Thermodynamic limits of networks of rate (Hopfield, Wilson-Cowan) neurons I: independent weights

Olivier Faugeras

MathNeuro Laboratory - INRIA Sophia/UNSA LJAD

ICMNS 2016
May 29, 2016

Isolated rate neuron

The dynamics

$$\mathcal{S} := \begin{cases} dV_t & = & -\alpha V_t dt + \sigma dB_t, \; 0 \leq t \leq T \\ \text{Law of } V_0 & = & \mu_0, \end{cases}$$

- B_t is a standard Wiener/Brownian process
- σ is the intensity of the noise
- α controls how fast the neuron relaxes to the initial condition

The solution of $\mathcal{S}$ is

$$V_t = \exp(-\alpha t) V_0 + \sigma \int_0^t \exp(\alpha(s-t)) dB_s.$$

Coupling N neurons

$$\mathcal{S}^N(J,\chi) :=$$
$$\begin{cases} dV_t^i & = & -\alpha V_t^i dt + \chi_i dt + \sigma dB_t^i + \sum_{j=1}^N J_{ij} f(V_t^j) dt \\ \text{Law of } V_N(0) & = & \mu_0^{\otimes N} \end{cases}$$

- f is a sigmoid: Lipschitz continuous.
- The χ_is are i.i.d. $\mathcal{N}(\bar{\chi}, \sigma_\chi^2)$ Gaussian random variables: injected currents.

Coupling N neurons

- The J_{ij}s are i.i.d $\mathcal{N}(\frac{\bar{J}}{N}, \frac{J^2}{N})$ Gaussian random variables: synaptic weights
- The B^i_ts are independent standard Wiener/Brownian processes
- We note $(\Omega, \mathcal{A}, \gamma)$ the underlying probability space, e.g.

$$\bar{\chi} = \int_\Omega \chi^i(\omega)\, d\gamma(\omega)$$

Well-posedness of the finite size problem

- The function f is Lipschitz continuous, hence
- so is in $\mathbb{R}^N$ the drift term of $\mathcal{S}^N(J, \chi)$, hence
- there is a unique solution to this set of N coupled stochastic differential equations (SDEs).
- The solution $V_N(t) = (V_t^1, \cdots, V_t^N)$ is continuous on $[0, T]$. We note $\mathcal{T}$ the set $\mathcal{C}([0, T]; \mathbb{R})$.

Well-posedness of the finite size problem

- We note $P^N(J,\chi)$ the law of this solution on $[0,T]$.
- It is a (random) element of $\mathcal{P}(\mathcal{T}^N)$ where $\mathcal{T}^N$ is the set of elements $u_N = (u^1, \cdots, u^N)$ where each u^i, $i = 1, \cdots, N$ is an element of $\mathcal{T}$.
- The law of the solution to $\mathcal{S}$ is an element of $\mathcal{P}(\mathcal{T})$ noted P.

Empirical measure

Define the empirical measure:

$$\hat{\mu}_N(V_N) = \frac{1}{N} \sum_{i=1}^{N} \delta_{V^i}$$

It is a probability measure on $\mathcal{T}$
Example:

$$\int_{\mathcal{T}} v_t v_s \, \hat{\mu}_N(V_N)(dv) = \frac{1}{N} \sum_{i=1}^{N} V_t^i V_s^i$$

Goal

To study the law of $\hat{\mu}_N(V_N)$ under $P^N(J,\chi)$
Consider $\Pi^N(J,\chi)$ the probability on $\mathcal{P}(\mathcal{T})$ defined by

$$\Pi^N(J,\chi)(B) = P^N(J,\chi)(\hat{\mu}_N(V_N) \in B) \ \forall B \in \mathcal{B}(\mathcal{P}(\mathcal{T}))$$

This is too complicated.

Goal

We first study the law of $\hat{\mu}_N(V_N)$ under Q^N the annealed law of $P^N(J, \chi)$ w.r.t. the synaptic weights J and the injected currents χ. Consider Π^N the probability on $\mathcal{P}(\mathcal{T})$ defined by

$$\Pi^N(B) = Q^N(\hat{\mu}_N(V_N) \in B) = \int_\Omega P^N(J(\omega), \chi(\omega))(\hat{\mu}_N(V_N) \in B)\, d\gamma(\omega)$$

The Cameron-Martin-Girsanov theorem

Theorem

Assume $B_t = (B_t^i)_{i=1\cdots N}$ *is an N-dimensional Brownian defined on* $(\Omega, \mathcal{A}, \mathcal{A}_t, P)$ *and let* $\Phi = (\Phi^1, \cdots, \Phi^N)$ *be in* $\mathbb{L}^2([0, T], \mathbb{R}^N)$. *Define*

$$\zeta_0^T = \int_0^T \langle \Phi(t), dB_t \rangle - \frac{1}{2} \int_0^T |\Phi(t)|^2 \, dt$$

$$\tilde{B}_t = B_t - \int_0^t \Phi(u) \, du$$

$$d\tilde{P}(\omega) = \exp(\zeta_0^T) dP(\omega)$$

If $\tilde{P}(\Omega) = 1$ *then* $\tilde{B}_t$ *is a N-dimensional Brownian motion on* $(\Omega, \mathcal{A}, \tilde{P})$.

The Cameron-Martin-Girsanov theorem

- $\exp(\zeta_0^T)$ is the Radon-Nikodym derivative of $\tilde{P}$ with respect to P.
- $\tilde{P}$ is said to be absolutely continuous with respect to P

$$\tilde{P} << P$$

The Cameron-Martin-Girsanov theorem

A sufficient condition for $\tilde{P}(\Omega) = 1$ to hold is that there exists two positive constants μ and C such that

$$\mathbb{E}\left[e^{\mu\|\Phi(t)\|^2}\right] \leq C \ \forall t \in [0, T]$$

The Girsanov theorem

Theorem

Let $B_t = (B_t^i)_{i=1\cdots N}$ be an N-dimensional Brownian defined on $(\Omega, \mathcal{A}, \mathcal{A}_t, P)$. Let $x(t)$ be the N-dimensional Itô process given by

$$x(t) = x(0) + \int_0^t g(s)\, ds + \int_0^t h(s) dB(s)$$

with g in $\mathbb{L}^2([0, T], \mathbb{R}^N)$, and h in $\mathbb{L}^2([0, T], \mathbb{R}^{N\times N})$. Let $\Phi \in \mathbb{L}^2([0, T], \mathbb{R}^N)$. Let $\tilde{B}_t$ and $\tilde{P}$ be defined as before. If $\tilde{P}(\Omega) = 1$ then $x(t)$ is still an Itô process on $(\Omega, \mathcal{A}, \mathcal{A}_t, \tilde{P})$. More precisely

$$x(t) = x(0) + \int_0^t (g(s) + h(s)\Phi(s))\, ds + \int_0^t h(s) d\tilde{B}(s)$$

Annealed law

- Apply Girsanov theorem to $\mathcal{S}^N(J,\chi)$ in order to obtain N times the uncoupled system $\mathcal{S}$.
- It follows that $P^N(J,\chi)$ is absolutely continuous w.r.t. $P^{\otimes N}$ and

$$\frac{dP^N(J,\chi)}{dP^{\otimes N}} = \exp \sum_{i=1}^N \left\{ \frac{1}{\sigma} \int_0^T \left(\chi_i + \sum_{j=1}^N J_{ij} f(V_t^j) \right) dB_t^i - \frac{1}{2\sigma^2} \int_0^T \left(\chi_i + \sum_{j=1}^N J_{ij} f(V_t^j) \right)^2 dt \right\}$$

- The right hand side being a measurable function of (J,χ), $Q^N = \int_\Omega P^N(J(\omega),\chi(\omega))\, d\omega$ is well-defined.

Topologies

We endow $\mathcal{T}$ with the topology of uniform convergence:

$$\|u\| = \sup_{t\in[0,T]} |u_t|, \; d(u,v) = \|u-v\|$$

and $\mathcal{P}(\mathcal{T})$ with the Wasserstein-1 distance

$$D(\mu,\nu) = \inf_{\xi} \int_{\mathcal{T}\times\mathcal{T}} d(u,v)\, d\xi(u,v),$$

where ξ is a coupling between μ and ν.

Fundamental relation: I

For fixed values of $(V_{N,t})$ consider the N random processes

$$G_t^{N,i} = \chi_i + \sum_{j=1}^{N} J_{ij} f(V_t^j)$$

They are i.i.d. Gaussian processes with mean

$$\mathbb{E}\left[G_t^{N,i}\right] = \bar{\chi} + \frac{\bar{J}}{N}\sum_{j=1}^{N} f(V_t^j) = \bar{\chi} + \bar{J}\int_{\mathcal{T}} f(v_t)\,\hat{\mu}_N(V_N)(dv),$$

and covariance

$$\mathrm{cov}(G_t^{N,i} G_s^{N,i}) = \sigma_\chi^2 + \frac{J^2}{N}\sum_{j=1}^{N} f(V_t^j) f(V_s^j) =$$

$$\sigma_\chi^2 + J^2 \int_{\mathcal{T}} f(v_t) f(v_s)\,\hat{\mu}_N(V_N)(dv)$$

Fundamental relation: II

Notation: $\gamma_{\hat{\mu}_N(V_N)}$ is the probability law on $(\Omega, \mathcal{A})$ under which the Gaussian processes $G_t^{N,i}$ are i.i.d with the previous mean and covariance.

Fundamental relation: II

Lemma

$$\frac{dQ^N}{dP^{\otimes N}}(V_N) = \exp N\Gamma(\hat{\mu}_N(V_N)),$$

where

$$\Gamma(\hat{\mu}_N(V_N)) = \int \log \int \exp\left(\frac{1}{\sigma}\int_0^T G_t^N(\omega)\, dB_t(v) - \frac{1}{2\sigma^2}\int_0^T G_t^N(\omega)^2\, dt\right) d\gamma_{\hat{\mu}_N}(\omega)\, d\hat{\mu}_N(v),$$

and

$$\sigma dB_t(v) = dv_t + \alpha v_t dt$$

Fundamental relation: III

Proof:
By Girsanov theorem

$$\frac{dQ^N}{dP^{\otimes N}} =$$

$$\int \exp \sum_{i=1}^{N} \left\{ \frac{1}{\sigma} \int_0^T G_t^{N,i}(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^{N,i}(\omega)^2 \, dt \right\} d\gamma_{\hat{\mu}_N(V_N)}(\omega$$

i.i.d. of the $(G_t^{N,i})$:

$$\Pi_{i=1}^N \int \exp \left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2 \, dt \right\} d\gamma_{\hat{\mu}_N(V_N)}(\omega)$$

Fundamental relation: IV

Take the log:

$$\sum_{i=1}^{N} \log \int \exp\left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(V^i) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2\, dt \right\} d\gamma_{\hat{\mu}_N}(\omega) =$$

$$N \int \log \int \exp\left\{ \frac{1}{\sigma} \int_0^T G_t^N(\omega) dB_t(v) - \frac{1}{2\sigma^2} \int_0^T G_t^N(\omega)^2\, dt \right\} d\gamma_{\hat{\mu}_N}(\omega)\, \hat{\mu}_N(dv)$$

Definition of Γ: I

Given $\mu \in \mathcal{P}(\mathcal{T})$ define
Mean

$$c_\mu(t) = \bar{\chi} + \bar{J} \int f(v_t)\, d\mu(v)$$

Covariance

$$K_\mu(t,s) = \sigma_\chi^2 + J^2 \int f(v_t) f(v_s)\, d\mu(v)$$

γ_μ is the probability law on $(\Omega, \mathcal{A})$ such that the process G_t has mean c_μ and covariance K_μ

Definition of Γ: II

Define $\Gamma(\mu)$ by

$$\Gamma(\mu) = \int \log \int \exp\left(\frac{1}{\sigma}\int_0^T G_t(\omega)\, dB_t(v) - \frac{1}{2\sigma^2}\int_0^T G_t(\omega)^2\, dt\right) d\gamma_\mu(\omega)\, d\mu(v)$$

When $\mu << P$, B_t is a semi-martingale under μ and the stochastic integral is well-defined.

Definition of the rate function

Mutual entropy:

$$I(\mu|P) = \begin{cases} \int_{\mathcal{T}} \log \frac{d\mu}{dP}\, d\mu \text{ if } \mu << P \\ \infty \text{ otherwise} \end{cases}$$

Rate function:

$$H(\mu) = \begin{cases} I(\mu|P) - \Gamma(\mu) \text{ if } \mu << P \\ \infty \text{ otherwise} \end{cases}$$

Properties of Γ: I

Lemma

$\Gamma(\mu) \leq I(\mu|P)$

Hence $H \geq 0$

Properties of Γ: II

Lemma

$$\Gamma = \Gamma_1 + \Gamma_2$$

with

$$\Gamma_1(\mu) = \log \int \exp\left\{-\frac{1}{2\sigma^2}\int_0^T G_t(\omega)^2\,dt\right\} d\gamma_\mu(\omega) = -\frac{1}{2}\det(\sigma \mathrm{Id} + \bar{K}_\mu),$$

and

$$\Gamma_2(\mu) = \frac{1}{2}\left(\langle \bar{L}_\mu^T c_\mu, c_\mu\rangle_{L^2([0,T])} - \|c_\mu\|^2_{L^2([0,T])}\right) + \int_0^T L_\mu^t(t,t)\,dt - \int_0^T\int_0^t (L_\mu^t(t,s))^2\,ds\,dt$$

Properties of Γ: III

Proposition

Γ_1 and Γ_2 are bounded and Lipschitz continuous for the Wasserstein-1 distance.

Large deviation principle: I

Proposition

For all open sets $\mathcal{O}$ of $\mathcal{P}(\mathcal{T})$

$$-\inf_{\mu\in\mathcal{O}} H(\mu) \leq \liminf_{N\to\infty} \frac{1}{N} \log \Pi^N(\mathcal{O})$$

Large deviation principle: II

Proposition (1)

The sequence Π^N is exponentially tight.

Large deviation principle: III

Proposition (2)

For every compact set F of $\mathcal{P}(\mathcal{T})$

$$\limsup_{N\to\infty} \frac{1}{N} \log \Pi^N(F) \leq - \lim_{\mu \in F} H(\mu)$$

A tightness result

Proposition

For all $\varepsilon > 0$ there exists a compact K_ε of $\mathcal{P}(\mathcal{T})$ such that

$$Q^N(\hat{\mu}_N \in K_\varepsilon^c) \leq \varepsilon$$

for all integers N.

Proof.

For any bounded continuous f on $\mathcal{T}^N$

$$\int f\, dQ^N \leq I(Q^N | P^{\otimes N}) + \log \int \exp f\, dP^{\otimes N}$$

Choose $f = \log(1 + P^{\otimes N}(A)^{-1})\mathbf{1}_A$:

$$Q^N(A) \leq \frac{\log 2 + I(Q^N | P^{\otimes N})}{\log(1 + P^{\otimes N}(A)^{-1})}$$

A tightness result

Because the law of the empirical measure under $P^{\otimes N}$ is exponentially tight we can find a compact K_ε such that

$$P^{\otimes N}(\hat{\mu}_N \in K_\varepsilon^c) \leq \exp -\frac{N}{\varepsilon}$$

$$I(Q^N | P^{\otimes N}) = \int \log \frac{dQ^N}{dP^{\otimes N}}\, dQ^N = N \int \Gamma(\hat{\mu}_N)\, dQ^N \leq CN,$$

since Γ is bounded. Hence

$$Q^N(\hat{\mu}_N \in K_\varepsilon^c) \leq \frac{\log 2 + NC}{\log(1 + \exp \frac{N}{\varepsilon})}$$

Uniqueness and characterization of the minimum of H

Theorem

H achieves its minimum $(=0)$ at the unique point of $\mathcal{P}(\mathcal{T})$ given by

$$\left\{ Q \in \mathcal{P}(\mathcal{T}) \,|\, Q << P,\ \frac{dQ}{dP} = \int \exp\left\{ \frac{1}{\sigma} \int_0^T G_s\, dB_s - \frac{1}{2\sigma^2} \int_0^T G_s^2\, ds \right\} d\gamma_Q \right\}$$

Characterization of the minimum of H as the solution of an SDE

In the case $c_\mu(t) = 0$, i.e. $\bar{J} = \bar{\chi} = 0$.

Theorem

Q is the solution to the non-Markovian stochastic system defined on $[0, T]$ by

$$(\mathcal{S}) \begin{cases} V_t & = & V_0 - \alpha \int_0^t V_s \, ds + \sigma B_t \\ \sigma B_t & = & W_t + \int_0^t \int_0^s L_Q^s(s, u) \, dB_u \, ds \\ \text{Law of } V & = & Q, \; Q_{|\mathcal{A}_0} = \mu_0 \end{cases}$$

or

$$dV_t = -\alpha V_t \, dt + \sigma dB_t$$

$$\sigma dB_t = \int_0^t L_Q^t(t, s) \, dB_s \, dt + dW_t,$$

where W_t is a Q-Brownian motion.

Convergence of Π^N

Theorem

The law of the empirical measure $\hat{\mu}_N$ under Q^N converges to δ_Q.

Proof.

For all $\delta > 0$ prove, using propositions (1) and (2), that $\lim_{N\to\infty} Q^N(\hat{\mu}_N \in B(Q,\delta)^c) = 0$. □

Annealed propagation of chaos

Definition

Q^N is said to be Q-chaotic if for all $m \geq 2$ and f_i, $i = 1, \cdots, m$ in $C_b(\mathcal{T})$

$$\lim_{N \to \infty} \int_{\mathcal{T}^N} f_1(v^1) \cdots f_m(v^m) \, dQ^N(v^1, \cdots, v^N) = \Pi_{i=1}^m \int_{\mathcal{T}} f_i(v) \, dQ(v)$$

Theorem

Q^N is Q-chaotic.

Proof.

This follows from lemma 3.1 in [Szn84]. □

Note that this has nothing to do with the notion of chaotic solutions to a system of ODEs.

Extension to replicae

Let r be an integer and consider $Q^{r,N}$ the annealed law of the replicated neuronal dynamics

$$Q^{r,N} = \int_{\Omega} P^N(J(\omega), \chi(\omega))^{\otimes r}\, d\omega$$

It is a probability law on $(\mathcal{T}^r)^N$.

Definition of the empirical measure

$$
\begin{aligned}
\hat{\mu}_N^r \; : \; (\mathcal{T}^r)^N & \rightarrow \mathcal{P}(\mathcal{T}^r) \\
(V_1^i, \cdots, V_r^i)_{1 \leq i \leq N} & \rightarrow \frac{1}{N} \sum_{i=1}^N \delta_{V_1^i \cdots V_r^i}
\end{aligned}
$$

Definition of the limit law

Define Q_r to be

$$Q_r << P^{\otimes r} \; \frac{dQ_r}{dP^{\otimes r}} = \int \exp\left\{ \frac{1}{\sigma} \int_0^T \langle G_t,\, dB_t \rangle - \frac{1}{2\sigma^2} \int_0^T \| G_t \|^2 \, dt \right\} d\gamma_{Q_r},$$

where, under γ_{Q_r} is an r-dimensional Gaussian process with mean

$$\mathbf{c}_{Q_r}(t)^i = \bar{\chi} + \bar{J} \int f(v_t^i) \, dQ_r(v^1, \cdots, v^r),$$

and covariance

$$\mathrm{cov}(G_t^i G_s^j) = \int f(v_t^i) f(v_s^j) \, dQ_r(v^1, \cdots, v^r)$$

Convergence of the empirical measure

Theorem

For any integer r, the law of the empirical measure $\hat{\mu}_N^r$ under $Q^{r,N}$ converges to δ_{Q_r}.

Propagation of chaos

As a consequence

Theorem

$Q^{r,N}$ is Q_r-chaotic.

For any bounded continuous functions $(F_1, \cdots, F_m)$ on $\mathcal{T}^r$

$$\lim_{N\to\infty} \int F_1(v_1^1, \cdots, v_r^1) \cdots F_m(v_1^m, \cdots, v_r^m)\, dQ^{r,N}(\mathbf{v}^1, \cdots, \mathbf{v}^N) = \Pi_{i=1}^m \int F_i(v^1, \cdots, v^r)\, dQ_r(v^1, \cdots, v^r)$$

Propagation of chaos

In particular, if $F_i(v^1, \cdots, v^r) = \Pi_{k=1}^r f_i(v^k)$ for m continuous bounded functions on $\mathcal{T}$

$$\lim_{N\to\infty} \mathbb{E}\left[\left(\int f_1(v^1)\cdots f_m(v^m)\, dP^N(J,\chi)\right)^r\right] = \Pi_{i=1}^m \int f_i(v^1)\cdots f_i(v^r)\, dQ_r$$

Quenched results

Rewrite the equation for the dynamics of a single neuron k

$$dV_t^k = -\alpha V_t^k + \sigma dB_t^k + \sum_{i=1}^{N} J_{ki} \mathbb{E}_{P^N(J,\chi)}[f(V_t^i)] + \sum_{i=1}^{N} J_{ki}(f(V_t^i) - \mathbb{E}_{P^N(J,\chi)}[f(V_t^i)])$$

Assume the (J_{ki}) are independent of the mean $\mathbb{E}_{P^N(J,\chi)}[f(V_t^i)]$. $\sum_{i=1}^N J_{ki} \mathbb{E}_{P^N(J,\chi)}[f(V_t^i)]$ is a Gaussian process with mean

$$\frac{\bar{J}}{N} \sum_{i=1}^{N} \mathbb{E}_{P^N(J,\chi)}[f(V_t^i)] = \bar{J} \int \langle f(v_t), \hat{\mu}_N(V_N) \rangle \, dP^N(J,\chi)(V_N)$$

which converges almost surely to $\mathbb{E}_Q[f(v_t)]$. We call H the corresponding Gaussian process.

Quenched results

The covariance is given by

$$\frac{J^2}{N}\sum_{i=1}^{N}\mathbb{E}_{P^N(J,\chi)}[f(V_t^i)]\mathbb{E}_{P^N(J,\chi)}[f(V_s^i)] = J^2\int\langle f(v_t^1)f(v_s^2),\hat{\mu}_N^2\rangle\, dP^N(J,\chi)^{\otimes 2}$$

which converges almost surely toward $\mathbb{E}_{Q_2}[f(v_t^1)f(v_s^2)]$

Quenched results

Similarly one shows that $\sum_{i=1}^N J_{ki}(f(V_t^i) - \mathbb{E}_{P^N(J,\chi)}[f(V_t^i)])$ converges to a centered Gaussian process G with covariance $\frac{1}{2}\mathbb{E}_{Q_2}[(f(v_t^1) - f(v_t^2))(f(v_s^1) - f(v_s^2))]$

External input

For $g \in \mathbb{L}^2([0, T])$ note $P(g)$ the law of the solution to the SDE

$$dV_t = -\alpha V_t \, dt + \sigma dB_t + g(t)dt$$

Note $\mathbb{E}^g$ the expectation over g and define

$$P_H = \mathbb{E}^G[P(G + H)],$$

Relation Q_r and P_H

Theorem

For any integer r

$$Q_r = \mathbb{E}^H[P_H^{\otimes r}]$$

Quenched result

Corollary

For any integer r, for any continuous bounded functions $f_1, \cdots, f_m$ on $\mathcal{T}$

$$\lim_{N \to \infty} \mathbb{E}\left[\left(\int f_1(v^1) \cdots f_m(v^m)\, dP^N(J, \chi)(v)\right)^r\right] = \Pi_{i=1}^m \mathbb{E}^H\left[\left(\int f_i(v)\, dP_H(v)\right)^r\right]$$

Quenched result

Since the random variables $\int f_1(v^1)\cdots f_m(v^m)\,dP^N(J,\chi)(v)$ are bounded, this is equivalent to the convergence in law of these random variables.

Theorem

For any continuous bounded functions $f_1,\cdots,f_m$ on $\mathcal{T}$, $\int f_1(v^1)\cdots f_m(v^m)\,dP^N(J,\chi)(v)$ converges in law, when N goes to infinity, to $\Pi_{i=1}^m \int f_i(v)\,dP_{H_i}(v)$, where H_i are independent copies of the Gaussian process described above.

This means that for all $\varepsilon > 0$

$$\lim_{N\to\infty} \gamma\left(\omega \,\middle|\, \left| \int f_1(v^1)\cdots f_m(v^m)\,dP^N(J(\omega),\chi(\omega))(v) - \Pi_{i=1}^m \int f_i(v)\,dP_{H_i}(v) \right|\right) = 0$$

P_H as a solution to a SDE

$$\begin{cases} dV_t & = & -\alpha V_t \, dt + \sigma dB_t \\ B_t & = & W_t + \int_0^t ds \int_0^s L^s_{Q_2}(s,u)(dB_u - H_u \, du) + \int_0^t H_s \, ds \end{cases}$$

P_H as a solution to a SDE

- The Law of V_0 is μ_0.
- The law of V_t is P_H.
- $Q_2 = \mathbb{E}^H[P_H^{\otimes 2}]$
- W_t is a Wiener process under P_H.
- $$L^t_{Q_2}(s,u) = \mathbb{E}^G\left[\left\{\frac{\exp -\frac{1}{2\sigma^2}\int_0^t G_s^2\,ds}{\mathbb{E}^G\left[\exp -\frac{1}{2\sigma^2}\int_0^t G_s^2\,ds\right]}\right\} G_s G_u\right]$$

  where G is a centered Gaussian process with covariance $\frac{1}{2}\mathbb{E}_{Q_2}[(S(v_s^1) - S(v_s^2))(S(v_t^1) - S(v_t^2))]$.
- H is a Gaussian process with mean $\mathbb{E}_{Q_1}[S(v_t)]$ and covariance $\mathbb{E}_{Q_2}[S(v_s^1)S(v_t^2)]$.

Complete annealing

Note that the result is in general random.

Proposition

$Q_r = Q_1^{\otimes r}$ *iff* $\int f(v_t)\, dQ_1(v) = 0 \ \forall t \leq T$

- Note that this cannot hold if f is the sigmoid between 0 and 1.
- It may hold for the sigmoid between -1 and 1 used by Crisanti, Sommers and Sompolinsky, [SCS88].
- In general for a "real" rate function (positive) the quenched limit is different from the annealed one.

Sanov's theorem

Theorem

Let V^k be an i.i.d. sequence of elements of $\mathcal{T}$ distributed as P. The law Π_0^N of the empirical measure under $P^{\otimes N}$ satisfies a LDP with good rate function $\mu \to I(\mu|P)$, $\mu \in \mathcal{P}(\mathcal{T})$.

Moreover, the sequence (Π_0^N) is exponentially tight:
For each $L > 0$, there exists a compact K_L of $\mathcal{P}(\mathcal{T})$ such that

$$\limsup_{N\to\infty} \frac{1}{N} \log \Pi_0^N(K_L^c) \leq -L$$

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