:s_i^*=v} \Delta sc_i \leq H(n_{s^*}(v)) - H(r)$. Using equality in (4) for $c_v(s^*)$, and $c_v(s) = c_v(s^*) + \sum_{i:s_i^*=v} \Delta sc_i$, we obtain the following ratio. Simplify using $\gamma + \ln m \leq H(m) \leq 1 + \ln m$ ($\gamma > 0.5$ is Euler's constant):

$$PoS \leq \frac{1 + \frac{1}{2}\Big(\frac{n_{s^*}(v)-r}{r} + H(n_{s^*}(v)) - H(r)\Big)}{1 + \frac{1}{2}\Big(\frac{n_{s^*}(v)-r}{r} - H(n_{s^*}(v)) + H(r)\Big)} \leq \frac{1.5 + \frac{n_{s^*}(v)}{r} + \ln \frac{n_{s^*}(v)}{r}}{0.5 + \frac{n_{s^*}(v)}{r} - \ln \frac{n_{s^*}(v)}{r}}$$

Let $y = \frac{n_{s^*}(v)}{r}$. The upper bound can be numerically maximized to $< 2.36$. □

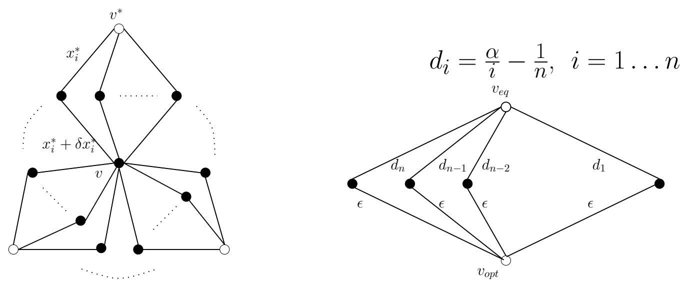
**Lower Bound** Take $2n$ agents; $n$ on a singe node $v$, the rest on a separate node each (black nodes in Fig. 1(a)). Facility costs are 1. In the social optimum $s^*$, $n$ agents on $v$ are serviced by $v$. The rest are equipartitioned to $v_l^*$, $l = 1 \ldots k$, $k = \sqrt{n}$. We analyze a single facility $v_l^*$, henceforth denoted by $v^*$ (same for the rest). By abusing notation, $c_{v^*}(s)$ is the cost of $v^*$-agents at equilibrium. Then:

$$PoS = \lim_{n \to \infty} \frac{1 + kc_{v^*}(s)}{1 + kc_{v^*}(s^*)} \geq \lim_{n \to \infty} \frac{c_{v^*}(s)}{(1/\sqrt{n} + c_{v^*}(s^*))} \quad (5)$$

In the least expensive equilibrium $s$, agents from each facility $v^*$ of $s^*$ miss-connect to $v$ in $s$. For some constant $p \in (0,1)$, only $r = \lceil (1-p)k \rceil$ of these agents increase the social cost significantly, by increasing their connection cost. Follow iterative best response of these $r$ agents starting from $s^*$. Assume $\lambda \geq n$ agents "play" $v$ before $r$ agents of $v^*$ deviate to $v$. Set the $i$-th deviating agent to increase its connection cost $x_i^*$ by $\delta x_i^* = \frac{1}{k-i+1} - \frac{1}{\lambda+i} - \epsilon$, $i = 1 \ldots r$; it decreases $c_i$ by $\epsilon$. By (1),(2), and because all $r$ agents deviate to $v$ (hence $\frac{\beta_{v''}}{\lambda_j} - \frac{\beta_{v'}}{\lambda_i} = 0$ in (2)), it is $x_i^* = d(u_i, v^*) = \max\{0, \frac{1}{2}(\max \delta x_j^* - \delta x_i^*)\}$, $\max \delta x_j^* = \delta x_r^*$. For the rest $k - r$ agents set $x_j^* = 0$, $d(u_j, v) = \frac{1}{k-r+1}$. Summing up as in (4) yields:

$$c_{v^*}(s^*) = 1 + \frac{1}{2}\Big[\frac{r}{k-r+1} - \Delta H(k, k-r)\Big] - \frac{1}{2}\frac{r}{n+r} + \frac{1}{2}\Delta H(\lambda + r, \lambda) \quad (6)$$

where $\Delta H(n,m) = H(n) - H(m)$. Then $\sum_{i=1}^r \delta x_i^* = \Delta H(k, k-r) - \Delta H(\lambda+r, \lambda)$, and *we can* set $c_{v^*}(s) = c_{v^*}(s^*) - 1 + \frac{k-r}{k-r+1} + \sum_{i=1}^r \delta x_i^*$ in (5). Simplify $\Delta H$ by logarithmic bounds and substitute $r = \lceil (1-p)k \rceil$ appropriately. Then limits of numerator and denominator in (5) exist (see [9] for details); the resulting simplified fraction can be maximized numerically to $> 1.45$ for $p \simeq 0.18$. Experimental




(a) (b)

Fig. 1: Lower bounds: (a) unweighted metric PoS, (b) unweighted non-metric SPoA.

evidence showed that $PoS > 1.77$. It is easy to verify that any configuration other than $s^*$ and $s$ is more expensive [9].

Strong equilibria do not always exist, even for unweighted agents on metric networks. We prove the following (see [9] for the proof, and for existence of 2.36-approximate strong equilibria with constant strong Price of Anarchy):

**Theorem 2.** *When strong equilibria exist in the unweighted metric Facility Location game, their Price of Anarchy is at most a constant.*

## 3 Approximate Strong Equilibria for Weighted Agents

The existence of pure equilibria for weighted agents is an open issue. We reduce the logarithmic approximation factor known for general network design [7, 6] to a constant. Our result is more general, as it concerns *strong* equilibria. We make use of the following remark.

*Remark 1.* If an instance of the Facility Location game does not have strong equilibria, then there is at least one cycle of deviations of particular coalitions that results in a circular sequence of configurations $\{s^j\}_{j=1}^{k}$ with $s^1 = s^k$.

Given such a sequence $\{s^j\}_{j=1}^{k}$, we denote the coalition that deviates from $s^j$ to form $s^{j+1}$ by $I_j$. Such a deviation causes a cost decrease of agents in $I_j$ and possibly a cost increase of agents in $A \setminus I_j$. Recall that $A$ is the set of all agents. We define two quantities, the *weighted improvement* $\mathrm{impr}(I_j)$ for agents in $I_j$ and the *weighted damage* $\mathrm{dam}(I_j)$ caused by agents in $I_j$ respectively:

$$\mathrm{impr}(I_j) = \prod_{i \in I_j} \left( \frac{c_i(s^j)}{c_i(s^{j+1})} \right)^{w_i} \qquad \mathrm{dam}(I_j) = \prod_{i \in A \setminus I_j} \left( \frac{c_i(s^{j+1})}{c_i(s^j)} \right)^{w_i}$$

We derive an approximation factor that eliminates cycles.

**Lemma 2.** *Let* $\{s^j\}_{j=1}^k$ *with* $s^1 = s^k$ *be a cycle of configurations in a Facility Location game instance, caused by consecutive deviations of coalitions. The game instance has an* $\alpha$*-approximate strong equilibrium if for all such sequences* $\alpha \geq \mathrm{dam}_{\max}(\{s^j\}_{j=1}^k)$*, where* $\mathrm{dam}_{\max}(\{s^j\}_{j=1}^k) = \max_{j=1\ldots k-1} \mathrm{dam}(I_j)^{1/W(I_j)}$.

*Proof.* If there is no $\alpha$-approximate strong equilibrium we know that there is at least one cycle $\{s^j\}_{j=1}^k$ such that $\forall j \in \{1,\ldots,k-1\} \forall i \in I_j : \frac{c_i(s^j)}{c_i(s^{j+1})} > \alpha$. Because $s^1 = s^k$ we have that $\prod_{j=1}^{k-1} \frac{c_i(s^j)}{c_i(s^{j+1})} = 1$ for every agent $i$. Then:

$$1 = \prod_{i=1}^{n} \left( \prod_{j=1}^{k-1} \frac{c_i(s^j)}{c_i(s^{j+1})} \right)^{w_i} = \prod_{j=1}^{k-1} \frac{\mathrm{impr}(I_j)}{\mathrm{dam}(I_j)} > \prod_{j=1}^{k-1} \frac{\alpha^{W(I_j)}}{\left(\mathrm{dam}(I_j)^{1/W(I_j)}\right)^{W(I_j)}}$$

It follows that $\mathrm{dam}_{\max}(\{s^j\}_{j=1}^k) > \alpha$. The lemma follows by contradiction. □

We derive an approximation factor as an upper bound of $\mathrm{dam}_{\max}(\{s^j\}_{j=1}^k)$ for any cycle.

**Theorem 3.** *For every* $\alpha \geq e$ *there exist* $\alpha$*-approximate strong equilibria in the Facility Location game.*

*Proof.* We prove that $\mathrm{dam}_{\max}(\{s^j\}_{j=1}^k) < e$ for every cycle $\{s^j\}_{j=1}^k$ of configurations and the result follows from Lemma 2. Let $I_j(v)$ be the set of agents going to $v$ in $s^j$, but not in $s^{j+1}$, and $A_j(v)$ be the set of agents going to $v$ in both $s^j$ and $s^{j+1}$. Note that $I_j = \bigcup_{v \in V} I_j(v)$, therefore:

$$\mathrm{dam}_{\max}(\{s^j\}_{j=1}^k) = \max_j \left( \prod_{v \in V} \left( \prod_{i \in A_j(v)} \left( \frac{c_i(s^{j+1})}{c_i(s^j)} \right)^{w_i} \right)^{\frac{W(I_j(v))}{W(I_j(v))}} \right)^{\frac{1}{W(I_j)}} \Rightarrow$$

$$\mathrm{dam}_{\max}(\{s^j\}_{j=1}^k) \leq \max_{j,v} \left( \prod_{i \in A_j(v)} \left( \frac{c_i(s^{j+1})}{c_i(s^j)} \right)^{w_i} \right)^{\frac{1}{W(I_j(v))}}$$

Hence, we need only consider what happens at the worst case node. For an agent $i$ in $A_j(v)$ we get that:

$$\frac{c_i(s^{j+1})}{c_i(s^j)} = \frac{w_i \left( d(u_i, v) + \frac{\beta_v}{W_{s^{j+1}}(v)} \right)}{w_i \left( d(u_i, v) + \frac{\beta_v}{W(I_j(v)) + W(A_j(v))} \right)} \leq 1 + \frac{W(I_j(v))}{W(A_j(v))}$$

It follows that:

$$\mathrm{dam}_{\max}(\{s^j\}_{j=1}^k) \leq \max_{j,v} \left( 1 + \frac{W(I_j(v))}{W(A_j(v))} \right)^{\frac{W(A_j(v))}{W(I_j(v))}} < \lim_{r \to \infty} \left( 1 + \frac{1}{r} \right)^r = e$$

□

**Corollary 1.** *The Facility Location game with non-uniform agent demands has $\alpha$-approximate pure strategy Nash equilibria for every $\alpha \geq e$.*

For the SPoA of $\alpha$-approximate strong equilibria we show [9]:

**Theorem 4.** *The Price of Anarchy of $\alpha$-approximate strong equilibria, for the Facility Location game is upper bounded tightly by $\alpha H(n)$ for unweighted and by $\alpha(1 + \ln W)$ for weighted agents, where $W$ is the sum of weights.*

Fig. 1(b) shows a tight (non-metric) example for $w_i = 1$. Facility opening costs are 1 and agents reside on $v_{opt}$. A single facility at $v_{eq}$ is the most expensive $\alpha$-approximate strong equilibrium, of cost $\alpha H(n)$: no coalition has incentive to deviate to $v_{opt}$, the sole optimum facility location of social cost 1, for $\epsilon = n^{-2} \to 0$

**Open Problems** Existence of pure equilibria for the weighted game merits further investigation. Extending our (unweighted) metric analysis of the PoS for weights (or proving a non-constant lower bound) appears to be quite challenging. This seems to apply for the lower bounding of the weighted SPoA on general networks as well. Lower bounding techniques of [7] do not seem applicable.

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