Thermodynamic limits of networks of rate (Hopfield, Wilson-Cowan) neurons
I: independent weights

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The Hopfield model

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Isolated rate neuron

The dynamics

\[ S := \begin{cases} 
  dV_t & = \ -\alpha V_t dt + \sigma dB_t, \ 0 \leq t \leq T \\
  \text{Law of } V_0 & = \mu_0,
\end{cases} \]

- \( B_t \) is a standard Wiener/Brownian process
- \( \sigma \) is the intensity of the noise
- \( \alpha \) controls how fast the neuron relaxes to the initial condition

The solution of \( S \) is

\[ V_t = \exp(-\alpha t) V_0 + \sigma \int_0^t \exp(\alpha (s - t)) dB_s. \]
Coupling $N$ neurons

\[
S^N(J, \chi) := \begin{cases}
    dV^i_t & = -\alpha V^i_t dt + \chi^i dt + \sigma dB^i_t + \sum_{j=1}^{N} J_{ij} f(V^j_t) dt \\
    \text{Law of } V_N(0) & = \mu_0^\otimes N
\end{cases}
\]

- $f$ is a sigmoid: Lipschitz continuous.
- The $\chi_i$s are i.i.d. $\mathcal{N}(\bar{\chi}, \sigma^2_\chi)$ Gaussian random variables: injected currents.
Coupling $N$ neurons

- The $J_{ij}$s are i.i.d $\mathcal{N}(\bar{J}/N, J^2/N)$ Gaussian random variables: synaptic weights
- The $B^i_t$s are independent standard Wiener/Brownian processes
- We note $(\Omega, \mathcal{A}, \gamma)$ the underlying probability space, e.g.
  $$\bar{\chi} = \int_{\Omega} \chi^i(\omega) \, d\gamma(\omega)$$
Well-posedness of the finite size problem

- The function $f$ is Lipschitz continuous, hence so is in $\mathbb{R}^N$ the drift term of $S^N(J, \chi)$, hence there is a unique solution to this set of $N$ coupled stochastic differential equations (SDEs).
- The solution $V_N(t) = (V_{1t}, \cdots, V_{Nt})$ is continuous on $[0, T]$. We note $\mathcal{T}$ the set $C([0, T]; \mathbb{R})$. 
Well-posedness of the finite size problem

- We note $P^N(J, \chi)$ the law of this solution on $[0, T]$.
- It is a (random) element of $\mathcal{P}(\mathcal{T}^N)$ where $\mathcal{T}^N$ is the set of elements $u_N = (u^1, \cdots, u^N)$ where each $u^i, i = 1, \cdots, N$ is an element of $\mathcal{T}$.
- The law of the solution to $S$ is an element of $\mathcal{P}(\mathcal{T})$ noted $P$. 
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Empirical measure

Define the empirical measure:

\[ \hat{\mu}_N(V_N) = \frac{1}{N} \sum_{i=1}^{N} \delta_{V^i} \]

It is a probability measure on \( T \)

Example:

\[ \int_{T} v_t v_s \hat{\mu}_N(V_N)(dv) = \frac{1}{N} \sum_{i=1}^{N} V^i_t V^i_s \]
Goal

To study the law of $\hat{\mu}_N(V_N)$ under $P^N(J, \chi)$
Consider $\Pi^N(J, \chi)$ the probability on $\mathcal{P}(\mathcal{T})$ defined by

$$\Pi^N(J, \chi)(B) = P^N(J, \chi)(\hat{\mu}_N(V_N) \in B) \quad \forall B \in \mathcal{B}(\mathcal{P}(\mathcal{T}))$$

This is too complicated.
Goal

We first study the law of $\hat{\mu}_N(V_N)$ under $Q^N$ the annealed law of $P^N(J, \chi)$ w.r.t. the synaptic weights $J$ and the injected currents $\chi$. Consider $\Pi^N$ the probability on $\mathcal{P}(T)$ defined by

$$\Pi^N(B) = Q^N(\hat{\mu}_N(V_N) \in B) = \int_{\Omega} P^N(J(\omega), \chi(\omega))(\hat{\mu}_N(V_N) \in B) d\gamma(\omega)$$
The Cameron-Martin-Girsanov theorem

**Theorem**

Assume $B_t = (B^i_t)_{i=1 \ldots N}$ is an $N$-dimensional Brownian defined on $(\Omega, \mathcal{A}, \mathcal{A}_t, P)$ and let $\Phi = (\Phi^1, \ldots, \Phi^N)$ be in $L^2([0, T], \mathbb{R}^N)$. Define

$$
\zeta_0^T = \int_0^T \langle \Phi(t), dB_t \rangle - \frac{1}{2} \int_0^T |\Phi(t)|^2 dt
$$

$$
\tilde{B}_t = B_t - \int_0^t \Phi(u) du
$$

$$
d\tilde{P}(\omega) = \exp(\zeta_0^T) dP(\omega)
$$

If $\tilde{P}(\Omega) = 1$ then $\tilde{B}_t$ is a $N$-dimensional Brownian motion on $(\Omega, \mathcal{A}, \tilde{P})$. 
The Cameron-Martin-Girsanov theorem

- \( \exp(\zeta_0^T) \) is the Radon-Nikodym derivative of \( \tilde{P} \) with respect to \( P \).
- \( \tilde{P} \) is said to be absolutely continuous with respect to \( P \)
  \[
  \tilde{P} \ll P
  \]
The Cameron-Martin-Girsanov theorem

A sufficient condition for $\tilde{P}(\Omega) = 1$ to hold is that there exists two positive constants $\mu$ and $C$ such that

$$\mathbb{E}\left[e^{\mu \|\Phi(t)\|^2}\right] \leq C \quad \forall t \in [0, T]$$
The Girsanov theorem

**Theorem**

Let \( B_t = (B^i_t)_{i=1}^N \) be an \( N \)-dimensional Brownian defined on \((\Omega, \mathcal{A}, \mathcal{A}_t, P)\). Let \( x(t) \) be the \( N \)-dimensional Itô process given by

\[
x(t) = x(0) + \int_0^t g(s) \, ds + \int_0^t h(s) \, dB(s)
\]

with \( g \) in \( L^2([0, T], \mathbb{R}^N) \), and \( h \) in \( L^2([0, T], \mathbb{R}^{N \times N}) \). Let \( \Phi \in L^2([0, T], \mathbb{R}^N) \). Let \( \tilde{B}_t \) and \( \tilde{P} \) be defined as before. If \( \tilde{P}(\Omega) = 1 \) then \( x(t) \) is still an Itô process on \((\Omega, \mathcal{A}, \mathcal{A}_t, \tilde{P})\). More precisely

\[
x(t) = x(0) + \int_0^t (g(s) + h(s)\Phi(s)) \, ds + \int_0^t h(s) \, d\tilde{B}(s)
\]
Annealed law

- Apply Girsanov theorem to $S^N(J, \chi)$ in order to obtain $N$ times the uncoupled system $S$.
- It follows that $P^N(J, \chi)$ is absolutely continuous w.r.t. $P^\otimes N$ and

$$
\frac{dP^N(J, \chi)}{dP^\otimes N} = \exp \left\{ \sum_{i=1}^{N} \left\{ \frac{1}{\sigma} \int_0^T \left( \chi_i + \sum_{j=1}^{N} J_{ij} f(V^j_t) \right) dB^i_t - \frac{1}{2\sigma^2} \int_0^T \left( \chi_i + \sum_{j=1}^{N} J_{ij} f(V^j_t) \right)^2 dt \right\} \right\}
$$

- The right hand side being a measurable function of $(J, \chi)$, $Q^N = \int_{\Omega} P^N(J(\omega), \chi(\omega)) \, d\omega$ is well-defined.
Topologies

We endow $\mathcal{T}$ with the topology of uniform convergence:

$$\|u\| = \sup_{t \in [0, T]} |u_t|, \ d(u, v) = \|u - v\|$$

and $\mathcal{P}(\mathcal{T})$ with the Wasserstein-1 distance

$$D(\mu, \nu) = \inf_{\xi} \int_{\mathcal{T} \times \mathcal{T}} d(u, v) \, d\xi(u, v),$$

where $\xi$ is a coupling between $\mu$ and $\nu$. 
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Fundamental relation: I

For fixed values of \((V_{N,t})\) consider the \(N\) random processes

\[ G_{t}^{N,i} = \chi_i + \sum_{j=1}^{N} J_{ij} f(V_{t}^j) \]

They are i.i.d. Gaussian processes with mean

\[ \mathbb{E} \left[ G_{t}^{N,i} \right] = \bar{\chi} + \frac{\bar{J}}{N} \sum_{j=1}^{N} f(V_{t}^j) = \bar{\chi} + \bar{J} \int_{\mathcal{T}} f(v_t) \hat{\mu}_N(V_N) (dv), \]

and covariance

\[ \text{cov}(G_{t}^{N,i}, G_{s}^{N,i}) = \sigma^2 \chi + \frac{J^2}{N} \sum_{j=1}^{N} f(V_{t}^j)f(V_{s}^j) = \sigma^2 \chi + J^2 \int_{\mathcal{T}} f(v_t)f(v_s) \hat{\mu}_N(V_N) (dv) \]
Fundamental relation: II

Notation: \( \gamma_{\hat{\mu}_N}(\nu_N) \) is the probability law on \( (\Omega, \mathcal{A}) \) under which the Gaussian processes \( G_{t,i}^N \) are i.i.d with the previous mean and covariance.
Fundamental relation: II

Lemma

\[
\frac{dQ^N}{dP \otimes N}(V_N) = \exp N \Gamma(\hat{\mu}_N(V_N)),
\]

where

\[
\Gamma(\hat{\mu}_N(V_N)) = \int \log \int \exp \left( \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(v) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 dt \right) d\gamma_{\hat{\mu}_N}(\omega) d\hat{\mu}_N(v),
\]

and

\[
\sigma dB_t(v) = dv_t + \alpha v_t dt
\]
Fundamental relation: III

Proof:
By Girsanov theorem

$$\frac{dQ^N}{dP \otimes N} = \int \exp \sum_{i=1}^{N} \left\{ \frac{1}{\sigma} \int_0^T G_t^{N,i}(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^{N,i}(\omega)^2 dt \right\} d\gamma_{\hat{\mu}_N(V_N)}(\omega)$$

i.i.d. of the $(G_t^{N,i})$:

$$\prod_{i=1}^{N} \int \exp \left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 dt \right\} d\gamma_{\hat{\mu}_N(V_N)}(\omega)$$
Fundamental relation: IV

Take the log:

\[
\sum_{i=1}^{N} \log \int \exp \left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 dt \right\} d\gamma \hat{\mu}_N(\omega) =
\]

\[
N \int \log \int \exp \left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(v) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 dt \right\} d\gamma \hat{\mu}_N(\omega) \hat{\mu}_N(dv)
\]
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Definition of $\Gamma$: I

Given $\mu \in \mathcal{P}(\mathcal{T})$ define

Mean

$$c_\mu(t) = \bar{\chi} + \bar{J} \int f(v_t) \, d\mu(v)$$

Covariance

$$K_\mu(t, s) = \sigma^2 \bar{\chi} + J^2 \int f(v_t)f(v_s) \, d\mu(v)$$

$\gamma_\mu$ is the probability law on $(\Omega, \mathcal{A})$ such that the process $G_t$ has mean $c_\mu$ and covariance $K_\mu$
Definition of $\Gamma$: II

Define $\Gamma(\mu)$ by

$$
\Gamma(\mu) = \int \log \int \exp \left( \frac{1}{\sigma} \int_0^T G_t(\omega) dB_t(v) - \frac{1}{2\sigma^2} \int_0^T G_t(\omega)^2 dt \right) d\gamma_\mu(\omega) d\mu(v)
$$

When $\mu \ll P$, $B_t$ is a semi-martingale under $\mu$ and the stochastic integral is well-defined.
Definition of the rate function

Mutual entropy:

\[ I(\mu | P) = \begin{cases} 
\int_{\mathcal{F}} \log \frac{d\mu}{dP} \, d\mu & \text{if } \mu \ll P \\
\infty & \text{otherwise}
\end{cases} \]

Rate function:

\[ H(\mu) = \begin{cases} 
I(\mu | P) - \Gamma(\mu) & \text{if } \mu \ll P \\
\infty & \text{otherwise}
\end{cases} \]
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Properties of $\Gamma$: I

**Lemma**

$\Gamma(\mu) \leq I(\mu|P)$

Hence $H \geq 0$
Properties of $\Gamma$: II

Lemma

$$\Gamma = \Gamma_1 + \Gamma_2$$

with

$$\Gamma_1(\mu) = \log \int \exp \left\{ -\frac{1}{2\sigma^2} \int_0^T G_t(\omega)^2 \, dt \right\} \, d\gamma_\mu(\omega) = -\frac{1}{2} \text{det}(\sigma \text{Id} + \bar{K}_\mu),$$

and

$$\Gamma_2(\mu) = \frac{1}{2} \left( \langle \bar{L}_\mu^T c_\mu, c_\mu \rangle_{L^2([0, T])} - \| c_\mu \|^2_{L^2([0, T])} \right) +$$

$$\int_0^T L_{\mu}^t(t, t) \, dt - \int_0^T \int_0^t (L_{\mu}^t(t, s))^2 \, ds \, dt$$
Properties of $\Gamma$: III

Proposition

$\Gamma_1$ and $\Gamma_2$ are bounded and Lipschitz continuous for the Wasserstein-1 distance.
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Large deviation principle: I

**Proposition**

*For all open sets* \( O \) *of* \( P(T) \)

\[
- \inf_{\mu \in O} H(\mu) \leq \liminf_{N \to \infty} \frac{1}{N} \log \Pi^N(O)
\]
Large deviation principle: II

Proposition (1)

*The sequence $\Pi^N$ is exponentially tight.*
Large deviation principle: III

Proposition (2)

For every compact set $F$ of $\mathcal{P}(T)$

$$\limsup_{N \to \infty} \frac{1}{N} \log \prod^N (F) \leq - \lim_{\mu \in F} H(\mu)$$
A tightness result

Proposition

For all $\varepsilon > 0$ there exists a compact $K_\varepsilon$ of $\mathcal{P}(\mathcal{T})$ such that

$$Q^N(\hat{\mu}_N \in K_\varepsilon) \leq \varepsilon$$

for all integers $N$.

Proof.

For any bounded continuous $f$ on $\mathcal{T}^N$

$$\int f \, dQ^N \leq I(Q^N|P^\otimes N) + \log \int \exp f \, dP^\otimes N$$

Choose $f = \log(1 + P^\otimes N(A)^{-1})\mathbf{1}_A$:

$$Q^N(A) \leq \frac{\log 2 + I(Q^N|P^\otimes N)}{\log(1 + P^\otimes N(A)^{-1})}$$
A tightness result

Because the law of the empirical measure under $P^\otimes N$ is exponentially tight we can find a compact $K_\varepsilon$ such that

$$P^\otimes N(\hat{\mu}_N \in K_\varepsilon^c) \leq \exp - \frac{N}{\varepsilon}$$

$$I(Q^N|P^\otimes N) = \int \log \frac{dQ^N}{dP^\otimes N} dQ^N = N \int \Gamma(\hat{\mu}_N) dQ^N \leq CN,$$

since $\Gamma$ is bounded. Hence

$$Q^N(\hat{\mu}_N \in K_\varepsilon^c) \leq \frac{\log 2 + NC}{\log(1 + \exp \frac{N}{\varepsilon})}$$
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Uniqueness and characterization of the minimum of $H$

**Theorem**

$H$ achieves its minimum ($=0$) at the unique point of $\mathcal{P}(\mathcal{T})$ given by

$$\left\{ Q \in \mathcal{P}(\mathcal{T}) \mid Q \ll P, \frac{dQ}{dP} = \right. \int \exp \left\{ \frac{1}{\sigma} \int_{0}^{T} G_s \, dB_s - \frac{1}{2\sigma^2} \int_{0}^{T} G_s^2 \, ds \right\} d\gamma Q \left. \right\}$$
Characterization of the minimum of $H$ as the solution of an SDE

In the case $c_{\mu}(t) = 0$, i.e. $\bar{J} = \bar{\chi} = 0$.

**Theorem**

$Q$ is the solution to the non-Markovian stochastic system defined on $[0, T]$ by

\[
(S) \left\{ \begin{array}{l}
V_t = V_0 - \alpha \int_0^t V_s \, ds + \sigma B_t \\
\sigma B_t = W_t + \int_0^t \int_0^s L_Q(s, u) \, dB_u \, ds \\
\text{Law of } V = Q, \quad Q_{|\mathcal{A}_0} = \mu_0
\end{array} \right.
\]

or

\[
dV_t = -\alpha V_t \, dt + \sigma dB_t
\]

\[
\sigma dB_t = \int_0^t L_Q^t(t, s) \, dB_s \, dt + dW_t,
\]

where $W_t$ is a $Q$-Brownian motion.
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Convergence of $\Pi^N$

**Theorem**

The law of the empirical measure $\hat{\mu}_N$ under $Q^N$ converges to $\delta_Q$.

**Proof.**

For all $\delta > 0$ prove, using propositions (1) and (2), that

$$
\lim_{N \to \infty} Q^N(\hat{\mu}_N \in B(Q, \delta)^c) = 0.
$$
Annealed propagation of chaos

Definition

$Q^N$ is said to be $Q$-chaotic if for all $m \geq 2$ and $f_i, \ i = 1, \cdots, m$ in $C_b(\mathcal{T})$

$$\lim_{N \to \infty} \int_{\mathcal{T}^N} f_1(v^1) \cdots f_m(v^m) \, dQ^N(v^1, \cdots, v^N) = \prod_{i=1}^m \int_{\mathcal{T}} f_i(v) \, dQ(v)$$

Theorem

$Q^N$ is $Q$-chaotic.

Proof.

This follows from lemma 3.1 in [Szn84].

Note that this has nothing to do with the notion of chaotic solutions to a system of ODEs.
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Thermodynamic limits
Extension to replicae

Let $r$ be an integer and consider $Q^{r,N}_N$ the annealed law of the replicated neuronal dynamics

$$Q^{r,N}_N = \int_{\Omega} P^N(J(\omega), \chi(\omega))^\otimes r \ d\omega$$

It is a probability law on $(\mathcal{T}^r)^N$. 
Definition of the empirical measure

\[ \hat{\mu}_N^r : (\mathcal{T}^r)^N \rightarrow \mathcal{P}(\mathcal{T}^r) \]

\[ (V_1^i, \cdots, V_r^i)_{1 \leq i \leq N} \rightarrow \frac{1}{N} \sum_{i=1}^{N} \delta_{V_1^i \cdots V_r^i} \]
Definition of the limit law

Define $Q_r$ to be

$$Q_r \ll P^r \frac{dQ_r}{dP^r} = \int \exp \left\{ \frac{1}{\sigma} \int_0^T \langle G_t, dB_t \rangle - \frac{1}{2\sigma^2} \int_0^T \| G_t \|^2 dt \right\} d\gamma_{Q_r},$$

where, under $\gamma_{Q_r}$ is an $r$-dimensional Gaussian process with mean

$$c_{Q_r}(t)^i = \bar{\chi} + \bar{J} \int f(v_t^i) dQ_r(v^1, \ldots, v^r),$$

and covariance

$$\text{cov}(G_t^i G_s^j) = \int f(v_t^i) f(v_s^j) dQ_r(v^1, \ldots, v^r)$$
Convergence of the empirical measure

Theorem

For any integer $r$, the law of the empirical measure $\hat{\mu}_N^r$ under $Q_r^N$ converges to $\delta_{Q_r}$. 
Propagation of chaos

As a consequence

**Theorem**

*Q_r,N* is *Q_r*-chaotic.

*For any bounded continuous functions* \((F_1, \cdots, F_m)\) *on* \(T^r\)

\[
\lim_{N \to \infty} \int F_1(v_1^1, \cdots, v_r^1) \cdots F_m(v_1^m, \cdots, v_r^m) \, dQ_r,N(v^1, \cdots, v^N) = \\
\prod_{i=1}^m \int F_i(v^1, \cdots, v^r) \, dQ_r(v^1, \cdots, v^r)
\]
Propagation of chaos

In particular, if $F_i(v^1, \cdots, v^r) = \prod_{k=1}^{r} f_i(v^k)$ for $m$ continuous bounded functions on $\mathcal{T}$

$$
\lim_{N \to \infty} \mathbb{E} \left[ \left( \int f_1(v^1) \cdots f_m(v^m) \, dP^N(J, \chi) \right)^r \right] = \\
\prod_{i=1}^{m} \int f_i(v^1) \cdots f_i(v^r) \, dQ_r
$$
Quenched results

Rewrite the equation for the dynamics of a single neuron $k$

$$dV^k_t = -\alpha V^k_t + \sigma dB^k_t + \sum_{i=1}^{N} J_{ki} \mathbb{E}_{P^N(J, \chi)}[f(V^i_t)] +$$

$$\sum_{i=1}^{N} J_{ki}(f(V^i_t) - \mathbb{E}_{P^N(J, \chi)}[f(V^i_t)])$$

Assume the $(J_{ki})$ are independent of the mean $\mathbb{E}_{P^N(J, \chi)}[f(V^i_t)]$. $\sum_{i=1}^{N} J_{ki} \mathbb{E}_{P^N(J, \chi)}[f(V^i_t)]$ is a Gaussian process with mean

$$\frac{\bar{J}}{N} \sum_{i=1}^{N} \mathbb{E}_{P^N(J, \chi)}[f(V^i_t)] = \bar{J} \int \langle f(v_t), \hat{\mu}_N(V_N) \rangle dP^N(J, \chi)(V_N)$$

which converges almost surely to $\mathbb{E}_Q[f(v_t)]$. We call $H$ the corresponding Gaussian process.
Quenched results

The covariance is given by

$$\frac{J^2}{N} \sum_{i=1}^{N} \mathbb{E}_{P^N(J,\chi)}[f(V^i_t)] \mathbb{E}_{P^N(J,\chi)}[f(V^i_s)] =$$

$$J^2 \int \langle f(v^1_t)f(v^2_s), \hat{\mu}_N^2 \rangle dP^N(J,\chi) \otimes 2$$

which converges almost surely toward $\mathbb{E}_{Q^2}[f(v^1_t)f(v^2_s)]$
Quenched results

Similarly one shows that \( \sum_{i=1}^{N} J_{ki} (f(V^i_t) - \mathbb{E}_{P^N(J,\chi)} [f(V^i_t)]) \) converges to a centered Gaussian process \( G \) with covariance

\[
\frac{1}{2} \mathbb{E}_{Q_2} [(f(v^1_t) - f(v^2_t))(f(v^1_s) - f(v^2_s))]
\]
External input

For \( g \in L^2([0, T]) \) note \( P(g) \) the law of the solution to the SDE

\[
dV_t = -\alpha V_t \, dt + \sigma dB_t + g(t) \, dt
\]

Note \( \mathbb{E}^g \) the expectation over \( g \) and define

\[
P_H = \mathbb{E}^G[P(G + H)],
\]
Relation $Q_r$ and $P_H$

**Theorem**

*For any integer $r$*

$$Q_r = \mathbb{E}^H[P_H^\otimes r]$$
Quenched result

Corollary

For any integer \( r \), for any continuous bounded functions \( f_1, \ldots, f_m \) on \( T \)

\[
\lim_{N \to \infty} \mathbb{E} \left[ \left( \int f_1(v^1) \cdots f_m(v^m) dP^N(J, \chi)(v) \right)^r \right] = \\
\prod_{i=1}^m \mathbb{E}^H \left[ \left( \int f_i(v) dP_H(v) \right)^r \right]
\]

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Thermodynamic limits
Quenched result

Since the random variables \( \int f_1(v^1) \cdots f_m(v^m) \, dP^N(J, \chi)(v) \) are bounded, this is equivalent to the convergence in law of these random variables.

**Theorem**

*For any continuous bounded functions \( f_1, \cdots, f_m \) on \( \mathcal{T} \), \( \int f_1(v^1) \cdots f_m(v^m) \, dP^N(J, \chi)(v) \) converges in law, when \( N \) goes to infinity, to \( \prod_{i=1}^m \int f_i(v) \, dP_{H_i}(v) \), where \( H_i \) are independent copies of the Gaussian process described above.*

This means that for all \( \varepsilon > 0 \)

\[
\lim_{N \to \infty} \gamma \left( \omega \left| \int f_1(v^1) \cdots f_m(v^m) \, dP^N(J(\omega), \chi(\omega))(v) - \prod_{i=1}^m \int f_i(v) \, dP_{H_i}(v) \right| \right) = 0
\]
$P_H$ as a solution to a SDE

\[
\begin{align*}
    dV_t &= -\alpha V_t \, dt + \sigma dB_t \\
    B_t &= W_t + \int_0^t ds \int_0^s L_{Q_2}^s(s, u)(dB_u - H_u \, du) + \int_0^t H_s \, ds
\end{align*}
\]
$P_H$ as a solution to a SDE

- The Law of $V_0$ is $\mu_0$.
- The law of $V_t$ is $P_H$.
- $Q_2 = \mathbb{E}^H[P_H^\otimes 2]$.
- $W_t$ is a Wiener process under $P_H$. 

$$
L^t_{Q_2}(s, u) = \mathbb{E}^G \left[ \begin{pmatrix} \exp - \frac{1}{2\sigma^2} \int_0^t G_s^2 \, ds \end{pmatrix} \begin{pmatrix} G_s \, G_u \end{pmatrix} \right]
$$

where $G$ is a centered Gaussian process with covariance $\frac{1}{2} \mathbb{E}_{Q_2}[(S(v_s^1) - S(v_s^2))(S(v_t^1) - S(v_t^2))]$.

- $H$ is a Gaussian process with mean $\mathbb{E}_{Q_1}[S(v_t)]$ and covariance $\mathbb{E}_{Q_2}[S(v_s^1)S(v_t^2)]$. 

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Thermodynamic limits
Complete annealing

Note that the result is in general random.

Proposition

\[ Q_r = Q_1^\otimes r \iff \int f(v_t) \, dQ_1(v) = 0 \quad \forall t \leq T \]

- Note that this cannot hold if \( f \) is the sigmoid between 0 and 1.
- It may hold for the sigmoid between -1 and 1 used by Crisanti, Sommers and Sompolinsky, [SCS88].
- In general for a "real" rate function (positive) the quenched limit is different from the annealed one.
Sanov’s theorem

Theorem
Let $V^k$ be an i.i.d. sequence of elements of $\mathcal{T}$ distributed as $P$. The law $\Pi_0^N$ of the empirical measure under $P^{\otimes N}$ satisfies a LDP with good rate function $\mu \rightarrow I(\mu | P)$, $\mu \in \mathcal{P}(\mathcal{T})$.

Moreover, the sequence $(\Pi_0^N)$ is exponentially tight: For each $L > 0$, there exists a compact $K_L$ of $\mathcal{P}(\mathcal{T})$ such that

$$\limsup_{N \to \infty} \frac{1}{N} \log \Pi_0^N(K_L^c) \leq -L$$
References I


References II


References III