Exo 1

1a/ $X_t = sh(W_t + t) = f(t, B_t)$

$$dX_t = \partial_t f\, dt + \partial_x f\, dB_t + \frac{1}{2} \partial^2_{xx} f\, (\overset{\overset{dt}{\shortparallel}}{dB_t})^2$$

$$= \Big[\underbrace{ch(W_t + t)}_{\sqrt{1+X_t^2}} + \frac{1}{2} X_t\Big] dt + \underbrace{ch(W_t + t)}_{\sqrt{1+X_t^2}}\, dB_t$$

1b/ $X_t = sh(W_t + t + Ash(x))$ est l'unique solution.

2/ $X_t = 0$ unique solut de l'EDS

3/ $d(s(X_t)) = s'\, dX_t + \frac{1}{2} s''\, dt = \left(s'F + \frac{s''}{2} G^2\right) dt + s' G\, dW_t$

CS pour martingale : $s'F + \frac{s''}{2} G^2 = 0$

$$s'' + \frac{\cancel{2} x}{\cancel{2}\, 1+x^2} s' = 0 \qquad s' = s^0 \cancel{\exp} \frac{1}{x\sqrt{1+x^2}}$$

$$s'(x) = A \frac{1}{\sqrt{1+x^2}} \qquad S = B\, Ash(x) + B$$

$$d(s(X_t)) = s' G\, dW_t = dW_t$$

$$Y_t = Y_0 + \cancel{W}B_t = Ash(X_t) - Ash(x)$$

d'où $X_t = sh(B_t + Ash(x))$

$\boxed{\text{Exo 3}}$

On suppose :

$T_a = \inf\{t > 0 \,/\, B_t = a\}$ $\qquad$ $T_a < \infty$ ps

$Z_t = \exp(\theta B_t - \theta^2/2t)$ martingale

Alors $\mathbb{E}(Z_{t \wedge T_a}) = \mathbb{E}(Z_0) = 1 \quad \forall t$

$$\mathbb{E} \exp\left(\theta B_{t \wedge T_a} - \frac{\theta^2}{2} t \wedge T_a\right) = 1 \quad \forall \theta \ (\geq 0)$$

Comme $B_{t \wedge T_a} \leq a$ on a :

$$\exp\left(\frac{-\theta^2 t \wedge T_a}{2}\right) \exp(\theta B_{t \wedge T_a}) \leq \exp(\theta a)$$

$$- \lim_{t \to \infty} Z_{t \wedge T_a} = \mathbb{1}_{\{T_a < \infty\}} e^{\theta B_{T_a} - \frac{\theta^2}{2} T_a} + \mathbb{1}_{\{T_a = +\infty\}} \times 0$$

TCVD : $\mathbb{E}\left(1(T_a < \infty)\, e^{\theta B_{T_a} - \frac{\theta^2}{2} T_a}\right) = 1$

ie $\mathbb{E}\left(1(T_a < \infty)\, e^{-\frac{\theta^2}{2} T_a}\right) = e^{-\theta a}$

Alors $P(T_a < \infty) = 1$

et $\mathbb{E}\left(e^{-\frac{\theta^2}{2} T_a}\right) = e^{-\theta a}$

et $p(T_a \in dt) = \dfrac{|a|}{\sqrt{2\pi t^3}} e^{-a^2/2t}$

Rappel :
(cours)

$$\begin{cases} \Delta u = 0 & \Omega \\ u = 1 & \Gamma_1 \\ u = 0 & \Gamma_2 \end{cases}$$

Alors $\forall x \in \mathcal{U}$ $u(x)$ est la probabilité que le MB commençant en x touche Γ_1 avant Γ_2

$$\Gamma_1 = \partial B(0, R_1) \qquad \Gamma_2 = \partial B(0, R_2)$$

On sait $\Delta \phi = 0$

$$\Phi = \frac{\alpha}{\|x\|^{n-2}} - \beta\alpha \quad \text{alors } \Delta\phi = 0$$

$$\Phi|_{\Gamma_1} = \frac{\alpha}{R_1^{n-2}} - \beta\alpha = 1$$

$$\Phi|_{\Gamma_2} = \frac{\alpha}{R_2^{n-2}} - \beta\alpha = 0$$

Alors $\alpha = \dfrac{1}{R_2^{2-n} - R_1^{2-n}}$; $\beta = \dfrac{1}{R_2^{2-n}}$

Proba :
$$\frac{R_1^{2-n} - |x_0|^{2-n}}{R_1^{2-n} - R_2^{2-n}}$$

$$f = \| \ \|$$

$$\frac{\partial f}{\partial x_i} = \frac{x_i}{f} \qquad \frac{\partial^2 f}{\partial x_i^2} = \frac{1}{f} - \frac{x_i^2}{f^3}$$

$$df(W_t) = \sum_i \frac{\partial f}{\partial x_i} dW_t^i + \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} dW_t^i dW_t^j$$

$$= \sum_i \frac{W_t^i \, dW_t^i}{R_t} + \sum_i \frac{\partial^2 f}{\partial x_i^2} dt$$

$$= \sum_i \frac{W_t^i \, dW_t^i}{R_t} + \sum_i \left(\frac{1}{R_t} - \frac{X_i^2}{R_t^3} \right) dt$$

$$= \sum_i \frac{W_t^i \, dW_t^i}{R_t} + \frac{n-1}{R_t} dt$$

Stratonovich

① $$\int_0^t W(s)\circ dW(s) = \int_0^t W\,dW_s + \frac{1}{2}t$$

Or dans le cas de $\mathcal{L}^2[0,t)$, on a par définition de l'intégrale de Ito : $\lim_{n} \sum_{i=1}^{p_n} W(t_i^n)(W_{t_{i+1}^n} - W_{t_i^n}) = \int_0^t W_s\,dW_s$

de m̂ pour la variation quadratique

$$\lim_{n\to\infty} \sum_{i=1}^{p_n} \left(W_{t_{i+1}^n} - W_{t_i^n}\right)^2 = t$$

Alors $$\int_0^t W_s\,dW_s + \frac{t}{2} = \sum_{i=0}^{p_n} \left(W_{t_{i+1}^n} - W_{t_i^n}\right)\frac{W_{t_{i+1}^n} + W_{t_i^n}}{2}$$
$$= \int_0^t W(s)\circ dW(s)$$

② $$dF(X_t) = F'(X_t)\,dX_t + \frac{1}{2}F''(X_t)\,\sigma^2 dt$$

on doit estimer $\int_0^t F'(X_s)\circ dX_s$, donc on applique Ito à $F'(X_t)$:

$$dF'(X_t) = F'(X_t)\,dX_t + \frac{1}{2}F'''(X_t)\,\sigma^2 dt$$

$$\int_0^t F'(X_s)\circ dX_s \equiv \underbrace{\int_0^t F'(X_s)\,dX_s + \frac{1}{2}\int_0^t F''\sigma^2\,ds}_{F(X_t)-F(X_0)}$$

(Formule de Ito)

③ $$X_t = X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s$$
$$= X_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\circ dW_s - \frac{1}{2}\int_0^t \sigma'(X)\sigma(X)\,d$$

ie

$$dX = b(X)dt + \sigma(X)dW \quad \text{ssi}$$

$$dX = \left[b(X) - \frac{\sigma'(X)\sigma(X)}{2}\right]dt + \sigma(X)\circ dW$$

Feynman Kac

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

① $$Y_t = e^{-\lambda t} v_\lambda(t, X_t) = f(t, X_t)$$

$$dY_t = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2 dt$$

$$= \left[-\lambda e^{-\lambda t} v_\lambda + \frac{\partial v_\lambda}{\partial x} b(X_t) e^{-\lambda t} + \frac{1}{2}\sigma^2 e^{-\lambda t}\frac{\partial^2 v}{\partial x^2} + e^{-\lambda t}\frac{\partial v_\lambda}{\partial t}\right]$$

$$+ e^{-\lambda t}\frac{\partial v_\lambda}{\partial x}\sigma dW_t$$

$$= e^{-\lambda t}\underbrace{\left[\mathcal{L} v_\lambda + \frac{\partial v_\lambda}{\partial t} - \lambda v_\lambda\right]}_{=0}dt + \sqcup dW_t$$

donc Y_t est une martingale

② $$\mathbb{E}_x[Y_t] = \mathbb{E}_x[Y_0] = v_\lambda(0, x)$$

$$= \mathbb{E}\left[\mathbb{E}(Y_t \mid \mathcal{F}_{t\wedge\tau_a}) \mid X_0 = x\right]$$

Or Y_t est une martingale $t\wedge\tau_a$ est un t.a.

$0 \leq \tau_a \wedge t \leq t < \infty$ donc

$$\mathbb{E}(Y_t \mid \mathcal{F}_{t\wedge\tau_a}) = Y_{t\wedge\tau_a}$$

et $$\mathbb{E}_x[Y_0] = \mathbb{E}[Y_{t\wedge\tau_a} \mid X_0 = x]$$

$\left(t\wedge\tau_a \xrightarrow[t\infty]{} \tau_a \text{ ps}\right)$ $$= \mathbb{E}[Y_{\tau_a} \mid X_0 = x]$$ th d'arrê

$t\wedge\tau_a \leq \tau_a$

$\leq t$